Lectures on Algebraic Geometry

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(Notes taken by Hang Yin)

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Preface

These lecture notes are for my graduate course Algebra Geometry in Fall 2020 at the University of the Chinese Academy of Sciences. The lectures were given in the Morningside Center of Mathematics. In addition to the original sources and classical textbooks, I have been much influenced by a course taught by Luc Illusie in Spring 2004 and the Stacks Project [SP].

These notes owe their existence to one student, Hang Yin, who painstakingly typed up a first draft. I am deeply indebted to him. I thank Yirong Hu, Luc Illusie, and Jiahao Niu for corrections and suggestions.

Chapters 3 through 5 and the end of Section 2.5 have not been checked for accuracy and are not included in the version online.

Weizhe Zheng
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Chapter 1

Schemes

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References:

(1) Atiyah-MacDonald [AM]

(2) Matsumura [M2, M1]

(3) Hartshorne [H], Ch. 2–4.

(4) Liu Qing [L], Ch. 1–7.

(5) Fu Lei [F]

(6) EGA [G, GD]

(7) Stacks Project [SP]

1.1 Algebraic subsets

Let k be an algebraically closed field, $\mathbb{A}^n(k) = \{(a_1, \ldots, a_n) \in k^n\}$ the affine n-space. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring.

Notation 1.1.1. For $f \in R$, $Z(f) = \{P \in \mathbb{A}^n(k), f(P) = 0\}$. Similarly, for $T \subseteq R$, $Z(T) = \{P \in \mathbb{A}^n(k) \colon f(P) = 0, \forall f \in T\}$.

Remark 1.1.2. $Z(T) = Z(I)$, where I is the ideal generated by T.

Theorem 1.1.3 (Hilbert Basis)**.** *R is a Noetherian ring.*

Remark 1.1.4. Every ideal $I \subseteq R$ is finitely generated. For $I = (f_1, \ldots, f_m)$, we have $Z(I) = \bigcap_i Z(f_i)$.

Proposition 1.1.5. *Some properties:*

(1) If $I_1 \subseteq I_2$, then $Z(I_1) \supseteq Z(I_2)$.

(2) $Z(\sum_i I_i) = \bigcap Z(I_i)$.

(3) $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$.

(4) $Z(0) = R$, $Z(R) = \emptyset$.

Proof. 3: it is easy to see $Z(I_1 \cap I_2) \supseteq Z(I_1) \cup Z(I_2)$. We see $I_1 I_2 \subseteq I_1 \cap I_2$. Thus we have $Z(I_1 \cap I_2) \subseteq Z(I_1 I_2)$. Let $P \notin Z(I_1), P \notin Z(I_2)$. By definition, we have $f \in I_1, f(P) \neq 0, g \in I_2, g(P) \neq 0$. But $fg \in I_1 I_2$ and $f(P)g(P) \neq 0$. □

Definition 1.1.6 (Zariski Topology and Algebraic Subsets)**.** The above properties ensure that the $Z(I)$ are the closed subsets of a topology on $\mathbb{A}^n(k)$, called the **Zariski topology**. Closed subsets in $\mathbb{A}^n(k)$ are called **algebraic subsets**.

Example 1.1.7. (1) $\mathbb{A}^0(k) = \text{pt}$.

(2) Consider $\mathbb{A}^1(k)$. Since $k[x]$ is a PID, every ideal $I = (f)$. Then $Z(I)$ consists of the roots of f. Therefore algebraic subsets $\subsetneq \mathbb{A}^1(k)$ are precisely finite subsets. This topology is called the **cofinite topology**. Since k is infinite, this topology is not Hausdorff.

(3) In $\mathbb{A}^n(k)$, every point is closed. $P = (a_1, ..., a_n)$ is defined by the ideal $\mathfrak{m}_P = (x_1 - a_1, ..., x_n - a_n)$. Actually $\mathfrak{m}_P$ is a maximal ideal. This shows $\mathbb{A}^n(k)$ is a T_1 space.

(4) Consider $\mathbb{A}^2(k)$. As a set, it is in bijection with $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$. But its Zariski topology does not agree with the product topology. For example $Z(x - y)$ is closed in $\mathbb{A}^2(k)$ but not in the product topology.

Notation 1.1.8. For $Y \subseteq \mathbb{A}^n(k)$, $I(Y) = \{f \in R : f(P) = 0, \forall P \in Y\}$. It is the same thing as $\bigcap_{P \in Y} \mathfrak{m}_P$.

Proposition 1.1.9. *Properties of* $I(Y)$*:*

(1) If $Y_1 \subseteq Y_2$*, then* $I(Y_1) \supseteq I(Y_2)$*;*

(2) $I(\bigcup_i Y_i) = \bigcap_i I(Y_i)$*;*

(3) $Z(I(Y)) = \overline{Y}$ *(closure in* $\mathbb{A}^n(k)$ *for the Zariski topology).*

Proof. 3: It is clear $Z(I(Y)) \supseteq Y$, hence contains its closure. Suppose $Y \subseteq Z(I)$, then $\forall f \in I$, f is zero on Y, thus $I \subseteq I(Y)$, hence $Z(I) \supseteq Z(I(Y))$. Hence $Z(I(Y))$ is the closure of Y. □

Theorem 1.1.10 (Hilbert's Nullstellensatz)**.** *For each ideal* I*, we have* $I(Z(I)) = \sqrt{I}$.

Corollary 1.1.11. *There is an order-reversing one-to-one correspondence between algebraic subsets of* $\mathbb{A}^n(k)$ *and radical ideals of* R*, given by*

$$\begin{aligned} Y &\mapsto I(Y) \\ Z(I) &\leftarrow\!\shortmid I \end{aligned}$$

Corollary 1.1.12. *Every maximal ideal in R has the form $\mathfrak{m}_P$ for some $P \in \mathbb{A}^n(k)$.*

Corollary 1.1.13. *For every ideal I in R, we have $\sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.*

A ring satisfying the above property is called a **Jacobson ring**.

Lemma 1.1.14. *Let K be a field (not necessarily algebraic closed). Let E be a finitely generated K-algebra. Suppose E is also a field. Then E/K is a finite field extension.*

For a proof, see [AM, Corollary 5.24].

Proof of the Nullstellensatz. We have $I(Z(I)) \supseteq \sqrt{I}$. Suppose $f \notin \sqrt{I}$, then in R_f, IR_f is not unit ideal. Choose a maximal ideal $\mathfrak{m} \supseteq IR_f$, then $R_f/\mathfrak{m}$ is a finitely generated k-algebra which is also a field, hence $R_f/\mathfrak{m} = k$. Now $R \to R_f \to R_f/\mathfrak{m} = k$ defines a ring homomorphism and a maximal ideal $\mathfrak{n}_P$ such that $\mathfrak{n}_P \supseteq I$ and $f \notin \mathfrak{n}_P$, hence $P \in I(Z(I))$ but f is not zero on P. Thus $f \notin I(Z(I))$. □

Notation 1.1.15. For a ring A, we let $\mathrm{Max}(A)$ denote the set of all maximal ideals of A. This set is called the **maximal spectrum** of A.

From the Nullstellensatz, there are bijections

$$\begin{aligned} \mathbb{A}^n(k) &\simeq \mathrm{Max}(R) \\ Z(I) &\simeq \mathrm{Max}(R/I) \end{aligned}$$

We find that an algebraic subset is in bijection with the maximal spectrum of a finitely generated k-algebra.

1.2 Spectrum of a ring

For a ring homomorphism $f\colon A \to B$, there is no natural map $\mathrm{Max}(B) \to \mathrm{Max}(A)$ in general. The pull back of a maximal ideal is a prime ideal but not necessarily maximal. This shows the maximal spectrum behaves badly.

Notation 1.2.1. For a ring A, we let $\mathrm{Spec}(A)$ denote the set of all prime ideals of A. We call $\mathrm{Spec}(A)$ the **(prime) spectrum** of A.

Every ring homomorphism $\phi\colon A \to B$ induces a map

$$\begin{aligned} \mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) &\to \mathrm{Spec}(A) \\ \mathfrak{p} &\mapsto \phi^{-1}(\mathfrak{p}) \end{aligned}$$

Notation 1.2.2. For $T \subseteq A$, let $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(A) : T \subseteq \mathfrak{p}\}$. For $f \in A$, let $D(f) = \mathrm{Spec}(A) \backslash V(f)$.

Remark 1.2.3. $V(T) = V(I)$, where I is the ideal generated by T.

Proposition 1.2.4. *(1) For $I_1 \subseteq I_2$, we have $V(I_1) \supseteq V(I_2)$;*

(2) $V(\sum_i I_i) = \bigcap_i V(I_i)$;

(3) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.

Proof. 3. It is clear $V(I_1 \cap I_2) \supseteq V(I_1) \cup V(I_2)$. Also, $V(I_1 I_2) \supseteq V(I_1 \cap I_2)$. Consider a prime ideal $\mathfrak{p}$ not in $V(I_1) \cup V(I_2)$, that is $I_1 \subsetneq p, I_2 \subsetneq \mathfrak{p}$, then there exists $f \in I_1, g \in I_2$ such that $f, g \notin \mathfrak{p}$, hence $fg \in I_1 I_2$ but $fg \notin \mathfrak{p}$. This shows $\mathfrak{p} \notin V(I_1 I_2)$. □

We equip $\mathrm{Spec}(A)$ with the topology for which the closed subsets are exactly subsets of the form $V(I)$. We call it the **Zariski topology**.

Notation 1.2.5. For $Y \subseteq \mathrm{Spec}(A)$, let $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. This is an ideal of A.

Proposition 1.2.6. *Properties of $I(Y)$:*

(1) If $Y_1 \subseteq Y_2$, then $I(Y_1) \supseteq I(Y_2)$

(2) $I(\bigcup_i Y_i) = \bigcap_i I(Y_i)$*;*

(3) $V(I(Y)) = \overline{Y}$*;*

(4) for an ideal I, $I(V(I)) = \sqrt{I}$.

Corollary 1.2.7. *There is an order-reversing one-to-one correspondence between closed subsets of* $\mathrm{Spec}(A)$ *and radical ideals of A, given by*

$$\begin{aligned} Y &\mapsto I(Y) \\ V(I) &\leftarrow\!\shortmid I \end{aligned}$$

Moreover, the closed points of $\mathrm{Spec}(A)$ *are the maximal ideals of A.*

Example 1.2.8. (1) $A = 0 \Leftrightarrow \mathrm{Spec}(A) = \emptyset$.

(2) Let k be a field. Then $\mathrm{Spec}(k) = \mathrm{pt}$.

(3) $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$. The closed points are of the form (f), where f is an irreducible polynomials. The generic point is (0). Closed subsets are either the whole space or a finite set of closed points. It is not even a T_1 space, but it is a T_0 space.

(4) Consider $\mathrm{Spec}(\mathbb{Z})$. The closed points are of the form (p), where p is a prime number. The generic point is (0). The topology is similar to that of $\mathbb{A}^1_k$.

Corollary 1.2.9. $\mathrm{Spec}(A)$ *is quasi-compact (namely, every open cover has a finite subcover).*

Proof. Suppose $\bigcap V(I_i) = \emptyset$. Then $\sqrt{\sum I_i} = A$, thus $1 \in \sum I_i$, hence there are some $i_1, \ldots, i_n$ such that $1 = a_1 + \cdots + a_n$ where $a_j \in I_{i_j}$, hence $I_{i_1}, \ldots, I_{i_n}$ generate A, hence $V(I_{i_1}) \cap \cdots \cap V(I_{i_n}) = \emptyset$. □

Notation 1.2.10. Let $I^e = IB$ and $J^c = \phi^{-1}(J)$ denote the extension and contraction ideals with respect to certain ring homomorphism ϕ.

Lemma 1.2.11. *Let* $\phi\colon A \to B$ *be a ring homomorphism and* $f = \operatorname{Spec}(\phi)\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. *Then*

(1) $f^{-1}(V(I)) = V(I^e)$ *for every ideal* I *of* A*;*

(2) $\overline{f(V(J))} = V(J^c)$ *for every ideal* J *of* B.

Proof. (1) For $\mathfrak{q} \in \operatorname{Spec}(B)$, $f(\mathfrak{q}) \in V(I)$ means $\mathfrak{q}^c \supseteq I$, which is equivalent to $\mathfrak{q} \supseteq I^e$. Hence $f^{-1}(V(I)) = V(I^e)$.

(2) We have $I(f(V(J))) = \bigcap_{J\subseteq\mathfrak{q}} \phi^{-1}(\mathfrak{q}) = \phi^{-1}(\bigcap_{J\subseteq\mathfrak{q}} \mathfrak{q}) = \phi^{-1}(\sqrt{J}) = \sqrt{\phi^{-1}(J)}$. Applying V, we get $\overline{f(V(J))} = V(J^c)$.

□

Proposition 1.2.12. *Let* $\phi\colon A \to B$ *be a ring homomorphism. Then* $\operatorname{Spec}(\phi)\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is continuous.*

Proof. This follows immediately from Part 1 of the above lemma. □

Example 1.2.13. (1) For I an ideal in A, the quotient map $\pi\colon A \to A/I$ induces to $\operatorname{Spec}(\pi)\colon \operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$, which is a closed embedding.

(2) Suppose S is a multiplicative subsets in A. The localization map $\phi\colon A \to S^{-1}A$ induces $\operatorname{Spec}(\phi)\colon \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$, which is also an embedding.

Lemma 1.2.14. *Let* $\phi\colon A \to B$ *be a ring homomorphism. Then* $\operatorname{Spec}(\phi)$ *identifies* $\operatorname{Spec}(B)$ *with a subspace of* $\operatorname{Spec}(A)$ *if and only if every ideal* $J \in B$ *satisfies* $\sqrt{J} = \sqrt{J^{ce}}$.

Proof. It is easy to see that $f = \operatorname{Spec}(\phi)$ is an embedding if and only if $f^{-1}(\overline{f(F)}) = F$ for every closed subset $F \subset \operatorname{Spec}(B)$, which translates to the corresponding condition on ideals. □

It is easy to verify that $A \to A/I$ and $A \to S^{-1}A$ satisfy the condition in the lemma.

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Recall we define $\mathrm{Spec}(A)$ as prime ideals of A with Zariski topology. A basis $D(f) = \{p \in \mathrm{Spec}(A) | f \notin p\}, f \in A$.
For a ring homomorphism $\phi \colon A \to B$, we have

$$\begin{aligned} \phi^* \colon \mathrm{Spec}(B) &\to \mathrm{Spec}(A) \\ \mathfrak{q} &\mapsto \mathfrak{q}^c \end{aligned}$$

Some examples: $\pi \colon A \to A/I$ and $\phi \colon A \to S^{-1}A$, where S is a multiplicative system in A. The image of π^* is $V(I)$ and the image of ϕ^* is $\bigcap_{f \in S} D(f)$. If A is an integral domain, $S = A\backslash\{0\}$, then $S^{-1}A = \mathrm{Frac}(A)$. $\mathrm{Spec}(\mathrm{Frac}(A))$ is a point, and its image in $\mathrm{Spec}(A)$ is the generic point of $\mathrm{Spec}(A)$, given by the zero ideal of A.

Example 1.2.15. Consider

$$\begin{tikzcd} & k[x,y] & \\ k[x] \arrow[ur] & & k[y] \arrow[ul] \end{tikzcd}$$

We have

$$\begin{tikzcd} & \mathrm{Spec}(k[x,y]) = \mathbb{A}_k^2 \arrow[dl] \arrow[dr] & \\ \mathrm{Spec}(k[x]) = \mathbb{A}_k^1 & & \mathrm{Spec}(k[y]) = \mathbb{A}_k^1 \end{tikzcd}$$

This defines a continuous map $\mathbb{A}_k^2 \to \mathbb{A}_k^1 \times^{\mathrm{top}} \mathbb{A}_k^1$ (product space). This is surjective but not injective, since the points (0) and $(x-y)$ in $\mathrm{Spec}(k[x,y])$ both map to $((0),(0))$ in $\mathbb{A}_k^1 \times^{\mathrm{top}} \mathbb{A}_k^1$.

Example 1.2.16 (Tangent Space)**.** Consider $k[x_1, \dots, x_n]/(f_1, \dots, f_m), I = (f_1, \dots, f_m)$. Let $P = (a_1, \dots, a_n) \in Z(I)$, then $, \forall i, f_i(P) = 0$. Consider $\mathfrak{m}_P = (x_1 - a_1, \dots, x_n - a_n)/I$. Define

$$T_P = \{(t_1, \dots, t_n) \in k^n \mid \sum_i \frac{\partial f}{\partial x_i}(P) t_i = 0\}$$

which is a linear subspace of k^n.

We can write it in algebraic form. Let $k[\epsilon]/(\epsilon^2) = \{a + b\epsilon, a, b \in k\}$. Consider the diagram of rings

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, dashed, "\phi"] & A/\mathfrak{m}_P = k \\ & k[\epsilon]/\epsilon^2 \arrow[u, two heads, "\rho"] \end{tikzcd}$$

where ρ sends ϵ to 0 and $\psi(x_i) = a_i$. A homomorphism ϕ is defined by $\phi(x_i) = a_i + t_i\epsilon$. It factors through I if and only if

$$0 = f_j(a_1 + t_1\epsilon, ..., a_n + t_n\epsilon) = f_j(a_1, ..., a_n) + \sum_i \frac{\partial f}{\partial x_i}(P) t_i \epsilon = \sum_i \frac{\partial f}{\partial x_i}(P) t_i \epsilon$$

in $k[\epsilon]/(\epsilon^2)$. This is the same thing as a tangent vector defined above. Thus we have a bijection

$$T_P \simeq \{\phi : A \to k[\epsilon]/(\epsilon^2) \mid \rho\phi = \psi\}.$$

However, T_P cannot be read off from the induced maps of topological spaces:

$$\begin{tikzcd} \mathrm{Spec}(A) & \mathrm{Spec}(k) \arrow[l, "\psi^*"'] \arrow[d, "\rho^*"] \\ & \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \arrow[ul, dashed, "\phi^*"] \end{tikzcd}$$

Indeed, ρ^* is a homeomorphism.

1.3 Sheaves

Let X be a topological space, $\mathcal{C}$ a category.

Definition 1.3.1. Let $\mathrm{Open}(X) = (\{\text{open subsets of } X\}, \subseteq)$. It is a poset and can be viewed as a category: there is a unique morphism $U \to V$ if $U \subseteq V$ and no such morphism otherwise.

(1) A **presheaf** on X with values in $\mathcal{C}$ is a contravariant functor $\mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$. Denote $\mathrm{PShv}(X, \mathcal{C}) = \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C})$.

(2) A morphism between two sheaf $\mathcal{F}, \mathcal{G}$ is a natural transformation $\phi\colon \mathcal{F} \to \mathcal{G}$.

In details, a presheaf $\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ consists of $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$ for each open sets U, and $\rho_{UV}\colon \mathcal{F}(V) \to \mathcal{F}(U)$ a morphism (called **restriction**) in $\mathcal{C}$ for $U \subseteq V$. We require them to satisfy:

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$,

(2) For $U \subseteq V \subseteq W$, we have $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$

A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ consists of $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U that satisfies for $U \subseteq V$:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[u, "\rho^{\mathcal{F}}_{UV}"] & \mathcal{G}(V) \arrow[u, "\rho^{\mathcal{G}}_{UV}"'] \end{tikzcd}$$

In the rest of this section we assume that $\mathcal{C}$ is Set, Ab, or Ring. Elements of $\mathcal{F}(U)$ are called **sections**. For the morphism $\rho_{UV}\colon \mathcal{F}(V) \to \mathcal{F}(U), s \in \mathcal{F}(V)$, we sometimes write $s|_U$ for $\rho_{UV}(s) \in \mathcal{F}(U)$.

Definition 1.3.2. A **sheaf** is a presheaf $\mathcal{F}$ satisfying the following gluing property: $\forall U \subseteq X$ open, $\{U_i\}$ an open cover of U,

$$\mathcal{F}(U) \longrightarrow \textstyle\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram. The latter two maps are induced respectively by two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$.

In other words, $\forall s_i \in \mathcal{F}(U_i)$, if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ then $\exists! s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Note that uniqueness is equivalent to the injectivity of $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$. A presheaf satisfying the uniqueness is called **separated**.

Remark 1.3.3. Consider the empty set $\emptyset$. The empty cover is a cover of $\emptyset$. Now by definition, empty product is the terminal object and the equalizer of a pair of endomorphism of the terminal object is terminal. This shows that for any sheaf $\mathcal{F}$, $\mathcal{F}(\emptyset)$ is a terminal object of $\mathcal{C}$.

Example 1.3.4. Let X, Y be topological spaces. Then $\mathcal{F}_Y(U) = \mathrm{Map}_{\mathrm{cont}}(U, Y)$ defines a presheaf $\mathcal{F}_Y$ on X. It is easy to see this is also a sheaf.

(1) If Y is discrete, then $Y_X = \mathcal{F}_Y$ is called the **constant sheaf**: $Y_X(U) = \{f \colon U \to Y \mid f \text{ locally constant}\}$

Example 1.3.5. Let $f : Z \to X$ be a continuous map. For $U \subset X$ open, define $h_Z(U)$ as the set of continuous sections s of $f|_U \colon f^{-1}(U) \to U$ (namely, continuous maps $s \colon U \to f^{-1}(U)$ satisfying $f|_U \circ s = \mathrm{id}$):

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, "f|_U"] & Z \arrow[d, "f"] \\ U \arrow[r, "j"] \arrow[u, bend left, "s"] & X \end{tikzcd}$$

Such sections correspond bijectively to continuous maps $s \colon U \to Z$ such that $f \circ s = j$. This defines a sheaf on X.

Take $Z = X \times Y$ and $p \colon Z \to X$ the projection, then $p^{-1}(U) = U \times Y$:

$$\begin{tikzcd} U \times Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

and a section $s \colon U \to U \times Y$ is determined by $U \to Y$. Thus $h_{X\times Y} = \mathcal{F}_Y$.

Example 1.3.6. Let X be a complex manifold. We have sheaves $\mathbb{C}_X \to \mathcal{O}_X \to \mathcal{F}_{\mathbb{C}}$ defined by

$$\begin{tikzcd} \mathbb{C}_X(U) \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_X(U) \arrow[r, hook] \arrow[d, equal] & \mathcal{F}_{\mathbb{C}}(U) \arrow[d, equal] \\ \{U \to \mathbb{C} \text{ locally constant}\} & \{U \to \mathbb{C} \text{ holomorphic}\} & \{U \to \mathbb{C} \text{ continuous}\} \end{tikzcd}$$

Example 1.3.7. Let $X = \mathrm{pt}$, then

$$\begin{aligned} \mathrm{Set} &\cong \mathrm{Shv}(\mathrm{pt}, \mathrm{Set}) \\ \mathcal{F}(\mathrm{pt}) &\hookleftarrow \mathcal{F} \\ S &\mapsto \begin{cases} \emptyset \mapsto \{*\} \\ \mathrm{pt} \mapsto S. \end{cases} \end{aligned}$$

Proposition 1.3.8. *Let $\mathcal{F}$ be a presheaf. Then $\exists$ a sheaf $\mathcal{F}$ and $\nu\colon \mathcal{F} \to \mathcal{F}^+$ such that $\forall$ sheaf $\mathcal{G}$ and a morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, there exists a unique $\phi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that*

$$\begin{tikzcd} & \mathcal{G} \\ \mathcal{F} \arrow[ur, "\phi"] \arrow[r, "\nu"] & \mathcal{F}^+ \arrow[u, "\phi^+"'] \end{tikzcd}$$

Definition 1.3.9. We call $\mathcal{F}^+$ the **sheafification** of the presheaf $\mathcal{F}$. It is also called the **sheaf associated to** $\mathcal{F}$ and sometimes denoted $a\mathcal{F}$.

Construction. For any open cover $\{U_i\}$ of an open subset U, consider

$$\mathrm{Eq}\left(\textstyle\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right).$$

We define a presheaf $\mathcal{F}'$ on X by

$$\mathcal{F}'(U) = \operatorname*{colim}_{\mathrm{Cov}(U)^{\mathrm{op}}} \mathrm{Eq}\left(\textstyle\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right)$$

The category $\mathrm{Cov}(U)$ of open covers of U is defined as follows. An object is an open cover $\{U_i\}_{i\in I}$. A morphism between two covers $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ is a map $f\colon I \to J$ such that $U_i \subseteq V_{f(i)}$.

It is easy to see that $\mathcal{F}'$ is a separated presheaf. Moreover, $\mathcal{F}'$ is a sheaf if $\mathcal{F}$ is separated. We take $\mathcal{F}^+ = (\mathcal{F}')'$. □

Categorical point of view: We have $\mathrm{Hom}(\mathcal{F}, \iota\mathcal{G}) \cong \mathrm{Hom}(a\mathcal{F}, \mathcal{G})$.

$$\mathrm{PShv} \underset{\iota}{\overset{a}{\rightleftarrows}} \mathrm{Shv} \qquad a \dashv \iota$$

Example 1.3.10. Let X be a topological space, A a set. Define the constant presheaf A^{psh} by $A_X^{\mathrm{psh}}(U) = A$. Then $(A_X^{\mathrm{psh}})^+ = A_X$ is the constant sheaf.

Definition 1.3.11 (Functoriality)**.** Let $f : X \to Y$ be a continuous map.

(1) For $\mathcal{F} \in \mathrm{PShv}(X)$, define $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, $f_*\mathcal{F} \in \mathrm{PShv}(Y)$. If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is also a sheaf. This is called **pushforward** or **direct image**.

(2) for $\mathcal{G} \in \mathrm{PShv}(Y)$, define

$$(f^{-1}_{\mathrm{psh}}\mathcal{G})(U) = \operatorname*{colim}_{f(U)\subset V} \mathcal{F}(V)$$

It is clear $f^{-1}_{\mathrm{psh}}(\mathcal{G}) \in \mathrm{PShv}(X)$. This is called **pullback** or **inverse image**. We have $\mathrm{Hom}(f^{-1}_{\mathrm{psh}}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$.

(3) Unfortunately, even if $\mathcal{G}$ is a sheaf on Y, $f^{-1}_{\mathrm{psh}}(\mathcal{G})$ may not be a sheaf. So we define $f^{-1}\mathcal{G} = (f^{-1}_{\mathrm{psh}}\mathcal{G})^+$. We have $f^{-1} \dashv f_*$. Form the commutative diagram

$$\begin{tikzcd} \mathrm{PShv}(X) \arrow[d, "f_*"] & \mathrm{Shv}(X) \arrow[l, "\iota"'] \arrow[d, "f_*"] \\ \mathrm{PShv}(Y) & \mathrm{Shv}(Y) \arrow[l, "\iota"'] \end{tikzcd}$$

Taking left adjoints, we obtain the following diagram, which commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathrm{PShv}(X) & \xrightarrow{a} & \mathrm{Shv}(X) \\ {\scriptstyle f_{\mathrm{psh}}^{-1}}\uparrow & & {\scriptstyle f^{-1}}\uparrow \\ \mathrm{PShv}(Y) & \xrightarrow{a} & \mathrm{Shv}(Y) \end{array} \tag{1.3.1}$$

Example 1.3.12. Consider $j\colon U \to X$ open. Then $j_{\mathrm{psh}}^{-1}\mathcal{F}(V) = \mathcal{F}(V \cap U)$. We usually denote it by $\mathcal{F}|_U$. We have $j^{-1}\mathcal{F} = j_{\mathrm{psh}}^{-1}\mathcal{F}$ if $\mathcal{F}$ is a sheaf.

Example 1.3.13. We have $f^{-1}A_Y \simeq A_X$ by (1.3.1) applied to A_Y^{psh}.

Example 1.3.14. Consider $i_x\colon \mathrm{pt} \to X,\ \mathrm{pt} \mapsto x \in X$.

$$\begin{aligned}(i_x)_{\mathrm{psh}}^{-1}(\mathcal{F})(x) &= \operatorname*{colim}_{x\in U} \mathcal{F}(U) \\ &= \{(U, s) | x \in U, s \in \mathcal{F}(U)\}/\sim\end{aligned}$$

where the equivalence relation is defined as follows: $(U, s) \sim (V, t)$ if and only if $\exists x \in W \subset U \cap V$ such that $s|_W = t|_W$. The same formula holds for i_x^{-1}.

This is also called the **stalk** of $\mathcal{F}$ at x and is denoted by $\mathcal{F}_x$. For each $x \in U$, we have

$$\begin{aligned}\mathcal{F}(U) &\to \mathcal{F}_x \\ s &\mapsto [(U, s)]\end{aligned}$$

The image of s is called the **germ** of s at x and denoted s_x. We have $\mathcal{F}_x^+ \simeq \mathcal{F}_x$ by (1.3.1).

Lemma 1.3.15. *Suppose $\mathcal{F}$ a sheaf, $s, t \in \mathcal{F}(U)$ such that $s_x = t_x \in \mathcal{F}_x, \forall x \in U$. Then $s = t$.*

Proof. By definition, for each $x \in U$, s, t agree on some neighborhood W_x of x. These W_x cover U when x varies in U, hence s, t agree on an open cover of U, hence they agree on U. □

Proposition 1.3.16. *Let $\mathcal{F}, \mathcal{G}$ be sheaves.*

(1) Suppose $\phi, \psi\colon \mathcal{F} \to \mathcal{G}$ morphisms of sheaves such that $\phi_x = \psi_x, \forall x \in X$. Then $\phi = \psi$.

(2) Suppose $\phi\colon \mathcal{F} \to \mathcal{G}$, ϕ_x is bijective for all $x \in X$. Then ϕ is an isomorphism.

Proof. (1) For U open, $s \in \mathcal{F}(U)$, then $\phi(U)(s)_x = \psi(U)(s)_x, \forall x \in U$, by above Lemma, we have $\phi(U)(s) = \psi(U)(s)$, hence $\phi = \psi$.

(2) We construct ψ to be the inverse of ϕ. For $t \in \mathcal{G}(U)$, and $x \in U$, since $\phi(U)_x$ is bijective, there exists open set $x \in V_x \subset U$ and $s_x \in \mathcal{F}(V_x)$ such that $\phi(V_x)(s_x) = t|_{V_x}$. Consider $x, y \in U$, $\forall z \in V_x \cap V_y$, we have $s_x|_z = t|_z = s_y|_z$, hence $s_x|_{V_x\cap V_y} = s_y|_{V_x\cap V_y}$, hence $s_x \in \mathcal{F}(V_x)$ glue to $s \in \mathcal{F}(U)$.

□

Consider continuous maps $f\colon X \to Y$ and $g\colon W \to X$. Then $g^{-1}f^{-1} = (fg)^{-1}$. In the case where $W = \mathrm{pt}$ and $g = i_x$, we get

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Limits and Colimits

Recall that $\mathcal{C} = \mathrm{Set}$, Ab, or Ring.

The category $\mathrm{PShv}(X, \mathcal{C})$ admits arbitrary small limits and colimits.

$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U)$$
$$(\operatorname{colim}_i^{\mathrm{psh}} \mathcal{F}_i)(U) = \operatorname{colim}_i^{\mathrm{psh}} \mathcal{F}_i(U)$$

It is easy to see the limit defined above takes sheaves to sheaves. But for colimits of sheaves, we need to sheafify: $\operatorname{colim}_i \mathcal{F}_i = (\operatorname{colim}_i^{\mathrm{psh}} \mathcal{F}_i)^+$. The category $\mathrm{Shv}(X, \mathcal{C})$ also admits small limits and colimits. The sheafification functor commutes with colimits and the functor ι commutes with limits.

Recall that filtered colimits in $\mathcal{C}$ commute with finite limits, hence finite limits of sheaves do not need to sheafification. The same remark shows that the sheafification functor is left exact, and hence exact. (Recall that a functor is called left (resp. right) exact if it commutes with finite limits (resp. finite colimits). A functor is called **exact** if it is left and right exact.)

The following special case will be used very often.

Definition 1.3.17. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian sheaves.

(1) $\ker(\varphi)$ is defined to be $(\ker(\varphi))(U) = \ker(\varphi(U))$. It is already a sheaf.

(2) $\operatorname{coker}(\varphi)$ is the sheafification of the presheaf $U \mapsto \operatorname{coker}(\varphi(U))$.

Proposition 1.3.18. $\mathrm{Shv}(X, \mathrm{Ab})$ *is an Abelian category.*

Proof. We first check that $\mathrm{Shv}(X, \mathrm{Ab})$ is an additive category:

(1) It has a zero object (namely, an object that is initial and final): the constant sheaf 0;

(2) Finite coproducts and finite products exist and coincide: we have $\mathcal{F} \times \mathcal{G} \simeq \mathcal{F} \oplus^{\mathrm{psh}} \mathcal{G} \simeq \mathcal{F} \oplus \mathcal{G}$.

(3) The commutative monoid $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ admits inverses: $(-\phi)_U(s) = -\phi_U(s)$.

Recall that an abelian category is an additive category admitting kernels, cokernels, and such that coimages coincides with images. The last property means that for every morphism $\phi\colon \mathcal{F} \to \mathcal{G}$, the canonical morphism $\psi\colon \operatorname{coker}(i) \to \ker(p)$ is an isomorphism, where

$$\ker(\phi) \xrightarrow{i} \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{p} \operatorname{coker}(\phi).$$

Since sheafification commutes with taking kernels, ψ is the sheafification of $\psi^{\mathrm{psh}}\colon \operatorname{coker}^{\mathrm{psh}}(i) \to \ker(p^{\mathrm{psh}})$, where $p^{\mathrm{psh}}\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{psh}}(\phi)$. Since ψ_U^{psh} is an isomorphism for every U, ψ is an isomorphism. $\square$

Let $f\colon X \to Y$ be a continuous map. The functor f^{-1} commutes with colimits and f_* commutes with limits.

Proposition 1.3.19. *Let $f\colon X \to Y$ be a continuous map between topological spaces. Then $f^{-1}\colon \mathrm{Shv}(Y, \mathcal{C}) \to \mathrm{Shv}(X, \mathcal{C})$ is an exact functor.*

In particular, taking stalks at a point is an exact functor.

Proof. It suffices to show that f^{-1} is left exact. By definition,

$$(f^{-1}_{\mathrm{psh}}\mathcal{G})(U) = \operatorname*{colim}_{f(U)\subset V} \mathcal{G}(V)$$

which is a filtered colimit, hence commutes with finite limits. The the sheafification functor also left exact, hence the result. □

Proposition 1.3.20. *A sequence $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\mathrm{Shv}(X, \mathrm{Ab})$ is exact if and only if it is exact on stalks: $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for every $x \in X$.*

Proof. This follows from the exactness of taking stalks. For the "if" part, we also need Proposition 1.3.16. □

Example 1.3.21. Consider $i\colon Y \subset X$ a closed embedding, $\mathcal{G}$ a sheaf on Y. Then $i_*\mathcal{G}(U) = \mathcal{G}(U \cap Y)$. For $x \in X$,

$$\begin{aligned}(i_*\mathcal{G})_x &= \operatorname*{colim}_{x\in U} \mathcal{G}(U\cap Y)\\ &= \begin{cases} *, & x \notin Y \\ \mathcal{G}_x, & x \in Y,\end{cases}\end{aligned}$$

where $*$ denotes a final object of $\mathcal{C}$. It follows that the functor $i_*\colon \mathrm{Shv}(Y, \mathrm{Ab}) \to \mathrm{Shv}(X, \mathrm{Ab})$ is exact. (The functor $i_*\colon \mathrm{Shv}(Y, \mathrm{Set}) \to \mathrm{Shv}(X, \mathrm{Set})$ does not preserve initial objects unless i is a homeomorphism.)

Let $\phi\colon i^{-1}i_*\mathcal{G} \to \mathcal{G}$ be the canonical morphism. Then ϕ_y can be identified with $\mathrm{id}_{\mathcal{G}_y}$. Hence ϕ is an isomorphism. In the other direction, for every an abelian sheaf $\mathcal{F}$ on X, the canonical morphism $\psi\colon \mathcal{F} \to i_*i^{-1}\mathcal{F}$ is an epimorphism. This is easy to check on stalks.

Warning 1.3.22. An epimorphism of sheaves is **not** surjective on sections in general. Let X be a connected topological space, $Y = \{x, y\}$ two distinct closed points in X, $\iota\colon Y \to X$. Consider the constant sheaf $\mathbb{Z}_X$ on X. Then $\mathbb{Z}_X(X) = \mathbb{Z}$ since X is connected. But $(\iota_*\iota^{-1}\mathbb{Z}_X)(X) \simeq \mathbb{Z}_Y(Y) \simeq \mathbb{Z}\times\mathbb{Z}$. The map $\psi_X\colon \mathbb{Z}_X(X) \to \iota_*\iota^{-1}\mathbb{Z}_X(X)$ is not surjective.

Let us describe epimorphisms of sheaves of sets or abelian groups. A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is an epimorphism if and only if $\forall U$ open in X, $s \in \mathcal{G}(U)$, $\exists\{U_i\}$ an open cover of U and $t_i \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(t_i) = s|_{U_i}$.

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Remark 1.3.23. We have defined for every continuous map $f\colon Z \to X$ between topological spaces, a sheaf of sections $\mathcal{F} = h_Z$ such that $h_Z(U)$ is the set of continuous sections $U \to Z$ of f over U. Conversely, every sheaf of sets has the form $\mathcal{F} \simeq h_Z$. Here $Z = \coprod_{x\in X} \mathcal{F}_x$. An element in Z has the form (x, s), where $x \in X$, $s \in \mathcal{F}_x$. We equip Z with the strongest topology such that for all $U \subset X$ open and $s \in \mathcal{F}(U)$, the map

$$\begin{aligned} \varphi_s \colon U &\to Z \\ x &\mapsto (x, s_x) \end{aligned}$$

is continuous. A basis for the topology is given by the subsets $\varphi_s(U)$. The space Z is called the **espace étalé** of $\mathcal{F}$.

1.4 Schemes

Let A be a ring and let $X = \operatorname{Spec}(A)$. We now proceed to equip X with a sheaf of rings $\mathcal{O}_X$ such that $\mathcal{O}_X(X) = A$ and for $f \in A$, $\mathcal{O}_X(D(f)) = A_f$. Recall $D(f) = \{\mathfrak{p} \in A \mid f \notin \mathfrak{p}\}$.

Consider the poset $\mathcal{B} = (\{D(f) \mid f \in A\}, \subseteq)$. Define a functor

$$\begin{aligned} \mathcal{B}^{op} &\to \mathrm{Ring} \\ D(f) &\mapsto A_f \end{aligned}$$

If $D(f) \subseteq D(g)$, we have $V(f) \supseteq V(g)$ hence $\sqrt{f} \subseteq \sqrt{g}$, which means that $f^n = ga$ for some $n \geq 1$ and $a \in A$. This implies that g is invertible in A_f and there is a natural ring morphism $A_g \to A_f$. This finishes the definition of functor.

Lemma 1.4.1. *Let X be a topological space, $\mathcal{B}$ an open basis such that $U, V \in \mathcal{B} \Rightarrow U \cap V \in \mathcal{B}$ and $\emptyset \in \mathcal{B}$. We let $\operatorname{Shv}(\mathcal{B}, \mathcal{C})$ denote the category of $\mathcal{B}$-sheaves, namely the full subcategory of $\operatorname{Fun}(\mathcal{B}^{op}, \mathcal{C})$ spanned by functors $\mathcal{F}$ satisfying the following gluing condition: for every open cover $\{U_i\}$ of $U \in \mathcal{B}$ with $U_i \in \mathcal{B}$,*

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram. Then the restriction functor

$$\Phi \colon \operatorname{Shv}(X, \mathcal{C}) \to \operatorname{Shv}(\mathcal{B}, \mathcal{C})$$

is an equivalence of categories, where $(*)$ denotes

Proof. We first prove that Φ is fully faithful, which means that for $\mathcal{F}, \mathcal{G}$ sheaves on X, we have $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}|_{\mathcal{B}}, \mathcal{G}|_{\mathcal{B}})$. This is clear by the gluing condition for sheaves on X, since $\mathcal{B}$ is a basis.

We next prove essential surjectivity. Let $\mathcal{G}$ be a $\mathcal{B}$-sheaf. We define a sheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) = \operatorname*{colim}_{\{U_i\} \in \operatorname{Cov}(U)^{\mathrm{op}}} \operatorname{Eq}\Big(\prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j). \Big)$$

Here $\mathrm{Cov}(U)$ is the category of open covers of U in $\mathcal{B}$. In more detailed words, an element $s \in \mathcal{F}(U)$ is an equivalence class of pairs $(\{U_i\}_{i\in I}, \{s_i\}_{i\in I})$, where $\{U_i\}$ is an open cover of U in $\mathcal{B}$ and $s_i \in \mathcal{G}(U_i)$. We require $\{s_i\}_{i\in I}$ to be compatible, namely $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Two pairs $(\{U_i\}_{i\in I}, \{s_i\}_{i\in I})$ and $(\{V_j\}_{j\in J}, \{t_j\}_{j\in J})$ are equivalent if there exists a common refinement $\{W_k\}_{k\in K}$ in $\mathcal{B}$ of $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ such that $\{s_i\}$ and $\{t_j\}$ restrict to the same family on $\{W_k\}$.

□

Proposition 1.4.2. *Let $X = \mathrm{Spec}(A)$, $\mathcal{B} = \{D(f) \mid f \in A\}$. Then the functor*

$$\mathcal{B}^{op} \to \mathrm{Ring}$$
$$D(f) \mapsto A_f$$

extends uniquely to a sheaf $\mathcal{O}_X$ on X up to isomorphism. Moreover, $\forall \mathfrak{p} \in X, \mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

Proof. The second assertion is clear. For the first assertion, let $U = D(f)$ be open and $\{D(f_i)\}_{i\in I}$ an open cover of U. Since $D(f) = \mathrm{Spec}(A_f)$, we may assume $U = X$. The gluing property in this case says that

$$A \xrightarrow{\lambda} \prod_i A_{f_i} \rightrightarrows \prod_{i,j} A_{f_if_j}$$

is an equalizer diagram.

Let us first show that the general case follows from the case of a finite cover. Since X is compact, there exists a subset $J \subset I$ such that $\{D(f_j)\}_{j\in J}$ covers X. The injectivity of λ follows from the case of a finite cover. Let $(s_i) \in \prod A_{f_i}$ such that $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$ for all $i, j \in I$. By the case of a finite cover, there exists $s \in A$ such that $s_j = s|_{D(f_j)}$ for all $j \in J$. Then, for all i, $s_i|_{D(f_if_j)} = s|_{D(f_if_j)}$ and $s_i = s|_{D(f_i)}$ by the injectivity of λ for the cover $\{D(f_if_j)\}_{j\in J}$ of $\{D(f_i)\}$.

Thus we may assume that I is finite. In this case λ is fully faithful and the result follows from Proposition 1.4.3 below. We also give a more direct proof as follows. Let $a \in A$ such that $a|_{D(f_i)} = 0$ for all i. Then for each i, there exists m_i such that $f_i^{m_i}a = 0$. But $\{D(f_i)\} = \{D(f_i^{m_i})\}$ cover X, so that $f_i^{m_i}$ generates the unit ideal. Therefore, 1 annihilates a and $a = 0$. It remains to check that every $(s_i) \in \prod_i A_{f_i}$ satisfying $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$ for all $i, j \in I$ is in the image of λ. Write $s_i = \frac{b_i}{f_i^{m_i}}$. We may multiply b_i with powers of f_i to assume $\forall i, m_i = m$. We have $\frac{b_i}{f_i^m} = \frac{b_j}{f_j^m} \in A_{f_if_j}$. Hence there exists r such that $(f_if_j)^r(b_if_j^m - b_jf_i^m) = 0$. Up to replacing b_i by $b_if_i^r$ and m by $m + r$, we may assume $b_if_j^m - b_jf_i^m = 0$. Since $D(f_i^m)$ cover X, we have $1 = \sum a_if_i^m$. Let $s = \sum a_ib_i$. Then, on $D(f_i)$, $sf_i^m = \sum_j a_jb_jf_i^m = \sum_j a_jb_if_j^m = b_i$, so that $s|_{D(g_i)} = s_i$. □

Proposition 1.4.3. *Let $\phi\colon A \to B$ be a faithfully flat ring homomorphism. Then*

$$A \xrightarrow{\phi} B \underset{i_2}{\overset{i_1}{\rightrightarrows}} B \otimes_A B$$

is an equalizer diagram in the category A-Mod. The morphism i_1, i_2 are defined by $i_1(b) = b \otimes 1$ and $i_2(b) = 1 \otimes b$.

Recall that $\phi\colon A \to B$ is called **faithfully flat** if for every sequence of A-modules

$$M \longrightarrow N \longrightarrow P,$$

it is exact if and only if it is exact after tensoring with B:

$$M \otimes_A B \longrightarrow N \otimes_A B \longrightarrow P \otimes_A B.$$

Note that ϕ is faithfully flat if and only if ϕ is flat and $\operatorname{Spec}(\phi)$ is surjective ([AM, Exercise 3.16], [M2, Theorem 7.3]).

Proof. Since ϕ is faithfully flat, we only need to prove that the diagram is an equalizer after tensoring with B on the right:

$$B \xrightarrow{\phi\otimes B} B \otimes_A B \underset{i_2\otimes B}{\overset{i_1\otimes B}{\rightrightarrows}} B \otimes_A B \otimes_A B$$

Define

$$\begin{aligned} f\colon B \otimes_A B &\to B \\ b_1 \otimes b_2 &\mapsto b_1 b_2 \\ g\colon B \otimes_A B \otimes_A B &\to B \otimes_A B \\ b_1 \otimes b_2 \otimes b_3 &\mapsto b_1 \otimes b_2 b_3 \end{aligned}$$

One readily checks that

$$\begin{aligned} f \circ \phi &= \mathrm{id} \\ g \circ (i_1 \otimes B) &= \mathrm{id} \\ \phi \circ f &= (i_2 \otimes B) \circ g \end{aligned}$$

This is called a **split equalizer** and one can show directly that a split equalizer is an equalizer. □

Next we consider the functoriality of the sheaf of rings defined above with respect to ring homomorphisms. Let $\phi\colon A \to B$ a ring homomorphism. We have the corresponding continuous map

$$\begin{aligned} \phi^*\colon \operatorname{Spec}(B) &\to \operatorname{Spec}(A) \\ \mathfrak{q} &\mapsto \mathfrak{q}^c \end{aligned}$$

Let $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$. For $g \in A$, we have

$$\begin{aligned} (\phi^*)^{-1}(D(g)) &= D(\phi(g)) \\ \mathcal{O}_Y(D(g)) &= A_g \\ \phi_*\mathcal{O}_X(D(g)) = \mathcal{O}_X(D(\phi(g)) &= B_{\phi(g)} \end{aligned}$$

The homomorphism ϕ naturally induces a homomorphism $A_g \to B_{\phi(g)}$. This defines a morphism of sheaves $f^\flat\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$, which corresponds by adjunction to $f^\sharp\colon \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. For $\mathfrak{p} \in \operatorname{Spec}(B)$, $f^\sharp$ induces $\mathcal{O}_{Y,\phi(\mathfrak{p})} \simeq (\phi^{-1}\mathcal{O}_Y)_\mathfrak{p} \to \mathcal{O}_{X,\mathfrak{p}}$.

Definition 1.4.4. A **ringed space** consists of a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A **locally ringed space** is a ringed space such that $\forall x \in X$, $\mathcal{O}_{X,x}$ is a local ring.

A morphism of ringed spaces is a pair $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, where $f\colon X \to Y$ is a continuous map and $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves of rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that $\forall x \in X$, $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism. (Recall that a homomorphism between local rings $\phi\colon B \to A$ is called **local** if $\phi(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$, or, equivalently, $\phi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_B$.) For two morphisms of locally ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{(f,f^\sharp)} (Y, \mathcal{O}_Y) \xrightarrow{(g,g^\sharp)} (Z, \mathcal{O}_Z)$$

the composition is $(gf, (gf)^\sharp)$, where $(gf)^\sharp$ is defined by

$$(g \circ f)^{-1}\mathcal{O}_Z \cong f^{-1}g^{-1}(\mathcal{O}_Z) \xrightarrow{f^{-1}(g^\sharp)} f^{-1}\mathcal{O}_Y \xrightarrow{f^\sharp} \mathcal{O}_X.$$

Definition 1.4.5. An **affine scheme** is a locally ringed space that is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for some ring A. A **scheme** X is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $\{U_i\}$ of X such that the restriction $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for all i. For schemes X and Y, a morphism of schemes $X \to Y$ is a morphism of locally ringed spaces.

We denote the category of schemes by Sch, which is a full subcategory of the category of locally ringed spaces.

Proposition 1.4.6. *The functor*

$$\begin{aligned}\mathrm{Spec}\colon \mathrm{Ring}^{\mathrm{op}} &\to \mathrm{Sch}\\ A &\to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})\end{aligned}$$

is fully faithful.

Proof. For A, B rings, we need to check that the map

$$\Psi\colon \mathrm{Hom}_{\mathrm{Ring}}(A, B) \to \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), \mathrm{Spec}(A))$$

is a bijection. Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. We define

$$\Phi\colon \mathrm{Hom}_{\mathrm{Sch}}(Y, X) \to \mathrm{Hom}_{\mathrm{Ring}}(A, B)$$

by $(f, f^\sharp) \mapsto f^\flat_Y\colon A = \mathcal{O}_X(X) \to \mathcal{O}_Y(Y) = B$. It is easy to see $\Phi \circ \Psi = \mathrm{id}$. It remains to show $\Psi \circ \Phi = \mathrm{id}$. Let $(f, f^\sharp)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism and let $\phi = \Phi(f, f^\sharp)\colon A \to B$. For $\mathfrak{q} \in \mathrm{Spec}(B)$, we have a natural commutative diagram defined by restricting to stalks:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{f(q)} \arrow[r, "f^\sharp_q"] & B_q \end{tikzcd}$$

Since $f^\sharp$ is a local ring morphism we have $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$. Moreover, by the universal property of localization, $f^\sharp_{\mathfrak{q}}$ must be the morphism induced by ϕ. This concludes that $\Psi \circ \Phi = \mathrm{id}$. □

If we did not require $f^\sharp$ to induce local homomorphisms, then the above proposition would fail to hold. For example, $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}, \mathbb{Q})$ has only one element, but for every $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$, we can define a morphism of ringed spaces $(f, f^\sharp)\colon \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ of image $\{\mathfrak{p}\}$ with $f^\sharp_{\mathrm{Spec}(\mathbb{Q})}$ given by $\mathbb{Z}_\mathfrak{p} \to \mathbb{Q}$.

Example 1.4.7. (1) $\mathrm{Spec}(0) = \emptyset$ is the initial object of Sch.

(2) For a field k, $\mathrm{Spec}(k)$ is a point equipped with the constant sheaf of value k.

(3) For $A = k[\epsilon]/(\epsilon^2)$, $\mathrm{Spec}(A)$ is a point equipped with the constant sheaf of value A.

(4) For a discrete valuation ring (DVR) A, $X = \mathrm{Spec}(A) = \{\eta, s\}$ where $\eta = (0)$ and $s = \mathfrak{m}$ is the unique maximal ideal of A. We have $\mathcal{O}_X(\eta) = \mathrm{Frac}(A)$, $\mathcal{O}_X(X) = A$.

(5) $\mathrm{Spec}(\mathbb{Z})$.

(6) For a ring A, $\mathbb{A}^n_A := \mathrm{Spec}(A[x_1, \dots, x_n])$ is called the **affine n-space** over A. For $n \geq 2$, not all opens are principal (see below).

Definition 1.4.8. Let X be a scheme, U an open subset of X. It is easy to see that $(U, \mathcal{O}_X|_U)$ is a scheme. This is called an **open subscheme** of X.

A morphism of scheme $f\colon Y \to X$ is called an **open immersion** if f identifies Y with an open subscheme of X, i.e. f is a composition $Y \xrightarrow{g} U \xrightarrow{j} X$, where g is an isomorphism and j is the inclusion of an open subscheme.

Not all schemes are affine.

Example 1.4.9. Let $X = \mathbb{A}^2_k$, $U = X \backslash V(x, y)$. Namely U is the open subset formed by removing the origin. We observe that $U = D(x) \cup D(y)$, so that $\mathcal{O}(U)$ is

$$\mathrm{Eq}\left(\mathcal{O}(D(x)) \times \mathcal{O}(D(y)) \rightrightarrows \mathcal{O}(D(x) \cap D(y))\right)$$

$$k[x, y, x^{-1}] \times k[x, y, y^{-1}] \qquad\qquad k[x, y, x^{-1}, y^{-1}]$$

The equalizer is $k[x, y, x^{-1}] \cap k[x, y, y^{-1}] = k[x, y]$. Thus the map

$$\Phi\colon \mathrm{Hom}_{\mathrm{Sch}}(X, U) \to \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{O}_U(U), \mathcal{O}_X(X))$$

defined by $(f, f^\sharp) \mapsto f^\flat_U$ is not surjective. In particular, U is not affine.

Example 1.4.10. For a family of schemes $\{X_i\}_{i \in I}$, the coproduct is $X = \coprod_i X_i$, equipped with $\mathcal{O}_X$ defined by $\mathcal{O}_X(\coprod_i U_i) = \prod_i \mathcal{O}_{X_i}(U_i)$. If I is infinite and X_i is non-empty for all i, then X is not quasi-compact, and hence not an affine scheme. On the other hand, if I is finite with $X_i = \mathrm{Spec}(A_i)$, then $X \cong \mathrm{Spec}(\prod_i A_i)$.

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Definition 1.4.11. Let X be a topological space, $\{U_i\}$ an open cover. A **Gluing Datum** consists of a family of sheaves $\mathcal{F}_i$ over U_i and a family of morphisms $\gamma_{ij} \colon \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$. such that

(1) $\gamma_{ii} = \mathrm{id}$ and

(2) $\gamma_{ik} = \gamma_{jk} \circ \gamma_{ij}$ on $U_i \cap U_j \cap U_k$.

A morphism of gluing data $(\mathcal{F}_i, \gamma_{ij}) \to (\mathcal{G}_i, \delta_{ij})$ is a family of morphisms of sheaves $\phi_i \colon \mathcal{F}_i \to \mathcal{G}_i$ such that

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "\phi_i"] \arrow[d, "\gamma_{ij}"] & \mathcal{G}_i \arrow[d, "\delta_{ij}"] \\ \mathcal{F}_j \arrow[r, "\phi_j"] & \mathcal{G}_j \end{tikzcd}$$

is commutative.

Lemma 1.4.12 (Gluing sheaves)**.** *We have an equivalence of categories* $\mathrm{Shv}(X, \mathcal{C}) \cong$ $\{$*gluing data*$\}$.

Proof. Let $(\mathcal{F}_i, \gamma_{ij})$ be a gluing datum. Define

$$\mathcal{F}(U) = \mathrm{Eq}\left(\prod_i \mathcal{F}_i(U \cap U_i) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \prod_{ij} \mathcal{F}_i(U \cap U_i \cap U_j) \right)$$

where π_1 is induced by the restriction $\mathcal{F}_i(U \cap U_i) \longrightarrow \mathcal{F}_i(U \cap U_i \cap U_j)$ and π_2 is induced by $\mathcal{F}_i(U \cap U_i) \longrightarrow \mathcal{F}_i(U \cap U_i \cap U_j) \xrightarrow{\gamma_{ij}} \mathcal{F}_j(U \cap U_j \cap U_i)$.

□

Lemma 1.4.13 (Gluing morphisms of schemes)**.** *Let X, Y be schemes, $\{U_i\}_{i \in I}$ an open cover of X. Then*

$$\mathrm{Hom}_{\mathrm{Sch}}(X, Y) \longrightarrow \prod_i \mathrm{Hom}_{\mathrm{Sch}}(U_i, Y) \rightrightarrows \prod_{ij} \mathrm{Hom}_{\mathrm{Sch}}(U_i \cap U_j, Y)$$

is an equalizer diagram. More generally, $U \mapsto \mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ *is a sheaf of sets on* X.

Proof. Let $(f_i \colon U_i \to Y)$ be a compatible family of morphism. We first glue them in the category of topological spaces and get a continuous map $f \colon X \to Y$. Then $f_i^\sharp \colon (f^{-1}\mathcal{O}_Y)|_{U_i} \to \mathcal{O}_X|_{U_i}$ is a compatible family of morphisms of sheaves, namely a morphism of gluing data and the previous lemma tells us that there exists a unique $f^\sharp \colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ that restricts to $f_i^\sharp$. □

Remark 1.4.14. The above lemma implies that if X is a scheme and $\{U_i\}$ is an open cover, $U_{ij} = U_i \cap U_j$, then

$$\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \longrightarrow X$$

is a coequalizer diagram in the category Sch.

Lemma 1.4.15 (Gluing schemes)**.** *Let $\{X_i\}_{i\in I}$ be a family of schemes. Let $X_{ij} \subseteq X_i$ be open sub-schemes and $f_{ij}\colon X_{ij} \to X_{ji}$ isomorphisms of schemes for all $i, j \in I$. We require*

(1) $f_{ii} = \mathrm{id}$

(2) $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$

(3) $f_{ik} = f_{jk} \circ f_{ij}$ *on* $X_{ij} \cap X_{ik}$.

Then there exists a scheme X and open immersions $f_i\colon X_i \to X$ such that

$$\begin{tikzcd} X_{ij} \arrow[r, hook] \arrow[d, "f_{ij}"] & X_i \arrow[dr, "f_i"] & \\ X_{ji} \arrow[r, hook] & X_j \arrow[r, "f_j"] & X \end{tikzcd}$$

and has the universal property: For every scheme Y and a family of morphisms of schemes $g_i\colon X_i \to Y$ satisfying

$$\begin{tikzcd} X_{ij} \arrow[r, hook] \arrow[d, "f_{ij}"] & X_i \arrow[dr, "g_i"] & \\ X_{ji} \arrow[r, hook] & X_j \arrow[r, "g_j"] & Y \end{tikzcd}$$

then there exists a unique $g\colon X \to Y$ such that

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[dr, "g_i"] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

is commutative.

Proof. Let $X = \coprod_i X_i/\sim$, where $x \in X_i \sim y \in X_j \Leftrightarrow y = f_{ij}x$. This makes a topological space X with open subsets $X_i \subseteq X$. We have a sheaf $\mathcal{O}_{X,i}$ on each X_i, and we glue them to get $\mathcal{O}_X$. $\square$

Example 1.4.16. Consider $\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \dots, x_n]), n \geqslant 1$ and the origin $O = V(x_1, \dots, x_n)$. Let $X_0 = X_1 = \mathbb{A}^n_k$, $X_{01} = X_{10} = \mathbb{A}^n_k \backslash \{O\}$. We then glue them by $X_{01} \xrightarrow{\mathrm{id}} X_{10}$. The resulting scheme X is called the affine n-space with doubled origin. We have

$$\mathcal{O}_X(X) = \mathrm{Eq}\Big(\mathcal{O}(X_0) \times \mathcal{O}(X_1) \rightrightarrows \mathcal{O}(X_0 \cap X_1) \Big) = k[x_1, \dots, x_n].$$

Since the two morphisms $f_i\colon \mathbb{A}^n_k = X_i \to X$, $i = 0, 1$ induce the same ring homomorphism on global sections, the map

$$\Phi\colon \mathrm{Hom}_{\mathrm{Sch}}(\mathbb{A}^n_k, X) \to \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{O}_X(X), \mathcal{O}_{\mathbb{A}^n_k}(\mathbb{A}^n_k))$$

is not an injection. This shows that X is not affine.

Example 1.4.17. Let $X_0 = X_1 = \mathbb{A}^1_k$, $X_{01} = X_{10} = \mathbb{A}^1_k \backslash \{O\}$. Write $X_{01} = \operatorname{Spec}(k[x, x^{-1}])$, $X_{10} = \operatorname{Spec}(k[y, y^{-1}])$. Gluing them by $x \mapsto y^{-1}$, we get the projective line $\mathbb{P}^1_k$ over k.

Example 1.4.18. More generally, let A be a ring, $X_i = \operatorname{Spec}(A[T_i^{-1}T_0, \dots, T_i^{-1}T_n]) \simeq \mathbb{A}^n_A$. Let $X_{ij} = D(T_i^{-1}T_j) \subset X_i$. Then

$$X_{ij} = \operatorname{Spec}(A[T_i^{-1}T_k, T_j^{-1}T_i]_{k=0}^n) = \operatorname{Spec}(A[T_j^{-1}T_k, T_i^{-1}T_j]_{k=0}^n) = X_{ji}.$$

Gluing them by the identity morphisms, we get $X = \mathbb{P}^n_A$, the projective n-space over A. It can be shown from the construction that $\mathcal{O}_X(X) = \bigcap_i A[T_i^{-1}T_0, \dots, T_i^{-1}T_n] = A$. For $A \neq 0$ and $n \geq 1$, $\mathbb{P}^n_A$ is not affine.

Proposition 1.4.19. *Let X be a scheme, $Y = \operatorname{Spec}(A)$ an affine scheme. Then the map $\operatorname{Hom}_{\mathrm{Sch}}(X, Y) \to \operatorname{Hom}_{\mathrm{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ sending f to $f^\flat_Y$ is a bijection.*

Proof. Let $X = \bigcup_i U_i$, U_i open affine. Then by gluing morphisms of schemes, we have

$$\operatorname{Hom}_{\mathrm{Sch}}(X, Y) = \operatorname{Eq}\Big(\textstyle\prod_i \operatorname{Hom}_{\mathrm{Sch}}(U_i, Y) \rightrightarrows \prod_{ij} \operatorname{Hom}_{\mathrm{Sch}}(U_i \cap U_j, Y) \Big)$$

Write $U_i \cap U_j = \bigcup_k U_{ijk}$ with U_{ijk} open affine, then

$$\operatorname{Hom}_{\mathrm{Sch}}(X, Y) = \operatorname{Eq}\Big(\textstyle\prod_i \operatorname{Hom}_{\mathrm{Sch}}(U_i, Y) \rightrightarrows \prod_{ijk} \operatorname{Hom}_{\mathrm{Sch}}(U_{ijk}, Y) \Big)$$

but for X affine, we have $\operatorname{Hom}_{\mathrm{Sch}}(X, Y) \cong \operatorname{Hom}_{\mathrm{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$. Therefore the above equalizer diagram is isomorphism to

$$\operatorname{Eq}\Big(\textstyle\prod_i \operatorname{Hom}_{\mathrm{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(U_i)) \rightrightarrows \prod_{ijk} \operatorname{Hom}_{\mathrm{Sch}}(\mathcal{O}_Y(Y), \mathcal{O}_X(U_{ijk})) \Big)$$

Since

$$\mathcal{O}_X(X) \longrightarrow \textstyle\prod_i \mathcal{O}_X(U_i) \rightrightarrows \prod_{ijk} \mathcal{O}_X(U_{ijk})$$

is an equalizer diagram by the sheaf condition, we get the desired equalizer diagram by applying $\operatorname{Hom}_{\mathrm{Ring}}(\mathcal{O}_Y(Y), -)$. $\square$

Remark 1.4.20. We have

$$\mathrm{Sch} \underset{\operatorname{Spec}}{\overset{\Gamma}{\rightleftarrows}} \mathrm{Ring}^{\mathrm{op}} \qquad \Gamma \dashv \operatorname{Spec},$$

where Γ is the functor sending X to $\mathcal{O}_X(X)$. It follows that Spec transforms colimits in Ring to limits in Sch. Moreover, Spec is fully faithful and equivalently $\Gamma \circ \operatorname{Spec} \cong \mathrm{id}$.

Example 1.4.21. (1) Since $\mathbb{Z}$ is initial in Ring, $\operatorname{Spec}(\mathbb{Z})$ is the final object of Sch.

(2) Pushouts in Ring are given by tensor product. Hence $\operatorname{Spec}(B \otimes_A C) \cong \operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(C)$.

Example 1.4.22.

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(\mathbb{Z}[T])) & \cong & \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T], \mathcal{O}_X(X)) & \cong & \mathcal{O}_X(X) \\ f & \longmapsto & \Gamma(f) & \longmapsto & \Gamma(f)(T) \end{array}$$

Recall the **Yoneda embedding**. Let $\mathcal{C}$ be a locally small category. For every object X, consider the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. The Yoneda embedding is the functor

$$h\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}),\ X \mapsto h_X,$$

which is fully faithful.

In the case of $\mathrm{Ring}^{\mathrm{op}}$ and Sch, we have functors

$$\begin{tikzcd} \mathrm{Ring}^{\mathrm{op}} \arrow[r, "\mathrm{Spec}"] \arrow[d, "h"] & \mathrm{Sch} \arrow[d, "h"] \\ \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) & \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set}) \arrow[l, "\circ\,\mathrm{Spec}"'] \end{tikzcd}$$

The diagram commutes up to isomorphism by the full faithfulness of Spec. The functor $\circ\,\mathrm{Spec}$ is not fully faithful. However, by gluing morphisms of schemes one obtains the following.

Proposition 1.4.23. *The functor*

$$\begin{aligned} \mathrm{Sch} &\longrightarrow \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) \\ Y &\longmapsto (B \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), Y)) \end{aligned}$$

is fully faithful.

Proof. Let X and Y be schemes. Denote the functor

$$B \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), X)$$

by F_X. We construct an inverse of the map $f \mapsto F_f$ as follows. Let φ be a natural transformation from F_X to F_Y. Cover X by open affine subsets $\{U_i\}$ and cover $U_i \cap U_j$ by open affine subsets U_{ijk}. Then by Remark 1.4.14,

$$\textstyle\coprod_{ijk} U_{ijk} \rightrightarrows \coprod_i U_i \longrightarrow X$$

is a coequalizer diagram. Apply φ we get a corresponding diagram involving Y and a unique morphism f making the diagram commutative:

$$\begin{tikzcd} \coprod_{ijk} U_{ijk} \arrow[r, shift left] \arrow[r, shift right] & \coprod_i U_i \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

One then checks that $\varphi \mapsto f$ is the desired inverse. $\square$

Remark 1.4.24. The proposition implies that a morphism of schemes $f\colon X \to Y$ is an isomorphism if and only if for every ring B, $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), X) \to \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), Y)$ is an isomorphism.

We sometimes regard Sch via these fully faithful functors as subcategories of $\mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ or $\mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$.

Definition 1.4.25. Let S be a scheme. The category Sch/S of S-schemes or schemes over S is defined as follows. An object of Sch/S is a scheme X equipped with a morphism of schemes $f\colon X \to S$. A morphism from $(X, f\colon X \to S)$ to $(Y, g\colon Y \to S)$ is a morphism of schemes $h\colon X \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

is commutative.

For two S-schemes $T \to S$ and $X \to S$, the set of T**-points** of X is defined by $X(T) = \mathrm{Hom}_{\mathrm{Sch}/S}(T, X)$. For $T = \mathrm{Spec}(A)$, we write $X(A) := X(\mathrm{Spec}(A))$ and we refer to $\mathrm{Spec}(A)$-points as A-points.

Example 1.4.26. Let $\mathbb{A}^n_A = \mathrm{Spec}(A[x_1, \dots, x_n])$ and let $a\colon \mathbb{A}^n_A \to \mathrm{Spec}(A)$ be the canonical morphism. In $\mathrm{Sch}/A := \mathrm{Sch}/\mathrm{Spec}(A)$, an A-point of $\mathbb{A}^n_A$ is a morphism $s\colon \mathrm{Spec}(A) \to \mathbb{A}^n_A$ that makes

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[rr, "f"] \arrow[dr, equal, "\mathrm{id}"] & & \mathbb{A}^n_A \arrow[dl, "a"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

commutative, namely a section of a. This corresponds to an A-algebra homomorphism $\phi\colon A[x_1, \dots, x_n] \to A$, which is uniquely determined by $(\phi(x_1), \dots, \phi(x_n)) \in A^n$. Thus the set $\mathbb{A}^n_A(A)$ can be identified with A^n.

1.5 Topology of schemes

Lemma 1.5.1. *Let X be a scheme. Then $\mathcal{O}_X(X) = 0$ if and only if $X = \emptyset$.*

Proof. Let X be a scheme such that $\mathcal{O}_X(X) = 0$. Take U open affine subset, since $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is a ring homomorphism, it sends $1 = 0$ to 1, hence $\mathcal{O}_X(U) = 0$ and $U = \mathrm{Spec}(0) = \emptyset$. $\square$

Definition 1.5.2. Let X be a scheme, $f \in \mathcal{O}_X(X)$. Define $X_f = \{x \in X \mid f_x \in \mathcal{O}_{X,x}^\times\}$.

Example 1.5.3. $(\mathrm{Spec}(A))_f = D(f)$. If $U = \mathrm{Spec}(A) \subset X$ is an open affine, then $X_f \cap U = D(f|_U)$. It follows that $X_f \subseteq X$ is open.

Remark 1.5.4. It is easy to see that $X_f \cap X_g = X_{fg}$, $X_{f+g} \subseteq X_f \cup X_g$, $X_0 = \emptyset$, and $X_f = X$ for $f \in \mathcal{O}(X)^\times$.

Proposition 1.5.5. *For any scheme X, we have a bijection*

$$\{\textit{Open and closed subsets of } X\} \cong \{\textit{idempotent elements in } \mathcal{O}_X(X)\}.$$

Proof. Let U be open and closed. Then $X = U \amalg U^c$, where U^c is the complement of U. Let $e_U \in \mathcal{O}_X(X)$ such that $e_U|_U = 1$ and $e_U|_{U^c} = 0$. Then e_U is an idempotent element.

Let $e \in \mathcal{O}_X(X)$ be an idempotent element and consider X_e and X_{1-e}. By the remark preceding the proposition, $X = X_e \amalg X_{1-e}$. Thus X_e is open and closed.

It is clear that $X_{e_U} = U$. Moreover, let $s = e_{X_e}$. The only idempotents in a local ring are 0 and 1. It follows that the germs of s and e agree at every point. This implies $s = e$. □

We say that a scheme is connected if its underlying topological space is connected.

Corollary 1.5.6. *A scheme X is connected if and only if the only idempotents of $\mathcal{O}_X(X)$ are $0, 1$.*

Definition 1.5.7. A topological space X is called **irreducible** if it is nonempty and if $X = F_1 \cup F_2$ with F_1, F_2 closed implies $X = F_1$ or $X = F_2$.

Remark 1.5.8.

- Irreducible $\Rightarrow$ connected.
- A Hausdorff space cannot be irreducible unless X is a point.

Lemma 1.5.9. *Let X be a topological space, $Y \subseteq X$.*

(1) Y is irreducible if and only if Y is nonempty and whenever $Y \subseteq F_1 \cup F_2$, for closed subsets F_1, F_2 in X, we have $Y \subseteq F_1$ or $Y \subseteq F_2$.

(2) Y is irreducible if and only if $\overline{Y}$ is irreducible.

Lemma 1.5.10. *Let X be a nonempty topological space. Then X is irreducible if and only if every non-empty open subset U is dense in X. In that case, U is irreducible as well.*

Example 1.5.11. Let $X = \mathrm{Spec}(k[x, y]), Y = V(xy)$. Then Y is not irreducible since $Y = V(x) \cup V(y)$.

Definition 1.5.12. Let X be a topological space. If $X = \overline{\{\eta\}}$, then we call η a **generic point** of X.

If X has a generic point, then X is irreducible.

Lemma 1.5.13. *Let A be a ring, $I \subseteq A$ an ideal. Then $V(I)$ is irreducible if and only if $\sqrt{I} = \mathfrak{p}$ is a prime ideal. In that case, $\mathfrak{p}$ is the only generic point of $V(I)$.*

Proof. Up to replacing A by $A/\sqrt{I}$ we may assume that $I = 0$ and is radical. Then $\mathrm{Spec}(A)$ is irreducible if and only if whenever $D(f), D(g) \neq \emptyset$, we have $D(f) \cap D(g) = D(fg) \neq \emptyset$ if and only if whenever $f, g \neq 0$, we have $fg \neq 0$. Moreover, if $\mathfrak{p} = (0)$ is a prime, then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \mathrm{Spec}(A)$, so that $\mathfrak{p}$ is the generic point. □

Definition 1.5.14. Let X be a topological space. We say that X is **sober** if every irreducible subset has a generic point. We say that X is a $\mathbf{T}_0$ **space** (or Kolmogorov space) if for all $x \neq y \in X$, there exists either an open neighborhood U of x such that $y \notin U$ or an open neighborhood V of y such that $x \notin V$.

Consider the map

$$\begin{aligned} F\colon X &\to \{\text{irreducible closed subsets of } X\} \\ x &\mapsto \overline{\{x\}} \end{aligned}$$

Observe that X is T_0 if and only if F is injective, and X is sober if and only if F is bijective. Thus we have sober $\Rightarrow T_0$.

Proposition 1.5.15. *The underlying topological space of every scheme is sober.*

This follows from Lemma 1.5.13 and the following.

Lemma 1.5.16. *Any locally closed subspace of a sober space is sober. A topological space admitting an open cover by sober spaces is sober.*

Proof. Exercise. □

Definition 1.5.17. Let X be a scheme. We say that X is **irreducible** if its underlying topological space is irreducible. We say that X is reduced **reduced** if for every open subset U, $\mathcal{O}_X(U)$ is reduced. (Recall that a ring A is called **reduced** if $\sqrt{(0)} = (0)$). We say that X is **integral** if $X \neq \emptyset$ and for every nonempty open subset U, $\mathcal{O}_X(U)$ is an integral domain.

Proposition 1.5.18. *Let X be a scheme, then*

(1) X is reduced if and only if $\forall x \in X, \mathcal{O}_{X,x}$ is reduced.

(2) X is integral if and only if X is irreducible and reduced.

Proof. (1) $\Rightarrow$ since localization preserves reduced.

$\Leftarrow$ Let $s \in \mathcal{O}_X(U)$ and $s^n = 0$. For all $x \in U$, since $\mathcal{O}_{X,x}$ is reduced, we have $s_x = 0$. It follows that $s = 0$.

(2) $\Rightarrow$ X is easily seen to be reduced. Suppose $U_1, U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$. Then $\mathcal{O}_X(U_1 \cup U_2) \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not a domain.

$\Leftarrow$ It suffices to show that $\mathcal{O}_X(X)$ is integral. Suppose $f, g \in \mathcal{O}_X(X), fg = 0$. Then $X_f \cap X_g = \emptyset$, and hence $X_f = \emptyset$ or $X_g = \emptyset$. Say $X_f = \emptyset$. Then for each open affine subset $V = \operatorname{Spec}(A)$, $V \cap X_f = D(f|_V) = \emptyset$. This implies that $f|_V$ is nilpotent. Since V is arbitrary, f must be 0.

□

Warning 1.5.19. It is **not** true in general without assuming X quasi-compact that a global section of $\mathcal{O}_X$ is nilpotent if and only if every germ of it is nilpotent.

Example 1.5.20. (1) $\operatorname{Spec}(A)$ is reduced if and only if A is reduced.

(2) $\mathrm{Spec}(A)$ is irreducible if and only if $\sqrt{0}$ is a prime ideal.

(3) $\mathrm{Spec}(A)$ is integral if and only if A is integral.

Definition 1.5.21. A **spectral space** is a sober, quasi-compact space such that

(1) quasi-compact opens form a basis.

(2) Finite intersections of quasi-compact opens are quasi-compact.

A continuous map between spectral spaces $f\colon X \to Y$ is called **spectral** if $\forall V$ quasi-compact open of Y, $f^{-1}(V)$ is quasi-compact.

We denote by Sp the category of spectral spaces and spectral maps.

Theorem 1.5.22 (Hochster)**.** *The essential image of the functor* $\mathrm{Spec}\colon \mathrm{Ring} \to \mathrm{Top}$ *is* Sp.

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Lemma 1.5.23. *Let X be an integral scheme with a generic point η. Then*

*(1) $\mathcal{O}_{X,\eta}$ is a field called the **function field** of X.*

(2) For $U \subseteq X$ open, the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ is injective.

Proof. To see that $\mathcal{O}_{X,\eta}$ is a field, we may take an arbitrary nonempty open affine subset $U = \mathrm{Spec}(A)$ and observe that $\mathcal{O}_{X,\eta} = A_{(0)}$ is the fraction field of A.

For the second statement, we may replace U by an nonempty open affine subset and reduce to the case where $U = \mathrm{Spec}(A)$ is affine. In this case $\mathcal{O}_X(U) = A \to \mathrm{Frac}(A) = \mathcal{O}_{X,\eta}$ is injective. □

Corollary 1.5.24. *For X integral and open subsets $\emptyset \neq U \subseteq V \subseteq X$, the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is injective.*

Recall that every topological space X is the disjoint union of connected components. Each connected component is closed but not necessarily open.

Definition 1.5.25. Let X be a topological space, an **irreducible component** of X is a maximal irreducible subset of X.

An irreducible component is necessarily closed. By Zorn's Lemma, every irreducible subset is contained in some irreducible component. Since every point is irreducible, every topological space X is the union of its irreducible components.

Lemma 1.5.26. *Let $X = \bigcup_{i=1}^n Y_i$ be a finite union of irreducible closed subsets. Then the irreducible components of X are the maximal elements of the family $\{Y_i\}_{i=1}^n$. In particular, if there are no inclusions among the Y_i's, then the irreducible components of X are $\{Y_i\}_{i=1}^n$.*

Proof. Indeed, every irreducible subset of X is contained in some Y_i. □

Example 1.5.27. For $A = k[x, y]/(xy)$, $\mathrm{Spec}(A) = V(x) \cup V(y)$. The irreducible components of $\mathrm{Spec}(A)$ are $V(x)$ and $V(y)$.

Example 1.5.28. Let S be a profinite set and k a field. Consider the constant sheaf k_S on S and $A = k_S(S) = \{f : S \to k \text{ locally constant}\}$. It is easy to see that $\mathrm{Spec}(A) \cong (S, k_S)$. Thus, for S an infinite profinite set (e.g. the Cantor set), $\mathrm{Spec}(A)$ has infinitely many irreducible components.

In the case $k = \mathbb{F}_2$, k_S can be identified with the Boolean algebra of open closed subsets of S.

Definition 1.5.29. Let X be a topological space, $x, y \in X$. We say that x **specializes** to y or y **generizes** to x and we write $x \rightsquigarrow y$, if $y \in \overline{\{x\}}$.

Let X be a T_0 space. Generization defines a partial order: $x \leqslant y \iff x \in \overline{\{y\}} \iff \overline{\{x\}} \subseteq \overline{\{y\}}$.

- The minimal points are the closed points.

- If X is sober, then the maximal points are the generic points of irreducible components.

Example 1.5.30. In Spec(A), $\overline{\{x_\mathfrak{p}\}} = V(\mathfrak{p})$. Here $x_\mathfrak{p} \in \operatorname{Spec}(A)$ denotes the point corresponding to the prime ideal $\mathfrak{p}$. We have

$$\begin{aligned} \mathfrak{p} \subseteq \mathfrak{q} &\Leftrightarrow \overline{\{x_\mathfrak{p}\}} \supseteq \overline{\{x_\mathfrak{q}\}} \\ &\Leftrightarrow x_p \rightsquigarrow x_q. \end{aligned}$$

Thus we have a bijection

$$\begin{aligned} \{\text{irreducible components of } \operatorname{Spec}(A)\} &\longleftrightarrow \{\text{minimal primes of } A\} \\ V(\mathfrak{p}) &\leftarrow\!\shortmid \mathfrak{p}. \end{aligned}$$

Warning 1.5.31. Schwede gave an example of a scheme without a closed point. The underlying topological space looks like $x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots$. Note that an affine scheme must have closed points which correspond to maximal ideals.

Noetherian Spaces

Definition 1.5.32. A topological space X is called **Noetherian** if its closed subsets satisfy the descending chain condition, i.e. if $Y_1 \supseteq Y_2 \supseteq \ldots$ is a descending chain of closed subsets, there exists N such that $Y_N = Y_{N+1} = \ldots$. Equivalently, any nonempty family of closed subsets admits a minimal element.

Example 1.5.33. If A is a Noetherian ring, then Spec(A) is Noetherian space.

Warning 1.5.34. If Spec(A) is a Noetherian space, A may not be a Noetherian ring. Let $A = \bigcup_n k[[x^{1/n}]]$ be a union of rings of formal power series. Then $\operatorname{Spec}(A) = \{\eta, s\}$ is a Noetherian space. Here η corresponds to the 0 ideal and s corresponds to the ideal generated by $x^{1/n}, n \in \mathbb{N}$. The ring A is not Noetherian.

Lemma 1.5.35. *Let X be a topological space. The following are equivalent:*

(1) X is Noetherian.

(2) Every open subset of X is quasi-compact.

(3) Every subset of X is quasi-compact.

Proof. (3) $\Rightarrow$ (2) is obvious.

For (2) $\Rightarrow$ (1), note that the union U of an ascending chain of open subsets $U_1 \subseteq U_2 \subseteq \ldots$ is open, hence is quasi-compact by assumption (2).

For (1) $\Rightarrow$ (3), let $Y \subseteq X$ be a subset and $Z_1 \supseteq Z_2 \supseteq \ldots$ be a descending chain of closed subsets in Y. Then $\overline{Z_i} \cap Y = Z_i$ where $\overline{Z_i}$ is the closure in X, and $\overline{Z_i}$ forms a descending chain of closed subsets in X. $\square$

Corollary 1.5.36. *If X is Noetherian, $Y \subseteq X$ with subspace topology. Then Y is Noetherian.*

Corollary 1.5.37. *X is Noetherian and sober $\Rightarrow$ X is a spectral space.*

Lemma 1.5.38. *X is Noetherian $\Rightarrow$ X has only finitely many irreducible components.*

Proof. Consider $\mathcal{F} = \{Y \subseteq X \mid Y \text{ is not a finite union of irreducible closed subsets}\}$. If it is not empty, we can find a minimal element Y by Noetherian hypothesis. Y cannot be irreducible, hence $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper closed subset. But at least one of Y_1, Y_2 must be in $\mathcal{F}$, hence there exists a smaller one, say $Y_1 \in \mathcal{F}$, violating the minimal property of Y. $\square$

Definition 1.5.39. Let X be a scheme.

- X is **quasi-compact** if its underlying space $\mathrm{sp}(X)$ is quasi-compact.
- X is **locally Noetherian** if X can be covered by open affine subsets $U_i = \mathrm{Spec}(A_i)$ with A_i Noetherian rings.
- X is **Noetherian** if X is quasi-compact and locally Noetherian.

Proposition 1.5.40. *Let X be a locally Noetherian scheme and $U = \mathrm{Spec}(A)$ is an open affine subset. Then A is a Noetherian ring. In particular, a ring A is Noetherian if and only if $\mathrm{Spec}(A)$ is a Noetherian scheme.*

Definition 1.5.41. Let $\mathcal{P}$ be a collection of rings. We say that $\mathcal{P}$ is **local** if it satisfies the following properties:

(1) $A \in \mathcal{P}$ implies for any $f \in A, A_f \in \mathcal{P}$.

(2) If there are $f_i \in A, 1 \leqslant i \leqslant n$ such that $\mathrm{Spec}(A) = \bigcup_{i=1}^n D(f_i)$ and $A_{f_i} \in \mathcal{P}$, then $A \in \mathcal{P}$.

Lemma 1.5.42. *Let $\mathcal{P}$ be a local collection of rings, and X a scheme with an open affine cover $\{U_i\}_{i \in I}$ with $U_i = \mathrm{Spec}(A_i)$, where each $A_i \in \mathcal{P}$. Then for every open affine subset $U = \mathrm{Spec}(A)$, we have $A \in \mathcal{P}$.*

The proof relies on the following technical result.

Lemma 1.5.43. *Let X be a scheme and $U = \mathrm{Spec}(A), V = \mathrm{Spec}(B)$ open affine subsets. Then $U \cap V$ can be written as a union of open affine subsets which are principal open subsets of both U and V.*

Proof. For $x \in U \cap V$, choose a principal open subset W of U that covers x and is contained in V. Up to replacing U by W, we may assume $U \subseteq V$. Choose $f \in B$ such that $V_f = \mathrm{Spec}(B_f) \subseteq U$. We observe that $V_f = U_{\bar{f}} = \mathrm{Spec}(A_{\bar{f}})$ is also a principal open subset of U. Here $\bar{f} = f|_U$. $\square$

Proof of Lemma 1.5.42. Let $U = \mathrm{Spec}(A)$ be an open affine subset. Then $U = \bigcup_i (U \cap U_i)$. By the previous lemma, $U \cap U_i$ can be covered by open affine subsets U_{ij} which are both principal in U and U_i. By hypothesis 1 of $\mathcal{P}$, each U_{ij} is the spectrum of a ring in $\mathcal{P}$. Since U is quasi-compact, we may choose finitely many of them and apply hypothesis 2 in the definition. $\square$

Proof of Proposition 1.5.40. It remains to show that $\mathcal{P} = \{\text{Noetherian rings}\}$ is a local collection. The first condition is easy to verify. For the second one, let $f_i \in A$, $1 \leqslant i \leqslant n$, satisfying $\mathrm{Spec}(A) = \bigcup_i D(f_i)$ with each A_{f_i} Noetherian. We will show that every ideal $I \subseteq A$ is finitely generated. For each i, the ideal $IA_{f_i} \subseteq A_{f_i}$ is finitely generated. Let $\{a_{ij}\}_{j=1}^{m_i}$ be a family of generators of IA_{f_i} in I. Then $\{a_{ij}\}_{i,j}$ generates I. Indeed, if $\phi\colon A^{m_1+\cdots+m_n} \to I$ denotes the homomorphism of A-modules given by $\{a_{ij}\}_{i,j}$, then ϕ_{f_i} is a surjection for every i, which implies that ϕ is a surjection. $\square$

Remark 1.5.44. If X is a Noetherian scheme, then its underlying space $\mathrm{sp}(X)$ is Noetherian.

Warning 1.5.45. There exists a Noetherian space which is not the underlying space of any Noetherian scheme.

In fact, it follows from Krull's principal ideal theorem that a Noetherian scheme of dimension ≥ 2 (see below for the definition of dimension) must have infinitely many points. Thus spaces such as $X = \{x \rightsquigarrow y \rightsquigarrow z\}$ cannot be the underlying space of a Noetherian scheme.

Warning 1.5.46. For a Noetherian scheme X, $\mathcal{O}_X(X)$ is not a Noetherian ring in general. Consider the projective space $\mathbb{P}^3_k$ over a field k, with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$. Let $D = V(x_0)$ and $E = V(x_1)$ be distinct planes of $\mathbb{P}^3_k$ and let $l = V(x_0, x_2) \neq D \cap E$ be a projective line on D. Let $Y = D \cup E$ and $X = Y \backslash l$. Then X is a Noetherian scheme. We have $X = (D\backslash l) \cup (E\backslash O)$, where $O = E \cap l$. We have $\mathcal{O}_X(D\backslash l) = k[x, y]$, where $x = \frac{x_1}{x_2}$ and $y = \frac{x_3}{x_2}$, and $\mathcal{O}_X(E\backslash O) = k$. The restriction map $\mathcal{O}_X(D\backslash l) \to \mathcal{O}_X((D\backslash l) \cap E)$ is given by substituting $x = 0$. One can deduce that $\mathcal{O}_X(X) \cong k + xk[x, y] \subseteq k[x, y]$. This is not a Noetherian ring: there is an ascending chain of ideals $(x) \subseteq (x, xy) \subseteq (x, xy, xy^2) \subseteq \ldots$.

Dimension

Definition 1.5.47. Let X be a topological space. The **dimension** of X, denoted by $\dim X$, is define to be

$$\sup_n \{n \mid \exists Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \text{ such that } Y_i \text{ are irreducible closed}\}$$

Let $Y \subseteq X$ be an irreducible closed subset. The **codimension** $\mathrm{codim}(Y, X)$ of Y is defined to be

$$\sup_n \{n \mid \exists Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \text{ such that } Y_i \text{ are irreducible closed}\}$$

If Y is an arbitrary closed subset, we define its codimension as

$$\inf\{\mathrm{codim}(Y', X) \mid Y' \subseteq Y \text{ irreducible and closed}\}$$

Example 1.5.48. $\dim \emptyset = -\infty$, $\mathrm{codim}(\emptyset, X) = \infty$.

Lemma 1.5.49. *Let X be a topological space.*

- $\dim X = \sup\{\dim X_i \mid X_i \subseteq X$ *are irreducible components*$\}$
- *If* $Z \subseteq X$, $\dim Z \leqslant \dim X$.
- *If* $\{U_i\}$ *is an open cover of* X, *then* $\dim X = \sup_i(\dim U_i)$.

If X is a sober space,

$$\dim X = \sup\{n \mid \exists x_n \rightsquigarrow x_{n-1} \rightsquigarrow \cdots \rightsquigarrow x_0 \text{ with all } x_i \text{ distinct}\}$$

$$\operatorname{codim}(\overline{\{x\}}, X) = \sup\{n \mid \exists x_n \rightsquigarrow x_{n-1} \rightsquigarrow \cdots \rightsquigarrow x_0 = x \text{ with all } x_i \text{ distinct}\}$$

Example 1.5.50. Let $X = \operatorname{Spec}(A)$.

- $\dim(X) = \dim(A) = \sup\{n \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.
- For a prime ideal $\mathfrak{p}$, $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{ht}(\mathfrak{p}) = \sup\{n \mid \exists \mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\}$

Theorem 1.5.51. *Let* A *be a Noetherian ring.*

- *For every prime ideal* $\mathfrak{p}$, $\operatorname{ht}(\mathfrak{p}) < \infty$.
- *If* A *is local, then* $\dim A < \infty$.

Warning 1.5.52. A Noetherian ring may have dimension ∞ (Nagata).

1.6 Morphisms and base change

In this section, we talk about properties between morphisms of schemes.

Definition 1.6.1. Let $: Y \to X$ be a morphism of schemes.

- f is called **locally of finite type** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine subset and for each i $f^{-1}(U_i) = \bigcup_j V_{ij}$ with each $V_{ij} = \operatorname{Spec}(B_{ij})$ open affine subset such that B_{ij} is a finitely generated A_i-algebra.
- f is called **quasi-compact** if $X = \bigcup_i U_i$ with each U_i open affine such that $f^{-1}(U_i)$ is quasi-compact.
- f is of **finite type** if f is locally of finite type and quasi-compact.
- f is called **affine** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine subset such that $f^{-1}(U_i)$ is also affine.
- f is called **finite** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine and for each i $f^{-1}(U_i) = \operatorname{Spec}(B_i)$ such that B_i is a finite A_i-algebra. (Recall that an A-algebra B is **finite** if B is finitely generated as an A-module.)

Remark 1.6.2. In the definition above, the existence of an open affine cover can be replaced by "for every open affine cover".

We clearly have the following implications:

$$\begin{tikzcd}
 & \text{affine} \arrow[dr, Rightarrow] & \\
\text{finite} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{quasi-compact} \\
 & \text{of finite type} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \text{locally of finite type}
\end{tikzcd}$$

Example 1.6.3. Let A be a DVR with fractional field K and residue field k. Then the natural morphisms $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ is finite but $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ is of finite type. Note that if π is a uniformizer of A, then $K = A[\pi^{-1}]$.

Example 1.6.4. $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ is not locally of finite type.

Example 1.6.5. Let A be a ring. Then $\mathbb{A}^n_A \to \mathrm{Spec}(A)$ and $\mathbb{P}^n_A \to \mathrm{Spec}(A)$ are both of finite type.

Definition 1.6.6. Let k be a field.

- An **affine k-variety** is an integral scheme equipped with an affine morphism of finite type to $\mathrm{Spec}(k)$.
- A **k-variety** is an integral scheme equipped with a separated morphism of finite type to $\mathrm{Spec}(k)$.

It is clear that we have an equivalence of categories

$$\{\text{affine } k\text{-varieties}\} \cong \{\text{finitely generated } k\text{-algebras that are domains}\}^{\mathrm{op}}.$$

Date: 10.6

We first supplement some results on dimension:

Fact 1.6.7. *Let A be a Noetherian ring.*

- $\dim A[x] = \dim A + 1$ *(*[AM, Exercise 11.7]*,* [M2, Theorem 15.4]*).*
- *If A is a finitely generated k-algebra which is also a domain, then* $\dim A = \operatorname{tr.deg}(\operatorname{Frac}(A)/k)$ [M2, Theorem 5.6]*.*
- *Krull's principal ideal theorem: Let $f \in A$. Then for each minimal prime $\mathfrak{p}$ containing f,* $\operatorname{ht}(\mathfrak{p}) \leqslant 1$ [AM, Corollary 11.16]*. Moreover, for $A \neq 0$,* $\operatorname{ht}(f) = 0$ *if and only if f is a zero divisor* [AM, Proposition 4.7]*.*

  Repeatedly applying Krull's principal ideal theorem, we get that for each minimal prime $\mathfrak{p}$ containing $(f_1, ..., f_r)$, $\operatorname{ht}(\mathfrak{p}) \leqslant r$*. In particular,* $\operatorname{ht}(f_1, ..., f_r) \leqslant r$*.*

Lemma 1.6.8. *Let X be a topological space and $Y \subseteq X$ a closed subset. Then* $\dim X \geqslant \dim Y + \operatorname{codim}(Y, X)$*.*

Proof. Take $Z \subseteq Y$ irreducible closed. By definition,

$$\dim X \geqslant \dim Z + \operatorname{codim}(Z, X) \geqslant \dim Z + \operatorname{codim}(Y, X).$$

We conclude by taking supremum over $Z \subseteq Y$. □

Example 1.6.9. Let A be a DVR and let $\mathfrak{m} = (\pi)$ be the maximal ideal. Consider the ideal $\mathfrak{p} = (\pi x - 1)$ in $B = A[x]$. This is a maximal ideal, since $B/\mathfrak{p} = A[1/\pi] = \operatorname{Frac}(A)$. From Fact 1.6.7, $\dim B = \dim A + 1 = 2$ and $\operatorname{ht}(\mathfrak{p}) = 1$, hence $\operatorname{ht}(\mathfrak{p}) + \dim B/\mathfrak{p} < \dim B$. In geometric form, we have $\dim \operatorname{Spec}(B) > \dim\{\mathfrak{p}\} + \operatorname{codim}(\{\mathfrak{p}\}, \operatorname{Spec}(B))$.

Definition 1.6.10. Let X be a topological space.

- We call X **equidimensional** if all irreducible components have the same dimension.
- Assume that X is T_0. We call X **equicodimensional** if all closed points have the same codimension.

Example 1.6.11. Suppose $\dim X = \dim Y + \operatorname{codim}(Y, X)$ holds for all $Y \subseteq X$ closed.

- Take Y to be an irreducible component of X. Then $\operatorname{codim}(Y, X) = 0$, hence $\dim Y = \dim X$. Thus X is equidimensional.
- Assume that X is T_0. Take $Y = \{x\}$ to be a closed point. Then $\dim\{x\} = 0$, hence $\operatorname{codim}(\{x\}, X) = \dim X$. Thus X is equicodimensional.

The example $B = A[x]$ in Example 1.6.9 is not equicodimensional.

By contrast, we have the following result.

Theorem 1.6.12. *Let $S = \mathrm{Spec}(k)$, k a field or let S be an integral Noetherian scheme of dimension 1 which has infinitely many points. For any equidimensional scheme X equipped with a finite type morphism $X \to S$, the equality $\dim X = \dim Y + \mathrm{codim}(Y, X)$ holds for every closed subset $Y \subseteq X$. In particular, X is equicodimensional.*

For a proof, see [G, IV 10.6.1].

Definition 1.6.13. Let $\phi\colon A \to B$ be a ring homomorphism.

- We call B a finite A-algebra if B is a finitely generated A-module.
- We call B an integral A-algebra if $\forall x \in B$, $\phi(A)[x]$ is a finitely generated A-module.

We have B is a finite A-algebra $\Leftrightarrow$ B is finitely generated and integral.

Definition 1.6.14. Let $f\colon Y \to X$ be a morphism of schemes. We say that f is **integral** if there exists an cover $X = \bigcup U_i$ with $U_i = \mathrm{Spec}(A_i)$ affine open such that $f^{-1}(U_i) = \mathrm{Spec}(B_i)$ and B_i integral over A_i.

For $f\colon Y \to X$, we have f finite $\Leftrightarrow$ f integral and locally of finite type.

Theorem 1.6.15. *An integral morphism is a closed map.*

Proof. Let $f\colon Y \to X$ be integral. Since a subset is closed if and only if its intersection with every member of an open cover is closed, we may assume $X = \mathrm{Spec}(A)$ is affine. In this case $Y = \mathrm{Spec}(B)$ is affine as well and f is induced by $\phi\colon A \to B$. Let $J \subseteq B$ be an ideal, $V(J) \subset \mathrm{Spec}(B)$ a closed subset. Let $I = \phi^{-1}(J)$. We have $A/I \to B/J$ is integral as well. From the fact that every prime ideal in A/I is a contracted ideal [AM, Theorem 5.10] (which implies the going-up theorem), we have $f(V(J)) = V(I)$. Therefore, f is closed. $\square$

Fiber Products

Recall a fiber product of a diagram $X \xrightarrow{a} S \xleftarrow{b} S$ is an object $X \times_S Y$ equipped with two morphisms p, q indicated below, which satisfies the following universal property: For any object Z equipped with two morphisms f, g such that $af = bg$, there exists a unique morphism h such that $ph = f$, $qh = g$.

$$\begin{tikzcd}
 & Z \arrow[d, dashed, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & X \times_S Y \arrow[dl, "p"] \arrow[dr, "q"] & \\
X \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\
 & S &
\end{tikzcd}$$

Proposition 1.6.16. *Fiber products exist in the category of schemes.*

Proof. Let $a\colon X \to S$ and $b\colon Y \to S$ be given.

Case 1: $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$ are all affine.

Define $X \times_S Y = \mathrm{Spec}(B \otimes_A C)$. For any scheme Z, we have $\mathrm{Hom}(Z, X) \simeq \mathrm{Hom}(B, \mathcal{O}_Z(Z))$ and similarly for Y and S. The universal property for $X \times_S Y$ translates into the universal property of $B \otimes_A C$ in the category of rings.

Case 2: $X = \bigcup X_i$ with $X_i \subseteq X$ open such that $X_i \times_S Y$ exists.

For any $U \subseteq X_i$, $U \times_S Y$ exists and can be identified with the inverse image of U along $X_i \times_S Y \to X_i$, as shown in the diagram with Cartesian squares

$$\begin{tikzcd} U \times_S Y \arrow[r] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

Let $X_{ij} = X_i \cap X_j$. Then $X_{ij} \times_S Y$ exists and we can glue $X_i \times_S Y$ along $X_{ij} \times_S Y$ and get $X \times_S Y$.

Case 3: S and Y are affine and X is general.

Cover X by affine open subsets and apply Cases 1 and 2.

Case 4: S affine and X, Y general.

Cover X by affine open subsets and apply Cases 2 and 3 (with X and Y swapped).

Case 5: The general case.

Let $S = \bigcup S_i$ be an affine open cover. Let $X_i = a^{-1}(S_i)$, $Y_i = b^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists by Case 4. But we have $X_i \times_{S_i} Y_i \cong X_i \times_S Y$ as shown in the diagram below

$$\begin{tikzcd} X_i \times_{S_i} Y_i \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & Y \arrow[d, "b"] \\ X_i \arrow[r] & S_i \arrow[r] & S \end{tikzcd}$$

Thus we can glue them to get $X \times_S Y$.

□

Warning 1.6.17. The natural map $\mathrm{sp}(X \times_S Y) \to \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ is not injective in general.

Definition 1.6.18. Let $f\colon X \to S$ be a morphism of schemes and let $s \in S$. Define the fiber X_s of f at s

$$\begin{tikzcd} X_s = X \times_S \mathrm{Spec}(\kappa(s)) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

Proposition 1.6.19. *The map $X_s \to f^{-1}(s)$ is a homeomorphism.*

Proof. Without loss of generality, we may assume $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, and f is induced by $\phi\colon A \to B$. Let $s \in S$ be defined by the prime ideal $\mathfrak{p}$. We have $\kappa(s) = \kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Hence $X_s = \operatorname{Spec}(B \otimes_A (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$. Elements of $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ correspond bijectively to primes of $\mathfrak{q}$ of B such that $\mathfrak{q} \supseteq \phi(\mathfrak{p})$ and $\mathfrak{q}$ does not intersect $\phi(A\backslash\mathfrak{p})$. This is equivalent to $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Thus the map $g\colon X_s \to f^{-1}(s)$ is a bijection. Since $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \to \operatorname{Spec}(B_\mathfrak{p}) \to \operatorname{Spec}(B)$ are successive embeddings and $f^{-1}(s)$ is endowed with the subspace topology, g is a homeomorphism. □

We may view a morphism $f\colon X \to S$ as a family of fibers X_s parameterized by $s \in S$.

Example 1.6.20. Consider $f\colon X = \operatorname{Spec}(k[t,y,x]/(xy-t)) \to S = \operatorname{Spec}(k[t])$. The fiber at a rational point $t = a$ of S is $\operatorname{Spec}(k[x,y]/(xy-a))$. For $a \neq 0$, the fiber is a hyperbola isomorphism to $\operatorname{Spec}(k[x,x^{-1}])$. For $a = 0$, the fiber is the union of the coordinate axes of the affine plane and, in particular, is not irreducible.

Definition 1.6.21. Let $\mathcal{P}$ be a class of morphisms. We call $\mathcal{P}$ **stable under base change** if for every $f\colon X \to S$ in $\mathcal{P}$ and every morphism $Y \to S$, the base change $f \times_S Y\colon X \times_S Y \to Y$ belongs to $\mathcal{P}$.

Example 1.6.22. The following classes of morphisms are stable under base change

- locally of finite type
- quasi-compact
- affine
- integral
- of finite type
- finite

Lemma 1.6.23. *Surjective morphisms are stable under base change.*

Proof. Let $f\colon X \to S$ be a surjective morphism and $S' \to S$ a morphism. Let $X' = S \times_S S'$. Take $s' \in S'$. We need to show that the fiber $X'_{s'} \neq \emptyset$.

$$\begin{tikzcd}
X'_{s'} \arrow[r] \arrow[d] \arrow[drr] & X' \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\
& & X_s \arrow[ur] \arrow[dd] & \\
\operatorname{Spec}(\kappa(s')) \arrow[r] \arrow[drr] & S' \arrow[rr, "f'"] & & S \\
& & \operatorname{Spec}(\kappa(s)) \arrow[ur] &
\end{tikzcd}$$

Since f is surjective, $X_s \neq \emptyset$. We are thus reduced to showing that for any k-scheme $X \neq \emptyset$ and any field extension k'/k, we have $X \otimes_k k' := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k') \neq \emptyset$. We may assume $X = \operatorname{Spec}(A)$ is affine. In this case it suffices to observe that $A \otimes_k k' \neq 0$. □

Warning 1.6.24. Injectivity and bijectivity are not stable under base change. For example, $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is bijective. After base change to $\mathbb{C}$, we have $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) = \mathbb{C}[x]/(x^2+1) \cong \mathbb{C} \times \mathbb{C}$, which has two prime ideals. Thus $\mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ is not injective or bijective.

Warning 1.6.25. Closed morphisms are not stable under base change. For example, $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is closed but $\mathbb{A}^2_k = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ is not closed, since the image of $V(xy-1)$ is the open subset $\mathbb{A}^1_k \backslash \{0\}$, which is not closed.

Definition 1.6.26. Let f be a morphism of schemes.

- f is called **universally closed** if every base change of f is a closed mapping.
- f is called a **universal homeomorphism** if every base change of f is a homeomorphism.
- f is called **universally injective** or **radiciel** if every base change of f is injective.

Example 1.6.27. An integral morphism is universally closed.

Proposition 1.6.28. *Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent*

(a) f is radiciel.

(b) f is injective and $\forall x \in X$, $\kappa(x)/\kappa(f(x))$ is purely inseparable.

(c) For every field K, $f(K)\colon X(K) \to Y(K)$ is injective, where $X(K) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(K), X)$.

Note that $X(K)$ can be identified with the set of pairs (x, ι), where $x \in X$ and $\iota\colon \kappa(x) \to K$ is a field embedding.

Proof. $(a) \Rightarrow (c)$. Let $t_1, t_2 \in X(K)$ such that $f(K)(t_1) = f(K)(t_2)$. Consider the Cartesian square in the following diagram

$$\begin{tikzcd}
X \times_Y \mathrm{Spec}(K) \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(K) \arrow[u, bend left, "s"] \arrow[r] \arrow[ur, shift left, "t_1"] \arrow[ur, shift right, "t_2"'] & Y
\end{tikzcd}$$

Each t_i corresponds to a section s_i of f' by the universal property of fiber product. f' is injective, the image of s_1 coincides with the image of s_2. For any morphism $g\colon Z \to \mathrm{Spec}(K)$, sections s of g are uniquely determined by the image of s. Thus $s_1 = s_2$ and hence $t_1 = t_2$.

$(c) \Rightarrow (a)$. For any $Y' \to Y$, if we write $X' = X \times_Y Y'$, then $X'(K) = X(K) \times_{Y(K)} Y'(K)$, which injects into $Y'(K)$.

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[rr, bend left] \arrow[dr] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y' \arrow[r] & Y
\end{tikzcd}$$

Therefore, it suffices to prove that f is injective itself. Let $x, x' \in X$ such that $f(x) = f(x') = y$. There exists a field K and field embeddings $\kappa(x) \xrightarrow{\iota} K \xleftarrow{\iota'} \kappa(x')$ making

$$\begin{tikzcd}[column sep=small] & K & \\ \kappa(x) \arrow[ur, "\iota"] & & \kappa(x') \arrow[ul, "\iota'"'] \\ & \kappa(y) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutative. This defines $(x, \iota), (x', \iota') \in X(K)$ satisfying $f(K)(x, \iota) = f(K)(x', \iota')$. Hence $(x, \iota) = (x', \iota')$ and in particular $x = x'$.

For the equivalence $(b) \Leftrightarrow (c)$, recall that, in the category of fields, $k \to k'$ is an epimorphism if and only if k'/k is purely inseparable.

$(c) \Rightarrow (b)$. We have already proven that f is injective. It suffices to show that $\phi\colon \kappa(f(x)) \to \kappa(x)$ is an epimorphism of fields. Let $\iota, \iota'\colon \kappa(x) \to K$ be field embeddings satisfying $\iota\phi = \iota'\phi$. Then $f(K)(x, \iota) = f(K)(x, \iota')$. Hence $(x, \iota) = (x, \iota')$, namely $\iota = \iota'$.

$(b) \Rightarrow (c)$. This is similar to the last step. Let $(x, \iota), (x', \iota') \in X(K)$ such that $f(K)(x, \iota) = f(K)(x', \iota')$. In other words, $f(x) = f(x')$ and $\iota\phi = \iota'\phi$. Since f is injective, we have $x = x'$. Since ϕ is an epimorphism of fields, we have $\iota = \iota'$. □

Remark 1.6.29. We have integral + surjective + radiciel ⇒ universal homeomorphism. The converse also holds by a result of Deligne [G, IV 18.12.11].

Example 1.6.30. Let k'/k be a purely inseparable field extension. Then $\mathrm{Spec}(k) \to \mathrm{Spec}(k')$ is integral, surjective, radiciel, and hence a universal homeomorphism.

Definition 1.6.31. Let $\mathcal{P}$ be a class of morphisms. We say $\mathcal{P}$ is **stable under composition** if whenever $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f, g \in \mathcal{P}$, we have $gf \in \mathcal{P}$.

Example 1.6.32. The classes in Example 1.6.22 are stable under composition.

Lemma 1.6.33. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$.

(1) If gf is locally of finite type, then so is f.

(2) If gf is quasi-compact and f is surjective, then g is quasi-compact.

Proof. The first statement boils down to the following property of rings: if the composition $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is of finite type, then so is ψ.

For the second statement, let V be a quasi-compact open subset of Z. Then $(gf)^{-1}(V)$ is quasi-compact and $g^{-1}(V) = f((gf)^{-1}(V))$ is quasi-compact. □

Date: 10.8

We first continue our discussion about topology.

Let $f\colon X \to Y$ be a continuous map. For $x, x' \in X$ and $x \rightsquigarrow x'$, we have $f(x) \rightsquigarrow f(x')$. Indeed, for every closed subset F of Y containing $f(x)$, we have $f^{-1}(F) \ni x$ and consequently $f^{-1}(F) \ni x'$ and $F \ni f(x')$.

Definition 1.6.34. Let $f\colon X \to Y$ be a continuous map.

- f is called **specilizing** if $\forall y \rightsquigarrow y' \in Y$, $\forall x \in f^{-1}(y)$, $\exists x' \in f^{-1}(y')$ such that $x \rightsquigarrow x'$.
- f is called **generizing** if $\forall y \rightsquigarrow y' \in Y$, $\forall x' \in f^{-1}(y')$, $\exists x \in f^{-1}(y)$ such that $x \rightsquigarrow x'$.

Example 1.6.35. f closed $\Rightarrow$ f specializing. This is easily deduced from $f(\overline{\{x\}}) \supseteq \overline{\{f(x)\}}$.

Let X be a scheme. Then $\mathrm{Spec}(\mathcal{O}_{X,x})$ maps homeomorphically onto the subspace $\{x' \in X \mid x' \rightsquigarrow x\}$ of X. To see this, we may assume that $X = \mathrm{Spec}(A)$ is affine. Let x correspond to a prime ideal $\mathfrak{p}$. Then

$$\{x' \in X \mid x' \rightsquigarrow x\} = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \simeq \mathrm{Spec}(A_{\mathfrak{p}}).$$

From this, we deduce:

Lemma 1.6.36. *A morphism of schemes $f\colon X \to Y$ is generizing if and only if $\forall x \in X$, $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,f(x)})$ is surjective.*

Definition 1.6.37. A morphism of schemes $f\colon X \to Y$ is called **flat** if $\forall x \in X$, $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Recall that a ring homomorphism $\phi\colon A \to B$ is flat $\Leftrightarrow$ $\forall \mathfrak{q} \in \mathrm{Spec}(B), A_{\phi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is flat.

Flat morphisms are stable under composition and base change.

Lemma 1.6.38. *Every flat local homomorphism $\phi\colon A \to B$ of local rings is faithfully flat. In other words, ϕ induces a surjective map $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.*

Proof. This follows from the fact that a flat homomorphism of rings $A \to B$ is faithfully flat if and only if for every maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m}B \subsetneq B$ ([AM, Exercise 3.16], [M2, Theorem 7.2]). □

Corollary 1.6.39. *Every flat morphism of schemes is generizing.*

Remark 1.6.40. Let $f\colon X \to Y$ be a morphism of schemes.

(1) If f is generizing, then every maximal point of X lies above a maximal point of Y.

(2) Assume that Y is irreducible with generic point η. We have an injective map

$$\begin{aligned}\mathrm{IrrComp}(X_\eta) &\to \mathrm{IrrComp}(X),\\ Z &\to \overline{Z}\end{aligned}$$

whose image consists precisely of the irreducible components intersecting X_η. In particular, if f is generizing, then the above map is a bijection.

In particular:

Lemma 1.6.41. *Let $f\colon X \to Y$ be a morphism of schemes. Suppose Y is irreducible with generic point η. Then*

(1) X irreducible $\Rightarrow X_\eta$ irreducible or empty.

(2) If f is generizing, then X irreducible $\iff X_\eta$ irreducible.

Consider k'/k a field extension, X/k a k scheme, denote $X \otimes_k k = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$.

Remark 1.6.42. Let k'/k be a field extension.

(1) $X \otimes_k k$ connected $\Rightarrow X$ connected.

(2) $X \otimes_k k$ irreducible $\Rightarrow X$ is irreducible.

(3) $X \otimes_k k$ reduced $\Rightarrow X$ reduced.

(4) $X \otimes_k k$ integral $\Rightarrow X$ integral.

(1) and (2) follow from the surjectivity of $X \otimes_k k' \to X$. To see (3), we may assume $X = \mathrm{Spec}(A)$ is affine. Then $A \hookrightarrow A \otimes_k k'$ and the latter is assumed to be reduced. For (4), combine (2) and (3).

Definition 1.6.43. Let X be a scheme over a field k and let $\overline{k}$ be an algebraic closure of k.

- X is called **geometrically connected** if $X \otimes_k \overline{k}$ is connected.
- X is called **geometrically irreducible** if $X \otimes_k \overline{k}$ is irreducible.
- X is called **geometrically reduced** if $X \otimes_k \overline{k}$ is reduced.
- X is called **geometrically integral** if $X \otimes_k \overline{k}$ is integral.

Remark 1.6.44. If k is separably closed, then

$$\begin{aligned}X \text{ connected } &\iff X \text{ geometrically connected}\\ X \text{ irreducible } &\iff X \text{ geometrically irreducible}\end{aligned}$$

Indeed, in this case $\overline{k}/k$ is purely inseparable and $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ is a universal homeomorphism.

Proposition 1.6.45. *Let X/k be a scheme over a field. The following are equivalent.*

(1) For every finite separable extension k'/k, $X \otimes_k k'$ is irreducible.

(2) X is geometrically irreducible.

(3) X is irreducible with generic point η and the separable closure of k in $\kappa(\eta)$ is k.

Proof. (2) $\Rightarrow$ (1) is clear, since $X \otimes_k \overline{k} \to X \otimes_k k'$ is surjective.

(1) $\Rightarrow$ (3). X is clearly irreducible. For every finite separable extension k'/k, since $X \otimes_k k'$ is irreducible, $(X \otimes_k k')_\eta = \mathrm{Spec}(\kappa(\eta) \otimes_k k')$ is irreducible. Let $\alpha \in \kappa(\eta)$ be a separable algebraic element over k with minimal polynomial $P(x)$. Let $k' = k(\alpha)$. Then $\kappa(\eta) \otimes_k k' = (\kappa(\eta) \otimes_k k'[x]/(P(x)) = k(\eta)[x]/(P(x))$. We have $P(x) = (x-\alpha)Q(x)$ with $Q(x) \in k(\eta)[x]$. Then $k(\eta)[x]/(P(x)) = k(\eta)[x]/(x-\alpha) \oplus k(\eta)[x]/(Q(x))$, which implies $Q(x) = 1$ and $\alpha \in k$.

(3) $\Rightarrow$ (2). The projection $X \otimes_k \overline{k} \to X$ is a base change of $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ and hence is flat and generizing. Thus $X \otimes_k \overline{k}$ is irreducible if and only if $(X \otimes_k \overline{k})_\eta = \mathrm{Spec}(\kappa(\eta) \otimes_k \overline{k})$ is irreducible. Let k^{sep} be a separable closure of k. Since $\overline{k}/k^{\mathrm{sep}}$ is purely inseparable, it suffices to show that $\mathrm{Spec}(\kappa(\eta) \otimes_k k^{\mathrm{sep}})$ is irreducible. By the lemma below applied to the Galois extension, k^{sep}/k $\kappa(\eta) \otimes_k k^{\mathrm{sep}}$ is a field. □

Lemma 1.6.46. *Let k'/k be a field extension and K/k a Galois extension. Assume $k' \cap K = k$ in the composite field $k' \cdot K$. Then $k' \otimes_k K$ is a field.*

Proof. Since K is a union of Galois extensions of k, we may assume K/k is a finite Galois extension of degree d. Consider the surjection $\phi\colon k' \otimes_k K \to k' \cdot K$ is surjective. Note that $k' \cdot K/k'$ is a Galois extension of Galois group $\mathrm{Gal}(k' \cdot K/k') \simeq \mathrm{Gal}(K/k' \cap K) = \mathrm{Gal}(K/k)$. Thus $\dim_{k'}(k' \cdot K) = d = \dim_{k'}(k' \otimes_k K)$. It follows that ϕ is an isomorphism. □

We give some examples which are not geometrically irreducible.

Example 1.6.47. $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is not geometrically connected, since we have shown its base change to $\mathbb{C}$ is two points. This can also be seen from criterion (3), since $\mathbb{C}/\mathbb{R}$ is a separable algebraic extension.

Example 1.6.48. Let $A = \mathbb{R}[x,y]/(x^2+y^2)$ and $X = \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{R})$. Since x^2+y^2 is irreducible in $\mathbb{R}[x,y]$, X is irreducible. Since x^2+y^2 factors as $(x+iy)(x-iy)$ in $\mathbb{C}$, the base change of X to $\mathbb{C}$ is $\mathbb{C}[x,y]/(x+iy)(x-iy)$, which is the union of two lines intersecting at a point. Thus X is geometrically connected but not geometrically irreducible.

Let η be the generic point of η. In $\kappa(\eta) = \mathrm{Frac}(A)$, we have $(x/y)^2 + 1 = 0$. Thus the separable closure of $\mathbb{R}$ in $\kappa(\eta)$ can be identified with $\mathbb{C}$.

Next, we study geometrically reduced schemes. We start with the case of field extensions.

Definition 1.6.49. A field extension K/k is said to be **separable** if $\mathrm{Spec}(K)$ is geometrically reduced over $\mathrm{Spec}(k)$.

Let $\overline{k}$ be the algebraic closure of k. By definition, K/k is separable if and only if $K \otimes_k \overline{k}$ is reduced.

Since $K = \bigcup_{\alpha \in K} k(\alpha)$, K/k is separable if and only if $k(\alpha)/k$ is separable for all $\alpha \in K$. We have

$$k(\alpha) \otimes_k \overline{k} = \begin{cases} \overline{k}(\alpha) & \alpha \text{ transcendental} \\ \overline{k}[x]/(P(x)) & \alpha \text{ algebraic with minimal polynomial P(x)} \end{cases}$$

Note that $\overline{k}[x]/(P(x))$ is reduced if and only if $P(x)$ is a separable polynomial (namely, a polynomial with only simple roots in $\overline{k}$). We have proved the following.

Lemma 1.6.50. *K/k is separable if and only if $\forall \alpha \in K$, α is either transcendental over k or separable algebraic over k.*

Remark 1.6.51. In particular,

(1) Definition 1.6.49 extends the usual notion of separable algebraic extensions.

(2) Any purely transcendental extension is separable.

(3) If k is a perfect field, then any field extension K/k is separable.

Lemma 1.6.52. *Let $L/K/k$ be a tower of field extensions.*

(1) K/k is separable $\iff$ for every finite field extension k'/k, $K \otimes_k k'$ is reduced.

(2) L/K and K/k separable $\Longrightarrow$ L/k separable.

(3) L/k separable $\Longrightarrow$ K/k separable.

Proof. (1) and (3) are trivial.

(2) For any finite field extension k'/k, $L \otimes_k k' = L \otimes_K (K \otimes_k k')$. Since K/k is separable, $K \otimes_k k'$ is a finite direct sum of finite field extensions of K. We conclude by the assumption that L/K is separable. □

Warning 1.6.53. Unlike the case of separable algebraic extensions, for a tower $L/K/k$ of field extensions, L/k separable does **not** imply L/K separable. Here is an example: $L = k(x)$, $K = k(x^p)$, where $p = \mathrm{char}(k) > 0$. Then L/k is separable but L/K is purely inseparable.

Definition 1.6.54. Let K/k be a separable extension. A **separating transcendence basis** is a transcendence basis B for K/k such that $K/k(B)$ is separable.

Lemma 1.6.55. *Let $K = k(x_1, \dots x_n)/k$ be a finitely generated separable extension. Then K admits a separating transcendence basis contained in $\{x_1, \dots x_n\}$.*

Proof. This is proved in [M2, Theorem 26.2]. Note that the definition there *a priori* differs from ours. We give a proof here for completeness.

We may assume $\mathrm{char}(k) = p > 0$. We proceed by induction on n. We may assume that $x_1, \dots, x_r$ is a transcendence basis. If $r = n$, we are done. Suppose $r < n$. Then $x_1, \dots, x_{r+1}$ are algebraically dependent. There exists a nonzero $P \in k[X_1, \dots, X_{r+1}]$

with least degree such that $P(x_1, \dots, x_{r+1}) = 0$. The minimality of the degree implies that P is irreducible.

Let us prove $P \notin k[X_1^p, \dots, X_{r+1}^p]$. Assume otherwise. Then $P = Q^p$ with $Q \in k^{1/p}[X_1, \dots, X_{r+1}]$. Write $Q = \sum_\alpha c_\alpha X^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_{r+1})$ and $X^\alpha = X_1^{\alpha_1} \cdots X_{r+1}^{\alpha_{r+1}}$. Let $I = \{\alpha \mid c_\alpha \neq 0\}$. Since $(\sum c_\alpha \otimes x^\alpha)^p = 0$ in $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k K$ is reduced, we have $\sum_{\alpha \in I} c_\alpha \otimes x^\alpha = 0$ in $\overline{k} \otimes_k K$. This implies that $(x^\alpha)_{\alpha \in I}$ is linearly dependent over k. Thus there exists $R \in k[X_1, \dots, X_{r+1}]$ of degree $\leq \deg(Q) < \deg(P)$ such that $R(x_1, \dots, x_{r+1}) = 0$, a contradiction.

Thus we may assume $P \notin k[X_1^p, X_2, \dots, X_{r+1}]$. Then x_1 is separable over $k(x_2, \dots, x_{r+1})$, hence separable over $k(x_2, \dots, x_n)$. By assumption $k(x_2, \dots, x_n)$ has a separating transcendence basis $B \subseteq \{x_2, \dots, x_n\}$. Then B is a separating transcendence basis for K/k. □

Warning 1.6.56. A separating transcendence basis does not exist in general. For example, for $\mathrm{char}(k) = p > 0$, $K = \bigcup_{n \in \mathbb{N}} k(x^{1/p^n})$ is a separable extension of k of transcendence degree 1. However, for any $y \in K$ transcendental over k, $K/k(y)$ is not separable.

Now we come to the general case.

Proposition 1.6.57. *Let X/k be a k-scheme. The following are equivalent:*

(1) $X \otimes_k k'$ is reduced for every finite purely inseparable extension k'/k.

(2) X is geometrically reduced.

(3) $X \times_k Y$ is reduced for every reduced k-scheme Y.

(4) X is reduced and for every maximal point $x \in X$, $\kappa(x)/k$ is separable.

Recall that a maximal point of a scheme is the generic point of an irreducible component.

Proof. (3)⇒(2). Take $Y = \mathrm{Spec}(\overline{k})$.

(2)⇒(4). X is clearly reduced. Let $x \in X$ be a maximal point. Since X is reduced, we have $\kappa(x) = \mathcal{O}_{X,x}$. Since $X \otimes_k \overline{k}$ is reduced, $\mathcal{O}_{X,x} \otimes_k \overline{k}$ is reduced. In other words, $\kappa(x)/k$ is separable.

(4)⇒(3). We may assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine. It suffices to show that $A \otimes_k B$ is reduced.

Since B is a union of finitely generated k-algebras, we may assume that B itself is finitely generated and reduced. In this case, B is Noetherian and has finitely many minimal prime ideals $\mathfrak{q}$. Since B is reduced, we have $B \hookrightarrow \prod_{\mathfrak{q}} B/\mathfrak{q} \hookrightarrow \prod_{\mathfrak{q}} \kappa(\mathfrak{q})$ and the product is finite. Tensoring with A, we get $A \otimes_k B \hookrightarrow \prod A \otimes_k \kappa(\mathfrak{q})$. We are reduced to proving that $A \otimes_k k'$ is reduced for any field extension k'/k.

Since A is reduced, we have

$$A \hookrightarrow \prod_{\mathfrak{p}} (A/\mathfrak{p}) \hookrightarrow \prod_{\mathfrak{p}} \kappa(\mathfrak{p}),$$

where the product is taken over all minimal prime ideals. Tensoring with k', we get

$$A \otimes_k k' \hookrightarrow (\prod_{\mathfrak{p}} \kappa(\mathfrak{p})) \otimes_k k' \hookrightarrow \prod_{\mathfrak{p}} (\kappa(\mathfrak{p}) \otimes_k k').$$

(To see the injectivity of the last map, take a k-linear basis of k'.)

Thus it suffices to show that for any separable extension K/k, $K\otimes_k k'$ is reduced. Since $K = \bigcup_{\alpha\in K} k(\alpha)$, we may assume $K = k(\alpha)$. If α is separable algebraic over k of minimal polynomial $P(x)$, then $k(\alpha)\otimes_k k' = k'[x]/(P(x))$ is reduced. If α is transcendental over k, then $k(\alpha)\otimes_k k'$ is a localization of $k'[x]$ and hence reduced.

(2)⇒(1). Clear.

(1)⇒(2). Let k^{perf} be the perfection of k. By assumption, $Y = X\otimes_k k^{\mathrm{perf}}$ is reduced. Now $\bar{k}/k^{\mathrm{perf}}$ is separable. Applying (4)⇒(3) to $\mathrm{Spec}(\bar{k})$, we get that $X\otimes_k \bar{k} \simeq Y\otimes_{k^{\mathrm{perf}}}\bar{k}$ is reduced. □

Corollary 1.6.58. *If k is a perfect field, then a k-scheme X is reduced if and only if X is geometrically reduced.*

Immersions

Recall that a morphism of schemes $f\colon Z\to X$ is an open immersion if and only if $\mathrm{sp}(f)$ is an open embedding and $f^\sharp\colon f^{-1}\mathcal{O}_X \cong \mathcal{O}_Z$.

Definition 1.6.59. Let $f\colon Z\to X$ be a morphism of schemes.

(1) f is called a **closed immersion** if f is a closed embedding and $f^\sharp\colon f^{-1}\mathcal{O}_X\to\mathcal{O}_Z$ is epimorphism of Abelian sheaves, i.e. $\forall z\in Z$, $f^\sharp_z\colon \mathcal{O}_{X,f(z)}\to\mathcal{O}_{Z,z}$ is surjective.

(2) f is called an **immersion** if f factorizes as $Z\to U\to X$ where $Z\to U$ is a closed immersion and $U\to X$ is an open immersion.

Lemma 1.6.60.

- *A morphism of schemes $f\colon Z\to X$ is an immersion if and only if f is a locally closed embedding and $\forall z\in Z$, $f^\sharp_z\colon \mathcal{O}_{X,f(z)}\to\mathcal{O}_{Z,z}$ is surjective.*
- *Immersions are stable under composition.*
- *Immersions are monomorphisms.*

Example 1.6.61. Let A be a ring, I an ideal. Then $\mathrm{Spec}(A/I)\to\mathrm{Spec}(A)$ induced by $A\to A/I$ is a closed immersion. Indeed, $\forall\mathfrak{p}\supseteq I$, $A_\mathfrak{p}\to A_\mathfrak{p}/IA_\mathfrak{p}\simeq (A/I)_\mathfrak{p}$ is surjective.

Definition 1.6.62. Let X be a scheme.

- A **closed subscheme** of X is an equivalence class of pairs (Z,f), where Z is a scheme and $f\colon Z\to X$ is a closed immersion. Two pairs (Z,f) and (Z',f') are said to be equivalent if $\exists\varphi\colon Z\xrightarrow{\sim} Z'$ making

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\varphi"'] & X \\ Z' \arrow[ur, "f'"'] & \end{tikzcd}$$

commutative. φ is necessarily unique.

- A **subscheme** of X is an equivalence class of pairs (Z, f), where Z is a scheme and $f\colon Z \to X$ is an immersion. Two pairs (Z, f) and (Z', f') are said to be equivalent if $\exists \varphi\colon Z \xrightarrow{\sim} Z'$ making

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\varphi"] & X \\
Z' \arrow[ur, "f'"] &
\end{tikzcd}$$

  commutative.

By Lemma 1.6.60, we get:

Lemma 1.6.63. *An immersion that is closed is a closed immersion.*

Warning 1.6.64. An immersion that is open is **not** an open immersion in general. For example, if A is a non-reduced ring, then $\mathrm{Spec}(A/\sqrt{(0)}) \to \mathrm{Spec}(A)$ is a homeomorphism and a closed immersion, but not an open immersion.

Warning 1.6.65. If $I \neq J$ are ideals of A such that $\sqrt{I} = \sqrt{J}$, then $\mathrm{Spec}(A/I)$ and $\mathrm{Spec}(A/J)$ have the same underlying subspace of $\mathrm{Spec}(A)$, but are not the same as closed subscheme.

Warning 1.6.66. Let $f\colon Z \to X$ be an immersion. It is **not** possible in general to factorize f as $Z \to Y \to X$ where $Z \to Y$ is an open immersion and $Y \to X$ is a closed immersion. See [SP, 078B] for an example. For a positive result, see Lemma 1.9.27 later.

Warning 1.6.67. Not all monomorphisms are immersions. For example, the monomorphism $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ is not an immersion since it is not locally closed. In the same vein, a subobject of a scheme is **not** a subscheme in general.

An important class of immersions is given by the diagonal construction.

Diagonals, Separation axioms

For any morphism of schemes $f\colon X \to Y$, consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

We sometimes write Δ or $\Delta_{X/Y}$ for Δ_f.

Proposition 1.6.68. *Δ_f is an immersion.*

Before proving the proposition, we first consider an illuminating example.

Example 1.6.69. Let $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, and $f = \operatorname{Spec}(\phi)$, where $\phi\colon A \to B$. Then $\Delta_f\colon X \to X \times_Y X$ corresponds to

$$\begin{aligned}\nabla_\phi\colon B \otimes_A B &\to B\\ b_1 \otimes b_2 &\mapsto b_1 b_2\end{aligned}$$

This is clearly surjective. Hence Δ_f is a closed immersion.

Proof of Proposition 1.6.68. Let $Y = \bigcup V_i$, V_i affine open subsets. Let $f^{-1}(U_i) = \bigcup U_{ij}$, U_{ij} affine open. Let $W_{ij} = p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) \simeq U_{ij} \times_{V_i} U_{ij}$ and $W = \bigcup W_{ij}$. We have $\Delta_f(U_{ij}) \subseteq p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = W_{ij}$. Thus Δ_f factorizes as $X \xrightarrow{\delta} W \subseteq X \times_Y X$. Now $\Delta_f^{-1}(W_{ij}) = \Delta_f^{-1}(p_1^{-1}(U_{ij})) \cap \Delta_f^{-1}(p_2^{-1}(U_{ij})) = U_{ij}$ and the restriction of δ to $U_{ij} \to W_{ij}$ can be identified with $\Delta_{f_{ij}}$, where $f_{ij}\colon U_{ij} \to V_i$ is the restriction of f. By the example above, each $\Delta_{f_{ij}}$ is a closed immersion. Thus δ is a closed immersion. It follows that $\Delta_f\colon X \to Y$ is an immersion. ☐

Definition 1.6.70. Let $f\colon X \to Y$ be a morphism of schemes.

- f is called **separated** if Δ_f is a closed immersion.
- f is called **quasi-separated** if Δ_f is quasi-compact.

It is clear that we have affine ⇒ separated ⇒ quasi-separated.

The following graph construction will be very useful in the sequel. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

be a morphism of S-schemes. The graph of f, denoted Γ_f, is defined as follows:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[rr, bend left, "f"] \arrow[dr, "\mathrm{id}_X"'] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

From the functorial point of view, we have $\Gamma_f(x) = (x, f(x))$.

We have a commutative diagram with Cartesian squares

$$\begin{tikzcd} & X \arrow[rr, "\Gamma_f"] \arrow[dl, "f"'] \arrow[rrrr, bend left, "f"] & & X \times_S Y \arrow[rr, "q"] \arrow[dl, "f\times\mathrm{id}_Y"'] \arrow[dr, "p"] & & Y \arrow[dr, "g"] & \\ Y \arrow[rr, "\Delta_g"] & & Y \times_S Y & & X \arrow[rr, "gf"] & & S \end{tikzcd}$$

Thus, we get the following Lemma:

Lemma 1.6.71. *If $\mathcal{P}$ is a class of morphisms that $\mathcal{P}$ is stable under base change and composition, then $gf, \Delta_g \in \mathcal{P}$ implies $f \in \mathcal{P}$.*

In particular, we have

Corollary 1.6.72. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If gf is quasi-compact and g is quasi-separated, then f is quasi-compact.*

Definition 1.6.73.

- A scheme X is said to be **separated** if the morphism $X \to \operatorname{Spec}(\mathbb{Z})$ is separated.
- A scheme X is said to be **quasi-separated** if the morphism $X \to \operatorname{Spec}(\mathbb{Z})$ is quasi-separated.

Corollary 1.6.74. *Let $f\colon X \to Y$ be a morphism of schemes with X quasi-compact and Y quasi-separated. Then f is quasi-compact.*

Proposition 1.6.75. *A scheme X is quasi-separated if and only if for all quasi-compact opens U and V of X, $U \cap V$ is quasi-compact.*

Proof. $\Longrightarrow$. Since U is quasi-compact and X is quasi-separated, the open immersion $j\colon U \to X$ is quasi-compact by Corollary 1.6.74. Therefore, $U \cap V = j^{-1}(V)$ is quasi-compact.

$\Longleftarrow$.

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr] & X \times_{\mathbb{Z}} X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d] \\ & X \arrow[r] & \operatorname{Spec}(\mathbb{Z}) \end{tikzcd}$$

Let $X = \bigcup_i U_i$ be an affine open cover. Let $W_{ij} = p_1^{-1}(U_i) \cap p_2^{-1}(U_j)$. Then $\bigcup_{ij} W_{ij} = X \times_{\operatorname{Spec}(\mathbb{Z})} X$. Each $W_{ij} \simeq U_i \times_{\operatorname{Spec}(\mathbb{Z})} U_j$ is an affine scheme, and $\Delta^{-1}(W_{ij}) = U_i \cap U_j$ is quasi-compact. Thus Δ is a quasi-compact morphism. $\square$

Example 1.6.76. Let X be a scheme

- If the underlying space of X is locally Noetherian, then X is quasi-separated.
- Let X be a scheme and let $U \subseteq X$ be an open subset that is not closed. Let $Y = X \coprod_U X$ be the scheme obtained by gluing two copies of X along U. Then Y is not separated.

  To see this, let j_0 and j_1 denote the two open immersions from X to Y. We have an immersion $f = (j_0, j_1)\colon X \to Y \times_{\operatorname{Spec}(\mathbb{Z})} Y$. Let $\Delta = \Delta_{Y/\operatorname{Spec}(\mathbb{Z})}$. The inclusion $f(X) \cap \Delta(X) \subseteq f(X)$ can be identified with the inclusion $U \subseteq X$, which is not closed.
- Let X be a quasi-compact scheme and let $U \subseteq X$ be an open subset that is not quasi-compact (e.g. X is the Cantor set and U is the complement of a point). Then the same argument as above shows that $Y = X \coprod_U X$ is not quasi-separated.

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Warning 1.6.77. If α is transcendental over k and k'/k a field extension, then $k(\alpha) \otimes_k k'$ is the localization of $k'[\alpha]$ with respect to the multiplicative set $S = k[\alpha]\backslash\{0\}$. It is **not** a field in general.

We have seen that $f\colon X \to Y$ is separated if and only if $\Delta_f(X) \subseteq X \times_Y X$ is closed. This is analogous to the fact in general topology that a topological space X is Hausdorff if and only if $\Delta_X(X) \subseteq X \times X$ is closed.

1.7 Quasi-coherent sheaves

The properties of a ring A are often reflected by the category of A-modules. In order to better study a sheaf of rings $\mathcal{O}_X$, we now introduce the notion of $\mathcal{O}_X$-module.

Definition 1.7.1. Let $(X, \mathcal{O}_X)$ be a ringed space.

- An $\mathcal{O}_X$-**module** or **sheaf of** $\mathcal{O}_X$-**modules** is consists of
  - a sheaf of sets $\mathcal{F}$ on X;
  - $\forall U \subseteq X$ open, a structure of $\mathcal{O}_X(U)$-module on $\mathcal{F}(U)$

  such that for all $U \subseteq V$, the restriction map

  $$\mathcal{F}(V) \xrightarrow{\rho} \mathcal{F}(U)$$

  is a homomorphism of $\mathcal{O}_X(V)$-modules. Here $\mathcal{F}(U)$ is viewed as an $\mathcal{O}_X(V)$-module via the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

- A **morphism** of $\mathcal{O}_X$-modules $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of sets such that $\forall U \subseteq X$, $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. The sheaf of local homomorphisms, or "sheaf hom" for short, denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, is defined as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}_U, \mathcal{G}_U)$$

It is easy to see that this is a sheaf of $\mathcal{O}_X$-module.

The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined as the sheafification of

$$U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

The tensor product is again a sheaf of $\mathcal{O}_X$-modules.

The following properties are easy to verify.

Lemma 1.7.2. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of $\mathcal{O}_X$-modules.*

(1) $\forall x \in X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.

(2) $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$.

Recall that a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of the following:

- a continuous map $f\colon X \to Y$;
- a morphism of sheaves of rings $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (or, equivalently by adjunction, $f^\flat\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$).

For an $\mathcal{O}_X$-module $\mathcal{F}$, $f_*(\mathcal{F})$ is then naturally an $f_*(\mathcal{O}_X)$-module. We regard $f_*(\mathcal{F})$ as an $\mathcal{O}_Y$-module via $f^\flat$.

For an $\mathcal{O}_Y$-module $\mathcal{G}$, $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. We define

$$f^*(\mathcal{G}) := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Then $f^*(\mathcal{G})$ is an $\mathcal{O}_X$-module.

Combining the adjunction $f^{-1} \dashv f_*$ and the adjunction between $\otimes$ and $\mathcal{H}om$, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*(\mathcal{F}).$$

In other words, we have $f^* \dashv f_*$ between the categories of $\mathcal{O}$-modules.

Warning 1.7.3. f^* is **not** exact in general. f^* is exact if f is flat, i.e. $\forall x \in X$, $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Definition 1.7.4. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module.

- $\mathcal{F}$ is said to be **free** if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. $\mathcal{O}_X^n$ is called a free $\mathcal{O}_X$-module of rank n. For I a set, we write $\mathcal{O}_X^{\oplus I} := \bigoplus_{i \in I} \mathcal{O}_X$.
- $\mathcal{F}$ is said to be **locally free** if there is an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module.
- $\mathcal{F}$ is said to be **locally free of rank** n if there is an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module of rank n.
- $\mathcal{F}$ is said to be **invertible** if it is locall free of rank 1.

Remark 1.7.5.
- For $\mathcal{F}$ locally of rank n, its **dual** $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is also locally free of rank n.
- For $\mathcal{F}$ is invertible, we have $\mathcal{F} \otimes \mathcal{F}^\vee \xrightarrow{\sim} \mathcal{O}_X$. We let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of invertible $\mathcal{O}_X$-modules. $(\mathrm{Pic}(X), \otimes)$ is an Abelian group, called the **Picard group** of X.

Definition 1.7.6. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **quasi-coherent** if there exists an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a cokernel of free $\mathcal{O}_X|_{U_i}$ modules. i.e.

$$\mathcal{F}|_{U_i} \cong \mathrm{coker}(\mathcal{O}_X|_{U_i}^{\oplus I_i} \to \mathcal{O}_X|_{U_i}^{\oplus J_i})$$

Let A be a ring and M an A-module. Let $X = \operatorname{Spec}(A)$ and $D(f) \subseteq X$, $f \in A$ be a principal open subset. Define $\widetilde{M}(D(f)) = M_f$, which is a module over $A_f = \mathcal{O}_X(D(f))$. If $D(f) \subseteq D(g)$, then the homomorphism $A_g \to A_f$ induces $M_g \to M_f$. Let $\mathcal{B}$ be the partially ordered set $\{D(f) \mid f \in A\}$.

Lemma 1.7.7. *The functor*

$$\begin{aligned} \mathcal{B}^{\mathrm{op}} &\to \mathrm{Ab} \\ D(f) &\mapsto M_f \end{aligned}$$

extends uniquely to a sheaf of $\mathcal{O}_X$-module.

Proof. By Lemma 1.4.1, it suffices to verify the gluing property for a cover in $\mathcal{B}$ of some $D(f)$. Up to replacing A by A_f, we may without loss of generality suppose that the cover has the form $X = \bigcup_{i \in I} D(f_i)$. Since X is quasi-compact, we may assume as in the proof of Proposition 1.4.2 that I is finite. It suffices to show that

$$M \to \bigoplus_i M_{f_i} \rightrightarrows \bigoplus_{i,j} M_{f_i f_j}$$

is an equalizer diagram. For this, one can repeat the arguments in either one of the two proofs of Proposition 1.4.2. $\square$

We have

$$\widetilde{M}_{\mathfrak{p}} = \operatorname*{colim}_{\mathfrak{p} \in D(f)} M_f = \operatorname*{colim}_{f \notin \mathfrak{p}} M_f = M_{\mathfrak{p}}$$

Proposition 1.7.8. *The functor*

$$\begin{aligned} F \colon A\text{-Mod} &\to \operatorname{Shv}(X, \mathcal{O}_X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

is exact, fully faithful and left adjoint to $\Gamma(X, -)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and M an A-module. We consider the map

$$\Psi \colon \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \to \operatorname{Hom}(M, \mathcal{F}(X))$$

carrying $\phi \colon \widetilde{M} \to \mathcal{F}$ to $\phi(X) \colon M = \widetilde{M}(X) \to \mathcal{F}(X)$. For $\psi \colon M \to \mathcal{F}(X)$, we define $\phi = \Phi(\psi) \colon \widetilde{M} \to \mathcal{F}$ by $\phi(D(f)) \colon \widetilde{M}(D(f)) = M_f \xrightarrow{\psi_f} \mathcal{F}(X)_f \to \mathcal{F}(D(f))$ for each $f \in A$. One checks Φ and Ψ are inverse to each other. This shows $F \dashv \Gamma(X, -)$.

Since $\Gamma(X, \widetilde{M}) = M$, F is fully faithful. Finally, F is exact since the functor $M \mapsto M_f$ is exact $\forall f \in A$. $\square$

One checks the following properties:

Lemma 1.7.9. *Let $\phi \colon A \to B$ be a ring homomorphism, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, and $f = \operatorname{Spec}(\phi) \colon X \to Y$.*

(1) For A-modules M and M', we have $\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N} \simeq (M \otimes_A N)^\sim$.

(2) For every A-module M, we have $f^*(\widetilde{M}) \simeq (M \otimes_A B)^\sim$.

(3) For every B-module N, we have $f_*(\widetilde{N}) = \widetilde{{}_AN}$, where ${}_AN$ is N considered as an A-module via ϕ.

Proof. (1) The canonical morphism $\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N} \to (M \otimes_A N)A^\sim$ is an isomorphism by taking stalks.

(2) Consider the canonical morphism $f^*(\widetilde{M}) \to (M \otimes_A B)^\sim$. Let $\mathfrak{q}$ be a prime in B and $\mathfrak{p} = f(\mathfrak{q}) = \phi^{-1}(\mathfrak{p})$. The stalk of the morphism at $\mathfrak{q}$ is $(f^*\widetilde{M})_\mathfrak{q} \simeq (f^{-1}\widetilde{M})_\mathfrak{q} \otimes_{f^{-1}\mathcal{O}_{Y,\mathfrak{q}}} \mathcal{O}_{X,\mathfrak{q}} \simeq M_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q} \simeq (M \otimes_A B)^\sim_\mathfrak{q}$.

(3) Indeed, $\forall g \in B$, we have $f_*(\widetilde{N})(D(g)) = \widetilde{N}(D(\phi(g)) = N_{\phi(g)}$. □

Proposition 1.7.10. *Let $X = \operatorname{Spec}(A)$. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if $\mathcal{F} \cong \widetilde{M}$ for some A-module M.*

More generally, we have the following characterization of quasi-coherent sheaves on schemes.

Proposition 1.7.11. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then the following are equivalent*

(a) $\mathcal{F}$ is quasi-coherent.

(b) $\exists X = \bigcup U_i$ with $U_i = \operatorname{Spec}(A_i)$ affine open, such that for every i, $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i.

(c) $\forall U = \operatorname{Spec}(A) \subseteq X$ affine open, we have $\mathcal{F}|_U \cong \widetilde{M}$ for some A-module M.

Proof. (b)$\Rightarrow$ (a). It suffices to show for every A-module M, $\widetilde{M}$ quasi-coherent. There is a presentation of M using free modules:

$$A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0.$$

Taking the associated sheaves, we get an exact sequence

$$\exists\ \mathcal{O}_X^{\oplus I} \longrightarrow \mathcal{O}_X^{\oplus J} \longrightarrow \widetilde{M} \longrightarrow 0.$$

(c)$\Rightarrow$ (b). Trivial.

(a)$\Rightarrow$ (c). We may assume $X = \operatorname{Spec}(A)$ is affine. Let $M = \mathcal{F}(X)$. It suffices to show that the canonical map $M_f \to \mathcal{F}(D(f))$ is an isomorphism for every $f \in A$, which gives an isomorphism $\widetilde{M} \xrightarrow{\sim} \mathcal{F}$. The following lemma is a generalization of this assertion. □

Lemma 1.7.12. *Let X be a quasi-compact scheme, $f \in \mathcal{O}_X(X)$, and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module.*

(1) The map $\mathcal{F}(X)_f \to \mathcal{F}(X_f)$ is an injection.

(2) If X is quasi-separated, then $\mathcal{F}(X)_f \to \mathcal{F}(X_f)$ is bijective.

Proof. Let $X = \bigcup_{i=1}^n U_i$ be an affine open cover with $U_i = \mathrm{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i}$ is the cokernel of free module. Then $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i. Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X)_f \arrow[r] \arrow[d, "u"] & \prod_{i=1}^n \mathcal{F}(U_i)_f \arrow[r, "\varphi"] \arrow[d, "v"] & \prod_{i,j} \mathcal{F}(U_i \cap U_j)_f \arrow[d, "w"] \\
0 \arrow[r] & \mathcal{F}(X_f) \arrow[r] & \prod_{i=1}^n \mathcal{F}((U_i)_f) \arrow[r, "\varphi'"] & \prod_{i,j} \mathcal{F}((U_i)_f \cap (U_j)_f)
\end{tikzcd}$$

where φ and φ' are differences of the restrictions maps. $U_{i,f}$ means $U_i \cap X_f$. The rows are exact by the sheaf condition and by the exactness of localization.

(1) We have $\mathcal{F}(U_i)_f = (M_i)_f = \mathcal{F}((U_i)_f)$. Hence v is an isomorphism. This implies that u is injective.

(2) Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact and w is injective by (1). It follows that u is an isomorphism by a simple diagram chase. $\square$

Example 1.7.13. For an open immersion $j\colon U \hookrightarrow X$, $j_!\mathcal{O}_U$ is not a quasi-coherent $\mathcal{O}_X$-module in general. Recall

$$(j_!^{\mathrm{psh}}\mathcal{O}_U)(V) = \begin{cases} \mathcal{O}_U(V) & V \subseteq U \\ 0 & V \subsetneq U \end{cases} \qquad j_!\mathcal{O}_U = (j_!^{\mathrm{psh}}\mathcal{O}_U)^+.$$

Indeed, if X is irreducible and V is an affine open satisfying $V \subsetneq U$, then $j_!\mathcal{O}_U(V) = 0$. This implies that $j_!\mathcal{O}_U$ is not quasi-coherent. Otherwise we would have $(j_!\mathcal{O}_U)|_V = 0$, which is absurd.

Corollary 1.7.14. *Let X be a scheme. The full subcategory* $\mathrm{QCoh(X)} \subseteq \mathrm{Shv}(X, \mathcal{O}_X)$ *is stable under kernels and colimits.*

Lemma 1.7.15. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

(1) For quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$ is also a quasi-coherent $\mathcal{O}_Y$-module.

(2) For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, then $f^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.

Proof. We leave (1) as an exercise. For (2), note that f^* is right exact and preserves cokernels of free modules. $\square$

Example 1.7.16. Let A be a DVR, $K = \mathrm{Frac}(A)$, $X = \mathrm{Spec}(A)$. Then $\mathcal{O}_X^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathcal{O}_X$ is **not** quasi-coherent. Indeed, $\mathcal{F}(X) = A^{\mathbb{N}}$ and $\mathcal{F}(\eta) = K^{\mathbb{N}}$, and the map $A^{\mathbb{N}} \otimes_A K \to K^{\mathbb{N}}$ is not an isomorphism.

Let $f\colon Y = \coprod_{n\in\mathbb{N}} X \to X$. Then $\mathcal{F} = f_*(\mathcal{O}_Y)$. This shows f_* does note preserve quasi-coherent sheaves in general.

Proposition 1.7.17. *Let $f\colon X \to Y$ be a qcqs (quasi-coherent and quasi-separated) morphism of schemes and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $f_*(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-module.*

Proof. If $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, the assertion follows from Lemma 1.7.9.

In general, we may assume that Y is affine. Then X is quasi-compact and quasi-separated (See Lemma 1.7.18 below). Let $X = \bigcup_{i=1}^n U_i$ be an affine open cover. Then $U_i \cap U_j$ is quasi-compact and we can write $U_i \cap U_j = \bigcup U_{ijk}$ with k finite and U_{ijk} affine open. Let $u_i \colon U_i \hookrightarrow X$ and $u_{ijk} \colon U_{ijk} \hookrightarrow X$ be the inclusions. The sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i u_{i*}(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} u_{ijk*}(\mathcal{F}|_{U_{ijk}})$$

is exact by sheaf condition. Applying f_*, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i (fu_i)_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} (fu_{ijk})_*(\mathcal{F}|_{U_{ijk}}).$$

Since U_i is affine, $(fu_i)_*(\mathcal{F}|_{U_i})$ is quasi-coherent. Similarly $(fu_{ijk})_*(\mathcal{F}|_{U_{ijk}})$ is quasi-coherent. It follows that $f_*\mathcal{F}$ is quasi-coherent. $\square$

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We supplement some properties of quasi-separated morphisms.

Proposition 1.7.18.

(1) Quasi-separated morphisms are stable under composition and base change.

(2) If $X \xrightarrow{f} Y \xrightarrow{g} S$ *are morphisms such that* gf *is quasi-separated, then* f *is quasi-separated.*

Proof. For (1), we first consider base change. Suppose f is quasi-separated, and $g\colon Y' \to Y$ is another morphism. Form the Cartesian square on the left. Then the square on the right is also Cartesian.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "\Delta_{f'}"] \arrow[d] & X' \times_{Y'} X' \arrow[d] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \end{tikzcd}$$

Since quasi-compact morphisms are stable under base change, $\Delta_{f'}$ is quasi-compact.

For composition, let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Consider the diagram with pullback square:

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[rr, bend left, "\Delta_{gf}"] & X \times_Y X \arrow[r, "\Delta'"] \arrow[d] & X \times_S X \arrow[d] \\ & Y \arrow[r, "\Delta_g"] & Y \times_S Y \end{tikzcd}$$

Δ' is quasi-compact by base change. Thus Δ_{gf} is the composition of quasi-compact morphisms, and hence quasi-compact.

For (2), we apply Lemma 1.6.71. We have already proven the collection of quasi-separated morphisms are stable under base change and composition. Since Δ_g is an immersion and an immersion is clearly quasi-separated, we get that f is quasi-separated. □

1.8 Relative spectrum

Definition 1.8.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$**-algebra** or a sheaef of $\mathcal{O}_X$-algebra consists of

- a sheaf of sets $\mathscr{A}$ on X;
- $\forall U \subseteq X$ open, a structure of $\mathcal{O}_X(U)$-algebra on $\mathscr{A}(U)$

such that $\forall U \subseteq V$,
$$\begin{tikzcd} \mathscr{A}(V) \arrow[r] & \mathscr{A}(U) \\ \mathcal{O}_X(V) \arrow[r] \arrow[u] & \mathcal{O}_X(U) \arrow[u] \end{tikzcd}$$
commutes as ring homomorphisms. A **morphism** of $\mathcal{O}_X$-algebras $\phi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves of sets such that $\forall U \subseteq X$, $\phi_U\colon \mathcal{A}(U) \to \mathcal{B}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

We say that an $\mathcal{O}_X$-algebra $\mathscr{A}$ is **quasi-coherent** $\mathcal{O}_X$-algebra if it is quasi-coherent as an $\mathcal{O}_X$-module.

Example 1.8.2. If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then $f_*(\mathcal{O}_X)$ is an $\mathcal{O}_Y$-algebra via $f^\flat$.

Example 1.8.3. Let $X = \operatorname{Spec}(B)$. Then we have an equivalence of categories

$$B\text{-Alg} \xrightarrow{\sim} \{\text{quasi-coherent } \mathcal{O}_X\text{-algebras}\}$$
$$A \mapsto \widetilde{A}$$

Construction 1.8.4. Let S be a scheme and $\mathscr{A}$ a quasi-coherent $\mathcal{O}_S$-algebra. We construct a scheme $X = \underline{\operatorname{Spec}}(\mathscr{A})$ and an affine morphism $f\colon \underline{\operatorname{Spec}}(\mathscr{A}) \to S$ as follows.

For $U \subseteq S$ affine open, we consider $f_U\colon \operatorname{Spec}(\mathscr{A}(U)) \to \operatorname{Spec}(\mathcal{O}(U)) \simeq U$. For any inclusion $U \subseteq V$ of affine open subsets, we have a Cartesian square

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{A}(U)) \arrow[r] \arrow[d, "f_U"'] & \operatorname{Spec}(\mathcal{A}(V)) \arrow[d, "f_V"] \\ U \arrow[r] & V. \end{tikzcd}$$

One verifies that these data glue to a scheme $X = \underline{\operatorname{Spec}}(\mathscr{A})$ and an affine morphism $f\colon \underline{\operatorname{Spec}}(\mathscr{A}) \to S$.

By construction, $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U)) = \mathcal{A}(U)$. Thus $\mathcal{A} \simeq f_*\mathcal{O}_X$.

Example 1.8.5. Let $\mathcal{F} = \mathcal{O}_S^n$ be a free $\mathcal{O}_S$-module. Let $\mathscr{A} = \operatorname{Sym}_{\mathcal{O}_S}(\mathcal{F})$. If $U = \operatorname{Spec}(B) \subseteq S$ is affine, then $\mathscr{A}(U) = B[x_1, \dots, x_n]$. We call $\underline{\operatorname{Spec}}(\mathscr{A})$ the **affine n-space** of S. We have $\mathbb{A}^n_S \simeq \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} S$.

In general, for any quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$, we call $\mathbb{V}(\mathcal{F}) = \underline{\operatorname{Spec}}(\operatorname{Sym}_{\mathcal{O}_S}(\mathcal{F}))$ the **vector bundle** over S associated to $\mathcal{F}$. If $\mathcal{F}$ is locally free of rank n, $\mathbb{V}(\mathcal{F})$ is locally isomorphic to an affine n-space over S. For $n = 1$, we speak of line bundle instead of vector bundle.

Next we extend some properties of Spec to $\underline{\operatorname{Spec}}$.

Proposition 1.8.6. *Let $f\colon X \to S$ be a morphism of schemes and $\mathscr{A}$ a quasi-coherent $\mathcal{O}_S$-algebra. Then we have a canonical bijection*

$$\operatorname{Hom}_S(X, \underline{\operatorname{Spec}}(\mathscr{A})) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_S\text{-Alg}}(\mathscr{A}, f_*(\mathcal{O}_X)).$$

This is a relative analogue of the bijection

$$\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec}(A)) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X)).$$

Proof. Let $Y = \underline{\mathrm{Spec}}(\mathscr{A})$. The map is constructed as follows. To any morphism of S-schemes

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

we associate $\mathcal{A} \cong g_*\mathcal{O}_Y \xrightarrow{g_*h^\flat} g_*h_*\mathcal{O}_X \cong f_*\mathcal{O}_X$.

We will prove that for every open $U \subseteq S$,

$$\mathrm{Hom}_U(f^{-1}(U), \underline{\mathrm{Spec}}(\mathscr{A}|_U)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U\text{-alg}}(\mathscr{A}|_U, f_*(\mathcal{O}_X)|_U). \tag{1.8.1}$$

Since any morphism of S-schemes $f^{-1}(U) \to \underline{\mathrm{Spec}}(\mathscr{A})$ factors through $\underline{\mathrm{Spec}}(\mathscr{A}|_U)$, we have

$$\mathrm{Hom}_U(f^{-1}(U), \underline{\mathrm{Spec}}(\mathscr{A}|_U)) \cong \mathrm{Hom}_S(f^{-1}(U), \underline{\mathrm{Spec}}(\mathscr{A})).$$

Now both sides of (1.8.1) are sheaves when U runs through all open subsets of S. In order to show that the morphism of sheaves is an isomorphism, it suffices to check that (1.8.1) is an isomorphism for every affine open subset U.

Thus we may assume that $S = \mathrm{Spec}(B)$ is affine and thus $\mathscr{A} = \widetilde{A}$ with A a B-algebra. Then

$$\begin{aligned} \mathrm{Hom}_S(X, \underline{\mathrm{Spec}}(\mathscr{A})) &= \mathrm{Hom}_{\mathrm{Spec}(B)}(X, \mathrm{Spec}(A)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{B\text{-alg}}(A, \mathcal{O}_X(X)) = \mathrm{Hom}_{\mathcal{O}_S\text{-Alg}}(\widetilde{A}, f_*(\mathcal{O}_X)). \end{aligned}$$

□

Consider the functor

$$\begin{aligned} \{\text{schemes qcqs over } S\} &\to \{\text{quasi-coherent } \mathcal{O}_S\text{-algebras}\}^{\mathrm{op}} \\ (f\colon X \to S) &\mapsto f_*(\mathcal{O}_X). \end{aligned}$$

The above proposition shows $\underline{\mathrm{Spec}}$ is a right adjoint of this functor. Moreover, $\underline{\mathrm{Spec}}$ is fully faithful since $\mathscr{A} \simeq f_*\mathcal{O}_{\underline{\mathrm{Spec}}(\mathscr{A})}$.

Corollary 1.8.7. *Let S be a scheme. There is an equivalence of categories*

$$\begin{aligned} \{\textit{quasi-coherent } \mathcal{O}_S\textit{-algebras}\}^{\mathrm{op}} &\xrightarrow{\sim} \{\textit{schemes affine over } S\} \\ \mathscr{A} &\mapsto \underline{\mathrm{Spec}}(\mathscr{A}). \end{aligned}$$

Proof. It remains to check that for every affine morphism $f\colon X \to S$, the morphism $X \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_X)$ is an isomorphism. For this we may assume that S is affine and the assertion is then clear. □

Immersions

Definition 1.8.8. Let $(X, \mathcal{O}_X)$ be a ringed space. An **ideal sheaf** I of $\mathcal{O}_X$ is a $\mathcal{O}_X$-submodule of $\mathcal{O}_X$. This makes $\mathcal{O}_X/I$ into an $\mathcal{O}_X$-algebra.

Proposition 1.8.9. *Let X be a scheme. There is an order-reserving bijection*

$$\begin{aligned}\Phi\colon \{\textit{quasi-coherent ideal sheaves of } \mathcal{O}_X\} &\cong \{\textit{closed subschemes of } X\}\\ I &\mapsto \underline{\mathrm{Spec}}(\mathcal{O}_X/I)\\ I_Y = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y) &\leftarrow\!\shortmid (i\colon Y \to X)\end{aligned}$$

I_Y is called the **ideal sheaf** of Y.

Proof. Let $\Psi\colon (i\colon Y \to X) \mapsto I_Y$. Since a closed immersion is qcqs, $i_*\mathcal{O}_Y$ is a sheaf of $\mathcal{O}_X$ and I_Y is quasi-coherent ideal sheaf of $\mathcal{O}_X$.

It is clear that $\Psi\Phi = \mathrm{id}$. Indeed, for $Y = \underline{\mathrm{Spec}}(\mathcal{O}_X/I)$ and $i\colon Y \to X$, we have $i_*\mathcal{O}_Y = \mathcal{O}_X/I$. Thus Ψ is surjective.

It remains to prove that Ψ is injective. Since $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism of sheaves of $\mathcal{O}_X$-modules, we have $\mathcal{O}_X/I_Y \xrightarrow{\sim} i_*\mathcal{O}_Y$. Since i is a closed imbedding, we have $\mathrm{sp}(Y) = \mathrm{supp}(i_*\mathcal{O}_Y) = \mathrm{supp}(\mathcal{O}_X/I_Y)$. Thus $\mathrm{sp}(Y)$ is uniquely determined by I_Y. Furthermore, $\mathcal{O}_Y \simeq i^{-1}i_*\mathcal{O}_Y$ is also uniquely determined by I_Y. □

Corollary 1.8.10. *For $X = \mathrm{Spec}(A)$, we have an order-reversing bijection*

$$\begin{aligned}\{\textit{ideals of } A\} &\cong \{\textit{closed subschemes of } \mathrm{Spec}(A)\}\\ I &\mapsto \mathrm{Spec}(A/I)\end{aligned}$$

This is not so obvious without using ideal sheaves.

Corollary 1.8.11. *Closed immersions are finite and stable under base change.*

Corollary 1.8.12. *Immersions are stable under base change.*

Proof. Both open immersions and closed immersions are stable under base change. □

Proposition 1.8.13. *(1) Separated morphisms are stable under composition and base change.*

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. If gf is separated, then f is separated.

Proof. The proof of (1) is similar to that Proposition 1.7.18. We use the stability of closed immersions under base change and composition.

For (2), we can apply Lemma 1.6.71 as before. Let us give a more direct proof. Consider the diagram with pullback square:

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[rr, bend left, "\Delta_{gf}"] & X \times_Y X \arrow[r, "\Delta'"] \arrow[d] & X \times_S X \arrow[d] \\ & Y \arrow[r, "\Delta_g"] & Y \times_S Y \end{tikzcd}$$

Since $\Delta_{gf}(X)$ is closed in $X \times_S X$, $\Delta_f(X)$ is closed in $X \times_Y X$. This shows that f is separated. □

Proposition 1.8.14. *Let X be a scheme and $T \subseteq X$ a closed subset. Then there exists a unique reduced closed subscheme $Y \subseteq X$ whose underlying space is T.*

The closed subscheme structure of Y is called the **reduced induced closed subscheme structure** on T.

Proof. Existence. Let $I(U) = \{f \in \mathcal{O}_X(U) \mid f_x \in m_x, \forall x \in U \cap T\}$. Then $I \subseteq \mathcal{O}_X$ is clearly an ideal. We prove that I is quasi-coherent. Let $U = \operatorname{Spec}(A) \subseteq X$ be an affine open subset. Then $T \cap U = V(J)$ with J radical. We have $I(U) = \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p} = J$. This remains true if we replace U by any principal open subset of U: $\forall f \in A$, $I(D(f)) = JA_f$. This shows $I|_U = \widetilde{J}$ and I is quasi-coherent. Thus I gives rise to a closed subscheme $Y = \underline{\operatorname{Spec}}(\mathcal{O}_X/I)$. The scheme Y is reduced, since for every affine open $U = \operatorname{Spec}(A)$, $\mathcal{O}_Y(U) = A/J$ is reduced.

Uniqueness. Let Y' be another reduced closed subscheme with underlying space T. To check $Y = Y'$, it suffices to do so on each affine open subset. Thus we may assume $X = \operatorname{Spec}(A)$ is affine. In this case $Y = \operatorname{Spec}(A/J)$ and $Y' = \operatorname{Spec}(A/J')$ with J and J' radical and $V(J) = T = V(J')$. Thus $J = J'$. □

Example 1.8.15. Taking $T = X$, we get a unique reduced closed subscheme $X_{\mathrm{red}} \subseteq X$ whose underlying subspace is X. The scheme X_{red} is called the reduced scheme associated to X.

Normalization

Definition 1.8.16. A scheme X is said to be **normal** if for all $x \in X$, $\mathcal{O}_{X,x}$ is an integrally closed domain.

Proposition 1.8.17. [AM, Proposition 5.12] *Taking integral closure is compatible with localization: let $\phi\colon A \to B$ be a ring homomorphism and $S \subseteq A$ a multiplicative system. Let C be the integral closure of A in B. Then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Corollary 1.8.18. *Let A be a domain. Then A is integrally closed if and only if $\forall \mathfrak{p} \in \operatorname{Spec}(A)$, $A_{\mathfrak{p}}$ is integrally closed.*

Corollary 1.8.19. *Let $A \to B$ is a morphisms of ring homomorphism. Then $b \in B$ is integral over A if and only if $\forall \mathfrak{p} \in \operatorname{Spec}(A)$, b is integral over $A_{\mathfrak{p}}$.*

Construction 1.8.20. Let X be an integral scheme, K its function field, L/K a field extension. Define

$$\mathscr{A}(U) = \begin{cases} \{f \in L \mid f \text{ integral over } \mathcal{O}_{X,x}, \forall x \in U\} & U \neq \emptyset \\ 0 & U = \emptyset \end{cases}$$

This is clearly an $\mathcal{O}_X$-algebra. For $U = \operatorname{Spec}(A) \subseteq X$ affine open, $\mathscr{A}(U)$ is the integral closure A' of A in L by Corollary 1.8.19. For any principal open subset $D(f)$ of U, $\mathscr{A}(D(f)) = A'_f$. This shows that $\mathscr{A}|_U = \widetilde{A'}$. Thus $\mathscr{A}$ is quasi-coherent.

The scheme $X' = \underline{\operatorname{Spec}}(\mathscr{A})$ equipped with the morphism $X' \to X$ is called the **normalization** of X in L. If $L = K$, then $X^\nu := \underline{\operatorname{Spec}}(\mathscr{A})$ is called the **normalization** of X.

From the construction, we see X' is integral and normal and the canonical morphism $X' \to X$ is integral.

Example 1.8.21. $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3))$ has a cusp at the origin O. Since $(y/x)^2 = x$ in the function field, X is not normal. Let $z = y/x$. Then $X^\nu := \mathrm{Spec}(k[x,z]/(z^2 - x)) \simeq \mathrm{Spec}(k[z])$ is the normalization of X. In this case, $X^\nu \to X$ is a universal homeomorphism.

Example 1.8.22. $X = \mathrm{Spec}(k[x,y]/(y^2 - x^2(x+1)))$ has a node at O. Since $(y/x)^2 = x + 1$ in the function field, X is not normal. Let $y/x = z$. Then $X^\nu := \mathrm{Spec}([x,z]/(z^2 - (x+1)) \simeq \mathrm{Spec}(k[z])$ is the normalization of X. The fiber of $X^\nu \to X$ at the origin consists of the two rational points $z = \pm 1$.

Definition 1.8.23. An integral scheme X is said to be **Japanese** if for every finite extension L of the function field K of X, the normalization X' of X in L is finite over X.

A scheme X is said to be **universally Japanese** if every integral scheme locally of finite type over X is Japanese.

Theorem 1.8.24. *Let A be*

- *a field, or*
- *a Dedekind domain with fraction field K satisfying $\mathrm{char}(K) = 0$, or*
- *a Noetherian complete local ring.*

Then $\mathrm{Spec}(A)$ is universally Japanese.

1.9 Valuative criterion

Definition 1.9.1. A ring A is called a **valuation ring** if it is a domain and $\forall x \in \mathrm{Frac}(A)$, either $x \in A$ or $x^{-1} \in A$.

Definition 1.9.2. Let $f\colon X \to S$ be a morphisms of schemes.

- f is said to satisfy the existence part of the **valuation criterion** if for every valuation ring A with fraction field K, and all morphisms $i\colon \mathrm{Spec}(K) \to X$ and $j\colon \mathrm{Spec}(A) \to S$ making the following square commutative, there exists a morphism $t\colon \mathrm{Spec}(A) \to X$ making the two triangles below commutative:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, "j"] \arrow[ur, dashed, "t"] & S
\end{tikzcd}$$

- f is said to satisfy the uniqueness part of the valuation criterion if whenever given i and j as above, there exists at most one t making the triangles commutative. That is, for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(A) \arrow[r, "j"'] \arrow[ur, shift left, "t_1"] \arrow[ur, shift right, "t_2"'] & S
\end{tikzcd}$$

where A is a valuation ring with fraction field K, we have $t_1 = t_2$.

Remark 1.9.3. $f\colon X \to S$ satisfies the existence part of valuation criterion if and only if for all valuation ring A with fraction field K, $X(A) \to X(K) \times_{S(K)} S(A)$ is surjective.

$f\colon X \to S$ satisfies the uniqueness part of valuation criterion if and only if for all valuation ring A with fraction field K, $X(A) \to X(K) \times_{S(K)} S(A)$ is injective.

Remark 1.9.4. If we are given a diagram as below,

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \times_S S' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed, "t"] \arrow[urr, dashed, "t'"] & S' \arrow[r] & S
\end{tikzcd}$$

then the dotted arrow t exists if and only if t' exists and t is unique if and only if t' is unique. This follows immediately form the universal property of fiber product.

Definition 1.9.5. A morphism of schemes $f\colon X \to S$ is **universally specializing** if every base change of f is specializing.

Theorem 1.9.6. *Let $f\colon X \to S$ be a morphism of schemes.*

(1) f satisfies the existence part of valuation criterion $\iff$ f is universally specializing.

(2) f satisfies the uniqueness part of valuation criterion $\iff$ Δ_f is universally specializing.

Proof. (1) $\implies$ (2). We prove f satisfies uniqueness $\iff$ Δ_f satisfies existence.

Consider the following diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "\Delta_f"] \\
\operatorname{Spec}(A) \arrow[r, "{(a,b)}"'] \arrow[ur, dashed, "c"] & X \times_S X
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(A) \arrow[r] \arrow[ur, shift left, "a"] \arrow[ur, shift right, "b"'] & S
\end{tikzcd}$$

The morphism c exists if and only if $a = b$. Thus the existence of Δ_f corresponds to uniqueness of f.

$\square$

Date: 10.20

Definition 1.9.7. Let K be a field and let A, $B \subseteq K$ be local rings with maximal ideals $\mathfrak{m}_A$, $\mathfrak{m}_B$. We say that B **dominates** A if $A \subseteq B$ and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$. We say that A is a **valuation ring of** K if A is a valuation ring and $\mathrm{Frac}(A) = K$.

Fact 1.9.8. *(1)* [AM, Exercise 5.27] *Let $A \subseteq K$ be a local domain. Then A is a valuation ring of K if and only if A is maximal for the dominance relation among local rings in K.*

(2) [M2, Theorem 10.2] *For any local ring $B \subseteq K$, there exists a valuation ring A of K dominating B.*

Proof of Theorem 1.9.6 (1). $\Longleftarrow$. Since f is universally specializing, we may pull back and reduce to the following lifting problem:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, equal] \arrow[ur, dashed, "t"] & S
\end{tikzcd}$$

Write $\mathrm{Im}(g) = \{x'\}$, s the closed point of S. Since f is specializing, and $f(x') \rightsquigarrow s$, $\exists x' \rightsquigarrow x$ such that $f(x) = s$. Consider

$$\begin{tikzcd}
K & \kappa(x') \arrow[l] & \mathcal{O}_{X,x} \arrow[l] \arrow[ll, bend right, "\phi"'] \\
 & & A \arrow[u] \arrow[ull]
\end{tikzcd}$$

Since $f(x) = s$, $A \to \mathcal{O}_{X,x}$ is a local homomorphism. Thus $\phi(\mathcal{O}_{X,x})$ dominates A. Since A is a valuation ring, it is maximal for the dominance relation, hence $\phi(\mathcal{O}_{X,x}) = A$. Let ψ be ϕ regarded as a map $\mathcal{O}_{X,x} \to A$. Then $\mathrm{Spec}(A) \xrightarrow{\mathrm{Spec}(\psi)} \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ furnishes the desired morphism t.

$\Longrightarrow$. It suffices to show f specializing. Let $x' \in X$ and $f(x') = s' \rightsquigarrow s$. We need to find $x' \rightsquigarrow x$ such that $f(x) = s$. Consider

$$\begin{tikzcd}
K = \kappa(x') & \mathcal{O}_{X.x'} \arrow[l] & \\
 & \mathcal{O}_{S,s'} \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[ull, bend right, "\phi"']
\end{tikzcd}$$

Since $\phi(\mathcal{O}_{S,s})$ is a local ring in K, there is a valuation ring A of K dominating it. Thus we have

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{X,x'}) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[urr, dashed, "t"] & \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S
\end{tikzcd}$$

Since f satisfies the existence part of valuation criterion, there exists t making the two triangles commutative. Let η and σ be the generic and closed points of $\mathrm{Spec}(A)$, respectively. Let $x = t(\sigma)$. Then $f(x) = s$ and t maps the specialization $\eta \rightsquigarrow \sigma$ to $x' \rightsquigarrow x$ as desired. □

Proposition 1.9.9. *(1) A closed map is specializing.*

(2) Conversely, a specializing and quasi-compact morphism of schemes is closed.

Proof. (1) This was shown in Example 1.6.35.

(2) Let $f\colon X \to S$ be specializing and quasi-compact morphism of schemes. Let $Y \subseteq X$ be closed subset. Equip Y with the induced reduced subscheme structure. We observe that the composition $Y \hookrightarrow X \xrightarrow{f} S$ is specializing and quasi-compact. We conclude that $f(Y)$ is closed by the following lemma. □

Lemma 1.9.10. *Let $f\colon X \to S$ be a quasi-compact morphism of schemes. Then $f(X)$ contains every maximal point of $\overline{f(X)}$. In particular, $f(X)$ is closed if moreover $f(X)$ is closed under specialization.*

Proof. We may assume that $S = \mathrm{Spec}(B)$ is affine. Then X is quasi-compact. Take a finite affine open covering $X = \bigcup_i U_i$, $U_i = \mathrm{Spec}(A_i)$. Then $f(X) = \bigcup_i f(U_i) = \mathrm{Im}(\mathrm{Spec}(\prod_i A_i) \to \mathrm{Spec}(B))$. Thus we may assume $X = \mathrm{Spec}(A)$ is affine and $f = \mathrm{Spec}(\phi)$ where $\phi\colon B \to A$ is a ring homomorphism.

Factor ϕ as $B \twoheadrightarrow B/I \hookrightarrow A$, where $I = \mathrm{Ker}(\phi)$. Then $\overline{f(X)}$ is contained in $\mathrm{Spec}(B/I)$. Thus may assume that ϕ is injective. This case is the content of next Lemma. □

Lemma 1.9.11. *Let $\phi\colon B \to A$ is an injective ring homomorphism. Then the image of $f = \mathrm{Spec}(\phi)$ contains all maximal points of $\mathrm{Spec}(B)$.*

Compare with Lemma 1.2.11(2).

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(B)$ be a maximal point. Then $B_\mathfrak{p}$ has a unique prime ideal, $\mathfrak{p}B_\mathfrak{p}$. Since $B_\mathfrak{p} \to A_\mathfrak{p}$ remains injective by flatness, we have $A_\mathfrak{p} \neq 0$. Thus there exists a maximal ideal $\mathfrak{m}$ of $A_\mathfrak{p}$. Then $\mathfrak{m} \cap B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$. □

Corollary 1.9.12. *A universally specializing and quasi-compact morphism of schemes is universally closed.*

Example 1.9.13. $\coprod_p \mathrm{Spec}(\mathbb{Z}/p) \to \mathrm{Spec}(\mathbb{Z})$ is universally specializing but not closed. The image is the set of closed points of $\mathrm{Spec}(\mathbb{Z})$.

Corollary 1.9.14. *f is separated $\iff$ f is quasi-separated and satisfies the uniqueness part of the valuative criterion.*

Proof. f is separated $\iff$ Δ_f is closed $\iff$ Δ_f is quasi-compact and universally specializing $\iff$ Δ_f is quasi-compact and satisfies the existence part of the valuative criterion $\iff$ f is quasi-separated and satisfies the uniqueness part of the valuative criterion. □

Definition 1.9.15. We say that a morphism of schemes is **proper** if it is separated, of finite type and universally closed.

Corollary 1.9.16. *f is proper $\iff$ f is of finite type, quasi-separated and satisfies both parts of the valuative criterion.*

Proposition 1.9.17. *f is integral $\iff$ f is affine and universally closed.*

Corollary 1.9.18. *f is finite $\iff$ f is affine and proper.*

Proof of the proposition. We need only to prove $\Longleftarrow$. Let $f\colon X \to S$ be an affine morphism that is universally closed. We may assume that $S = \mathrm{Spec}(B)$ is affine. Then $X = \mathrm{Spec}(A)$ is affine and $f = \mathrm{Spec}(\phi)$, $\phi\colon A \to B$. In this case we have the following stronger result. □

Lemma 1.9.19. *Let $\phi\colon B \to A$ be a ring homomorphism such that* $\mathrm{Spec}(A[X]) \to \mathrm{Spec}(B[X])$ *is closed. Then ϕ is integral.*

Proof. Let $a \in A$. We will show a is integral over B. Consider

$$I = \mathrm{Ker}(B[X] \to A) \\ X \mapsto a$$

and

$$J = \mathrm{Ker}(B[X] \to A[X]/(aX-1) = A[a^{-1}]) \\ X \mapsto X$$

If $f = \sum_{i\geq 0} b_i X^i \in J$, then $f = (aX-1)g$ where $g = \sum_{i\geq 0} a_i X^i \in A[X]$. Expand the coefficients we have $b_i = aa_{i-1} - a_i$. Thus for $n \geq \max\{\deg(f), \deg(g)+1\}$, $h = \sum_i b_i X^{n-i} = \sum_i (aa_{i-1} - a_i) X^{n-i} = (a-x)\sum a_i X^{n-1-i} \in I$. The leading coefficient of h is b_0. Thus, to show that a is integral, it suffices to find $f \in J$ with constant term in $B^\times$.

Note that J contains a polynomial with constant term in $B^\times$ if and only if $J + XB[X] = B[X]$. Now consider the closed map $f = \mathrm{Spec}(\phi[X])\colon \mathrm{Spec}(A[x]) \to \mathrm{Spec}(B[x])$. By Lemma 1.2.11, $\overline{f(V(aX-1))} = V(J)$. Thus $f(V(aX-1)) = V(J)$. It follows that $g\colon \mathrm{Spec}(A[x]/(aX-1)) \to \mathrm{Spec}(B[x]/(J)$ is surjective. We have a Cartesian square

$$\begin{tikzcd}
\emptyset \arrow[r, "g'"] \arrow[d] & \mathrm{Spec}(B[X]/(J+XB[X])) \arrow[d] \\
\mathrm{Spec}(A[x]/(aX-1)) \arrow[r, "g"] & \mathrm{Spec}(B[x]/(J)).
\end{tikzcd}$$

Indeed, $A[X]/(X, aX-1) = 0$. Thus g' is surjective. In other words, $\mathrm{Spec}(B[x]/(J + XB[X]) = \emptyset$ and $J + XB[X] = B[X]$. □

As usual, proper morphisms behave well under composition and base change:

Proposition 1.9.20. *(1) Proper morphisms are stable under composition and base change.*

(2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ gf proper and g separated $\implies$ f proper.

Definition 1.9.21. Let S be a scheme. We call $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} S$ the **projective n-space** over S.

For $S = \mathrm{Spec}(A)$, we have $\mathbb{P}^n_A \simeq \mathbb{P}^n_{\mathrm{Spec}(A)}$.

We have seen that finite morphisms are proper. Another nontrivial example is the following.

Proposition 1.9.22. *$\mathbb{P}^n_S \to S$ is proper.*

It suffices to show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ is proper. Recall that $\mathbb{P}^n_{\mathbb{Z}} = \bigcup_{i=0}^n U_i$, $U_i \simeq \mathrm{Spec}(R_i)$, $R_i = \mathbb{Z}[x_j/x_i]_{j=0}^n$. Moreover, $U_i \cap U_j \simeq \mathrm{Spec}(R_{ij})$, $R_{ij} = \mathbb{Z}[\{x_k/x_i, x_k/x_j\}_{k=0}^n]$. Since each R_i is a finite type $\mathbb{Z}$-algebra and $U_i \cap U_j$ is quasi-compact, $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ is of finite type and quasi-separated. It remains to prove both parts of the valuative criterion.

We now describe the functor represented by $\mathbb{P}^n_{\mathbb{Z}}$ in a special case. For a ring A, let $\mathbb{P}^n_{\mathbb{Z}}(A) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(A), \mathbb{P}^n_{\mathbb{Z}})$.

Lemma 1.9.23. *Let A be a local domain and let $K = \mathrm{Frac}(A)$. Then*

$$\mathbb{P}^n_{\mathbb{Z}}(A) \cong W/K^\times$$

where $W = \{(a_0, \dots, a_n) \in K^{n+1} \backslash \{(0, \dots, 0)\} \mid \exists i, \forall j, a_j \in a_i A\}$ and $K^\times$ acts on W by scalar multiplication.

We will give a description of $\mathbb{P}^n_{\mathbb{Z}}(S)$ for a general scheme S later. The class of $(a_0, \dots, a_n)$ is denoted by $[a_0 : \cdots : a_n]$.

Proof. Let $f\colon \mathrm{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$ and let s be the closed point of $\mathrm{Spec}(A)$. Then there exists i such that $f(s) \in U_i$. It follows that $f\colon \mathrm{Spec}(A) \to U_i$. Thus $\mathbb{P}^n_{\mathbb{Z}}(A) = \bigcup_i U_i(A)$. Consider the subset $W_i = \{(a_0, a_1, \dots a_n) \in K^{n+1} \backslash \{(0, \dots, 0)\} \mid \forall j, a_j \in a_i A\} \subseteq W$. Note that $U_i(A) \simeq \mathrm{Hom}_{\mathrm{Ring}}(R_i \to A)$. The map $\phi_i\colon W_i/K^\times \to U_i(A)$ carrying $[a_0 : \cdots : a_n]$ to the homomorphism $x_j/x_i \mapsto a_j/a_i$ is a bijection. Indeed, the inverse carries $g\colon R_i \to A$ to $[g(x_j/x_i)]_{0 \le j \le n}$. Similarly, $(U_i \cap U_j)(A) \simeq \mathrm{Hom}_{\mathrm{Ring}}(R_{ij}, A)$. The maps ϕ_i and ϕ_j restrict to $(W_i \cap W_j)/K^\times \xrightarrow{\sim} (U_i \cap U_j)(A)$. Since $W = \bigcup_{i=0}^n W_i$, the maps ϕ_i patch together to a bijection $W/K^\times \xrightarrow{\sim} \mathbb{P}^n_{\mathbb{Z}}(A)$. $\square$

We next discuss the valuation defined by a valuation ring.

Definition 1.9.24. Let Γ be a totally ordered abelian group ($a \leqslant b \implies a + c \leqslant b + c$) and let K be a field. A **valuation** $v\colon K^\times \to \Gamma$ is a group homomorphism satisfying the strong triangle inequality:

$$v(x + y) \geqslant \max(v(x), v(y)).$$

We extend v to $v(0) = \infty$.

If $v\colon K^\times \to \Gamma$ is a valuation, then $\{x \in K \mid v(x) \geqslant 0\}$ is a valuation ring of K. Conversely, if A is a valuation ring of K, then the quotient map $v\colon K^\times \to K^\times/A^\times$ is a valuation. Here the total order on $K^\times/A^\times$ is defined as follows: $xA^\times \leqslant yA^\times$ if $x^{-1}y \in A$.

End of proof of Proposition 1.9.22. Since $\mathrm{Spec}(\mathbb{Z})$ is a final object, it suffices to show for every valuation ring A of fraction field K, the map

$$\varphi\colon \mathbb{P}^n_{\mathbb{Z}}(A) \to \mathbb{P}^n_{\mathbb{Z}}(K)$$

is a bijection. By the description in Lemma 1.9.23, this map can be identified with the inclusion

$$W/K^\times \subseteq (K^{n+1}\backslash\{O\})/K^\times,$$

where $O = (0,\dots,0)$. Let $v\colon K^\times \to \Gamma$ be the valuation given by v. Let $(a_0,\dots,a_n) \in K^{n+1}\backslash\{O\}$. We can find a (nonzero) a_i with the smallest valuation. Then $v(a_j/a_i) \geqslant 0$ for all $0 \leqslant j \leqslant n$. In other words, $a_j/a_i \in A$ for all j. This shows that $(a_0,\dots,a_n) \in W$. Thus $W = (K^{n+1}\backslash\{O\})/K^\times$ and φ is a bijection. □

GAGA (Algebraic Geometry and Analytic Geometry). Let $X/\mathbb{C}$ be a scheme of finite type. Any affine open $U \subseteq X$ is of the form $\mathrm{Spec}(\mathbb{C}[X_1,\dots,X_n]/(f_1,\dots f_m))$. Then $U(\mathbb{C}) = \{(a_1,\dots,a_n) \in \mathbb{C}^n \mid f_i(a_1,\dots,a_n) = 0\} \subseteq \mathbb{C}^n$. We equip $U(\mathbb{C})$ with the subspace topology induced from the usual topology on $\mathbb{C}^n$. One can show that this does not depend on the choice of the embedding $U \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ and there exists a topology τ on $X(\mathbb{C})$ such that each $U(\mathbb{C})$ is an open subspace. The space $X^{\mathrm{an}} = (X(\mathbb{C}),\tau)$ is called the **analytic space** associated to X.

Fact 1.9.25.
- *X separated $\iff$ X^{an} Hausdorff.*
- *$X/\mathbb{C}$ proper $\iff$ X^{an} Hausdorff and quasi-compact.*
- *For* $$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Spec}(\mathbb{C}) & \end{tikzcd}$$ *where both $X \to \mathbb{C}$ and $Y \to \mathbb{C}$ are separated and of finite type, f is proper $\iff$ $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper (i.e. for every quasi-compact subset $V \subseteq Y^{\mathrm{an}}$, $(f^{\mathrm{an}})^{-1}(V)$ is quasi-compact, or, equivalently, for every topological space Z, $f^{\mathrm{an}} \times Z\colon X^{\mathrm{an}} \times Z \to Y^{\mathrm{an}} \times Z$ is closed).*

Theorem 1.9.26 (Nagata compactification)**.** *Let S be a quasi-compact quasi-separated scheme and let $f\colon X \to S$ be a separated morphism of finite type. Then there exists an open immersion $j\colon X \hookrightarrow \overline{X}$ and a proper morphism $\bar{f}\colon \overline{X} \to S$ such that $\bar{f}j = f$.*

Nagata proved the theorem for Noetherian schemes and Deligne proved the general case.

In a couple of simple cases, we already know the result of Nagata compactification.

Lemma 1.9.27. *Let $f\colon X \to S$ be a quasi-compact immersion. Then there exists an open immersion $j\colon X \to \overline{X}$ and a closed immersion $\bar{f}\colon \overline{X} \to S$ such that $\bar{f}j = f$.*

The quasi-compactness assumption cannot be dropped. See Warning 1.6.66.

Proof. It suffices to take $\overline{X} = \underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I})$ with $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_S \to f_*\mathcal{O}_X)$. This is called the **scheme-theoretic closure** of X. □

Example 1.9.28. Let $X = \operatorname{Spec}(A)$, $S = \operatorname{Spec}(B)$, $\phi\colon B \to A$ and $f = \operatorname{Spec}(\phi)$. Assume that f is of finite type, namely A is a finitely generated B-algebra. Choosing a set of generators, we obtain a closed immersion $X \to \mathbb{A}^n_B$ over B. Choose an open immersion $\mathbb{A}^n_B \to \mathbb{P}^n_B$ over B.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbb{A}^n_B \arrow[r] \arrow[d] & \mathbb{P}^n_B \arrow[dl] \\ & S & \end{tikzcd}$$

Let $\overline{X}$ be the scheme-theoretic closure of X in $\mathbb{P}^n_B$. Then $\overline{X} \to S$ is proper.

Date: 10.22

1.10 Homogeneous spectrum

Let k be a field. We have a bijection

$$\begin{aligned}\mathbb{P}^n(k) &\xrightarrow{\sim} \{\text{lines in } \mathbb{A}^{n+1} \text{ through } O\}\\ [a_0 : \cdots : a_n] &\mapsto V(a_i x_j - a_j x_i).\end{aligned}$$

Closed subsets of $\mathbb{P}^n$ correspond to conical subsets of $\mathbb{A}^{n+1}$ of the form $V(f_1, \dots f_r)$, with each f_i homogeneous.

Example 1.10.1. The cone $V(x^2 - y^2 - z^2)$ in $\mathbb{A}^3$ corresponds to a curve in $\mathbb{P}^2$.

For every graded ring R, we will construct a scheme $\operatorname{Proj}(R)$, called the **homogeneous spectrum** of R. Recall that a **graded ring** is a ring equipped with a decomposition $R = \bigoplus_{d\geqslant 0} R_d$ as abelian groups, satisfying $R_d R_e \subseteq R_{d+e}$. In particular, $1 \in R_0$ and R_0 is both a sub-ring and a quotient ring of R. Elements in R_d are called **homogeneous of degree** d.

An ideal $I \subseteq R$ is said to be **homogeneous** if $I = \bigoplus_{d\geqslant 0}(I \cap R_d)$.

Example 1.10.2. $R_+ = \bigoplus_{d>0} R_d$ is a homogenous ideal.

Lemma 1.10.3. *Let $\mathfrak{p} \subseteq R$ be a homogeneous ideal. Then $\mathfrak{p}$ is a prime ideal if and only if $\forall x, y \in R$ homogeneous, $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.*

Proof. $\Longleftarrow$. Let $x = \sum x_d$, $y = \sum y_e$, $x, y \notin \mathfrak{p}$. Then there exists a smallest d_0 such that $x_{d_0} \notin \mathfrak{p}$. Similarly there exists a smallest e_0 such that $y_{e_0} \notin \mathfrak{p}$. Expanding xy, we see that $x_{d_0} y_{e_0} \notin \mathfrak{p}$. Thus $xy \notin \mathfrak{p}$. □

Definition 1.10.4. For a graded ring R, we define a subset $\operatorname{Proj}(R) \subseteq \operatorname{Spec}(R)$ by

$$\operatorname{Proj}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \text{ homogeneous and } \mathfrak{p} \not\supseteq R_+\} \subseteq \operatorname{Spec}(R).$$

We equip $\operatorname{Proj}(R)$ with the subspace topology.

Notation 1.10.5. For any subset $T \subseteq R$, we write $V_+(T) = V(T) \cap \operatorname{Proj}(R)$. For $f \in R$, we write $V_+(f) = V_+((f))$. For $f \in R_+$ homogeneous, we write $D_+(f) = \operatorname{Proj}(R)\backslash V_+(f)$.

Thus $V_+(T)$ is the set of homogeneous prime ideals of R satisfying $T \subseteq \mathfrak{p}$ and $R_+ \subsetneq \mathfrak{p}$. It is a closed subset of $\operatorname{Proj}(R)$. If $p_d \colon R \to R_d$ denotes the projection, then the homogeneous ideal generated by T is $I = \bigcup_{d\geqslant 0} p_d(T)$. It is clear that $V_+(T) = V_+(I)$. Thus every closed subset of $\operatorname{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal I.

Example 1.10.6. $V_+(R) = V_+(R_+) = \emptyset$, $V_+(0) = \operatorname{Proj}(R)$.

Lemma 1.10.7. *If $f \in R_0$, then*

$$D(f) \cap \operatorname{Proj}(R) = \bigcup_{g \in R_+ \text{ homogeneous}} D_+(fg)$$

Proof. $\supseteq$ is clear. For $\subseteq$, suppose there exists $\mathfrak{p} \in D(f) \cap \mathrm{Proj}(R)$ such that $\mathfrak{p} \notin \bigcup_g D_+(fg)$. Then $f \notin \mathfrak{p}$, but $\forall g \in R_+$ homogeneous, $fg \in \mathfrak{p}$, so that $g \in \mathfrak{p}$. It follows that $R_+ \subseteq \mathfrak{p}$, a contradiction. □

This is the reason why we only consider $D_+(f)$ with f homogeneous of positive degree.

Definition 1.10.8. $D_+(f)$ with $f \in R_+$ homogeneous are called **standard open subsets**.

Standard open subsets form an open basis for $\mathrm{Proj}(R)$ and $D_+(fg) = D_+(f) \cap D_+(g)$.

For any homogeneous $f \in R_+$, R_f is a $\mathbb{Z}$-graded ring. Let $R_{(f)}$ denote the degree 0 piece of R_f.

Lemma 1.10.9. *For f homogeneous of positive degree, we have*

$$D_+(f) \xleftarrow{\sim} \{\textit{homogeneous primes of } R_f\} \xrightarrow{\sim} \mathrm{Spec}(R_{(f)})$$

are homeomorphisms.

Proof. The isomorphism on the left is easy to establish. For the one on the right, we apply the following Lemma. □

Lemma 1.10.10. *Let S be a $\mathbb{Z}$-graded ring such that there exists $f \in S_d \cap S^\times$ for some $d > 0$. Then we have a homeomorphism*

$$\begin{aligned} j\colon G = \{\mathbb{Z}\textit{-graded primes of } S\} &\xrightarrow{\sim} \mathrm{Spec}(S_0) \\ \mathfrak{p} &\mapsto \mathfrak{p} \cap S_0 \\ \sqrt{\mathfrak{p}_0 S} &\leftarrow \mathfrak{p}_0 \end{aligned}$$

We remark that in a $\mathbb{Z}$-graded ring S, S_0 is a subring but typically not a quotient.

Proof. We need to prove that $\sqrt{\mathfrak{p}_0 S}$ is a prime ideal. Let $a, b \in \sqrt{\mathfrak{p}_0 S}$ homogeneous. There exists $n \geqslant 1$ such that $(ab)^n \in \mathfrak{p}_0 S$. We have $(a^d b^d / f^{\deg(a)+\deg(b)})^n \in \mathfrak{p}_0$, hence $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$. This shows $a \in \sqrt{\mathfrak{p}_0 S}$ or $b \in \sqrt{\mathfrak{p}_0 S}$.

It is easy to check that j is a bijection. It is continuous. We prove that it is open. Consider the open subset $G \cap D(g)$, where $g = \sum_i g_i$, $g_i \in S_i$. Then $j(G \cap D(g)) = \bigcup_i D(g_i^d / f^{deg(g_i)})$. Thus j is a homeomorphism. □

We now proceed to equip $X = \mathrm{Proj}(R)$ with a sheaf of rings $\mathcal{O}_X$. We take $\mathcal{O}_X(D_+(f)) = R_{(f)}$. The functoriality of this assignment is guaranteed by the following.

Lemma 1.10.11. *Assume $D_+(g) \subseteq D_+(f)$. Then there exist $n \geqslant 1$ such that $g^n = af$ with $a \in R$ homogeneous. Moreover, we have a commutative diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R_f \arrow[d] & R_{(f)} \arrow[l] \arrow[d] \\ & R_g & R_{(g)} \simeq (R_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \arrow[l] \end{tikzcd}$$

Proof. We first show that $f^{\deg(g)}/g^{\deg(f)} \in R_{(g)}$ is invertible. If $f^{\deg(g)}/g^{\deg(f)} \in \mathfrak{p}_0 \in \operatorname{Spec}(R_{(g)})$, then $f \in \sqrt{\mathfrak{p}_0 R_g} \cap R = \mathfrak{p} \in D_+(g) \subseteq D_+(f)$, a contradiction. Thus $f\frac{b}{g^m} = 1$ in $R_{(g)}$ with some $m \geq 1$. It follows that $af = g^n$ for some $n \geqslant 1$. The last part of the lemma is now clear. $\square$

Proposition 1.10.12. *For a graded ring R, the functor*

$$\begin{aligned}\{\textit{standard open subsets of } \operatorname{Proj}(R)\}^{\mathrm{op}} &\to \mathrm{Ring}\\ D_+(f) &\mapsto R_{(f)}\end{aligned}$$

extends to a sheaf $\mathcal{O}_X$ on $X = \operatorname{Proj}(R)$. Moreover, $(D_+(f), \mathcal{O}_X|_{D_+(f)}) \simeq \operatorname{Spec}(R_{(f)})$ and $(X, \mathcal{O}_X)$ is a scheme over $\operatorname{Spec}(R)$.

Proof. We first verify the gluing property. Let $D_+(f) = \bigcup_i D_+(g_i)$. Since $D_+(f) \simeq \operatorname{Spec}(R_{(f)})$ and

$$D_+(g_i) \simeq \operatorname{Spec}(R_{(g_i)}) \simeq \operatorname{Spec}(R_{(f)})_{g_i^{\deg(f)}/f^{\deg(g_i)}},$$

the gluing property for $\mathcal{O}_X$ follows from the gluing property for $\mathcal{O}_{\operatorname{Spec}(R_{(f)})}$. The last assertion is now clear. $\square$

Proposition 1.10.13. *For all $\mathfrak{p} \in \operatorname{Proj}(R)$, we have $\mathcal{O}_{X,\mathfrak{p}} = R_{(\mathfrak{p})}$. Here $R_{(\mathfrak{p})}$ is the degree 0 piece of $T^{-1}R$, where $T = \{f \in R\backslash\mathfrak{p}$ homogeneous$\}$.*

Proof. We have

$$\mathcal{O}_{X,\mathfrak{p}} = \operatorname*{colim}_{f \in R_+\backslash\mathfrak{p} \text{ homogeneous}} R_{(f)} = R_{(\mathfrak{p})}.$$

Here we used the fact that there exists $g \in R_+\backslash\mathfrak{p}$ homogeneous and $a/f = \frac{ag}{fg}$ in $R_{(\mathfrak{p})}$. $\square$

Example 1.10.14. $\mathbb{P}^n_A \simeq \operatorname{Proj}(R)$, where $R = A[x_0, \dots x_n]$. Indeed, $\operatorname{Proj}(R) = \bigcup D_+(x_i)$, $D_+(x_i) = R_{(x_i)} = A[x_j/x_i]_{j=0}^n$, $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j) = A[x_k/x_i, x_k/x_j]_{k=0}^n$.

In particular, $\operatorname{Spec}(A) = \mathbb{P}^0_A \simeq \operatorname{Proj}(A[x_0])$.

Example 1.10.15. Let $R = A[x_0, \dots, x_n]$ and let $d_0, \dots, d_n > 0$ be integers. We define a grading on R by $R_0 = A$ and $\deg(x_i) = d_i$. We call $\operatorname{Proj}(R) := \mathbb{P}_A(d_0, \dots, d_n)$ the **weighted projective n-space** of weights $(d_0, \dots, d_n)$. It is clear that $\mathbb{P}_A(d_0, \dots, d_n) = \mathbb{P}_A(dd_0, \dots, dd_n)$ for any $d \geqslant 1$.

Lemma 1.10.16. $\operatorname{Proj}(R)$ *is quasi-separated.*

Proof. $\operatorname{Proj}(R) = \bigcup_f D_+(f)$ and $D_+(f) \cap D_+(g) = D_+(fg)$. $\square$

In fact, $\operatorname{Proj}(R)$ is separated (exercise).

Proposition 1.10.17. $\operatorname{Proj}(R)$ *is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \dots f_r \in R_+$ such that $R_+ \subseteq \sqrt{(f_1, \dots, f_r)}$.*

Proof. $\operatorname{Proj}(R)$ is quasi-compact if and only if a finite number of standard opens cover $\operatorname{Proj}(R)$. In other words, there exist $f_1, \dots f_r \in R_+$ homogeneous such that $V_+(f_1, \dots, f_r) = \emptyset$. We conclude by the next Lemma. $\square$

Lemma 1.10.18. *Let $I \subseteq R$ be a homogeneous ideal. Then $V_+(I) = \emptyset \iff R_+ \subseteq \sqrt{I}$.*

Proof. $\Longleftarrow$. Clear.

$\Longrightarrow$. Assume $R_+ \not\subseteq \sqrt{I}$. Then there exists $f \in R_+ \backslash \sqrt{I}$ homogeneous. We have $(R/I)_f \neq 0$, so that $(R/I)_{(f)} \neq 0$ (since it contains $1 \in (R/I)_f$). Thus $\mathrm{Proj}(R/I) \neq \emptyset$. Then there exists a homogeneous prime $\mathfrak{q}$ of R/I satisfying $\mathfrak{q} \not\supseteq (R/I)_+$. The preimage of $\mathfrak{q}$ in R is a homogeneous prime $\mathfrak{p}$ of I satisfying $\mathfrak{p} \not\supseteq R_+$. Thus $\mathfrak{p} \in V_+(I)$, a contradiction. $\square$

Functoriality

Let $\phi\colon R \to S$ be a homomorphism of graded rings. For $\mathfrak{q} \in \mathrm{Proj}(S)$, $\phi^{-1}(\mathfrak{q})$ is a homogeneous prime ideal of R, but in general it may happen that $\phi^{-1}(\mathfrak{q}) \supseteq R_+$. Let

$$U(\phi) = \{\mathfrak{q} \in \mathrm{Proj}(S) \mid \phi^{-1}(\mathfrak{q}) \not\supseteq R_+\}$$

In other words, $U(\phi) = \mathrm{Proj}(S) \backslash f^{-1}(V(R_+))$, where $f = \mathrm{Spec}(\phi)$.

$$\begin{tikzcd}
 & U(\phi) \arrow[d, hook] \\
\mathrm{Proj}(R) \arrow[d, hook] & \mathrm{Proj}(S) \arrow[d, hook] \\
\mathrm{Spec}(R) & \mathrm{Spec}(S), \arrow[l, "f"']
\end{tikzcd}$$

Lemma 1.10.19.

$$U(\phi) = \bigcup_{\textit{homogenous } a \in R_+} D_+(\phi(a))$$

Proof. $\phi^{-1}(\mathfrak{q}) \in \mathrm{Proj}(R) \iff \exists\, a \in R_+,\, a \notin \phi^{-1}(\mathfrak{q}) \iff \exists\, a \in R_+,\, \phi(a) \notin \mathfrak{q}$. $\square$

The natural morphisms of schemes $D_+(\phi(a)) \to D_+(a)$ given by $\phi_{(a)}\colon R_{(a)} \to S_{(\phi(a))}$ glue to a morphism of schemes $\mathrm{Proj}(\phi)\colon U(\phi) \to \mathrm{Proj}(R)$.

We will give an example where $\mathrm{Proj}(\phi)$ is defined on $\mathrm{Proj}(S)$.

Let us start with a general remark on homogeneous localization. For $a \in R_+$ homogenous of degree d,

$$R_{(a)} = \mathrm{colim}(\ R_0 \xrightarrow{a} R_d \xrightarrow{a} R_{2d} \longrightarrow \dots\ )$$

can be computed using R_{nd} for n running through any unbounded subset of $\mathbb{N}$. In particular,

- If $\phi\colon R \to S$ is such that $\sup\{n | \phi_{nd} \text{ is surjective}\} = \infty$, then $\phi_{(a)}\colon R_{(a)} \to S_{(\phi(a))}$ is surjective.
- If $\phi\colon R \to S$ is such that $\sup\{n | \phi_{nd} \text{ is an isomorphism}\} = \infty$, then $\phi_{(a)}\colon R_{(a)} \to S_{(\phi(a))}$ is an isomorphism.

Proposition 1.10.20. *Let $\phi\colon R \to S$ be a graded homomorphism such that for all $d \geqslant 1$, there exists $n \geqslant 1$ such that ϕ_{nd} is surjective. Then $U(\phi) = \mathrm{Proj}(S)$ and $\mathrm{Proj}(\phi)\colon \mathrm{Proj}(S) \hookrightarrow \mathrm{Proj}(R)$ is a closed immersion.*

Proof. Let $\mathfrak{q} \in \mathrm{Proj}(S)$. There exists $b \in S_+$ homogeneous of degree $d > 0$, $b \notin \mathfrak{q}$. Then $b^n \notin \mathfrak{q}$ for all n. By assumption, there exists $n \geqslant 1$ such that ϕ_{nd} is surjective, and thus there exists $a \in R_+$ with $\phi(a) = b^n$. Thus $a \notin \phi^{-1}(\mathfrak{q})$ and $\phi^{-1}(\mathfrak{q}) \in \mathrm{Proj}(R)$. This shows $U(\phi) = \mathrm{Proj}(S)$. Moreover, for $a \in R_+$ homogeneous, $R_{(a)} \to S_{(\phi(a))}$ is surjective. Thus $\mathrm{Proj}(\phi)$ is a closed immersion. □

Example 1.10.21. For any homogeneous ideal $I \subseteq R$, $\mathrm{Proj}(R/I) \hookrightarrow \mathrm{Proj}(R)$ is a closed subscheme of image $V_+(I)$. We will give a partial converse later.

Proposition 1.10.22. *Let $\phi\colon R \to S$ be a graded homomorphism such that for all $d \geqslant 1$, there exists $n \geqslant 1$ such that ϕ_{nd} is an isomorphism. Then $\mathrm{Proj}(\phi)\colon \mathrm{Proj}(S) \xrightarrow{\sim} \mathrm{Proj}(R)$ is an isomorphism.*

Next we look at a different kind of functoriality.

Notation 1.10.23. For $d \geqslant 1$, we let $R^{(d)} := \bigoplus_n R_{nd}$ denote the graded ring with $R^{(d)}_n = R_{nd}$.

Proposition 1.10.24. *We have an isomorphism of schemes over* $\mathrm{Spec}(R_0)$

$$\begin{aligned} \mathrm{Proj}(R) &\xrightarrow{\sim} \mathrm{Proj}(R^{(d)}) \\ \mathfrak{p} &\mapsto \mathfrak{p} \cap R^{(d)}. \end{aligned}$$

Proof. We write $R_{+,\mathrm{homog}} = \bigcup_{i>0} R_i$. We have $\mathrm{Proj}(R) = \bigcup_{f\in R_{+,\mathrm{homog}}} D_{+,R}(f)$ and $\mathrm{Proj}(R^{(d)}) = \bigcup_{f\in R_{+,\mathrm{homog}}} D_{+,R^{(d)}}(f^d)$. Indeed, for $g \in R^{(d)}_{+,\mathrm{homog}}$, $D_{+,R^{(d)}}(g) = D_{+,R^{(d)}}(g)$. Observe that the inclusion $R^{(d)} \subseteq R$ induces an isomorphism $R^{(d)}_{(f^d)} \to R_{(f)}$, with inverse given by $a/f^n \mapsto af^{n(d-1)}/f^{nd}$. This gives $D_{+,R}(f) \xrightarrow{\sim} D_{+,R^{(d)}}(f^d)$, which patches together to an isomorphism of schemes $\mathrm{Proj}(R) \xrightarrow{\sim} \mathrm{Proj}(R^{(d)})$ over $\mathrm{Spec}(R_0)$. □

Remark 1.10.25. The underlying homeomorphism $\iota\colon \mathrm{Proj}(R) \xrightarrow{\sim} \mathrm{Proj}(R^{(d)})$ is compatible with the continuous map $\mathrm{Spec}(R) \to \mathrm{Spec}(R^{(d)})$ induced by the inclusion $R^{(d)} \subseteq R$, which is not graded for $d > 1$. The inverse of ι carries $\mathfrak{q}$ to $\mathfrak{p} = \{g \in R \mid g^d \in \mathfrak{q}\}$. To see this, we first need to show that $\mathfrak{p}$ is an ideal. If $g^d \in \mathfrak{q}$, $h^d \in \mathfrak{q}$, then $(g+h)^{2d} \in \mathfrak{q}$, hence $(g+h)^d \in \mathfrak{q}$. Thus $\mathfrak{p}$ is an ideal. It is graded, since otherwise, writing $g = \sum_i g_i$ with $g_i \in R_i$, there exists $g_i \notin \mathfrak{p}$ of lowest degree. Then $g^d \in \mathfrak{q}$ implies $g_i^d \in \mathfrak{q}$, a contradiction. It is clear that $\mathfrak{p}$ is a prime and $\mathfrak{q} \mapsto \mathfrak{p}$ is an inverse of ι.

More trivially we can also define a graded ring $R^{(1/d)}$ where

$$R^{(1/d)}_n = \begin{cases} R_{n/d} & d \mid n \\ 0 & d \nmid n. \end{cases}$$

We also have $\mathrm{Proj}(R^{(1/d)}) \xrightarrow{\sim} \mathrm{Proj}(R)$.

Example 1.10.26. $R = A[x_0, \dots, x_n]$ with $\deg(x_i) = 1$ for all i. Then $R^{(d)} = A[M_0, \dots, M_N]$, where $M_0, \dots M_N$ are the monomials of degree d. We have $N = \binom{d+n}{n} - 1$. We have a surjective graded homomorphism $S = A[y_0, \dots, y_N] \to R^{(d)}$ sending y_i to M_i. Let I be the kernel. Taking Proj, we get a closed immersion $\mathbb{P}^n_A = \mathrm{Proj}(R) \simeq \mathrm{Proj}(R^{(d)}) \hookrightarrow \mathrm{Proj}(S) = \mathbb{P}^N_A$. This is called the d-**uple embedding**. Here are some examples of low dimension and low degree.

- $n = 1$, $d = 2$, $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. This is a conic. $R = A[u, v]$,

$$S = A[x, y, z] \to R^{(2)} = A[u^2, uv, v^2].$$

  The kernel is $I = (y^2 - xz)$.

- $n = 1$, $d = 3$, $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. This is a twisted cubic curve.

$$S = A[w, x, y, z] \to R^{(3)} = A[u^3, u^2v, uv^2, v^3].$$

  We have $I = (x^2 - wy, y^2 - xz, wz - xy)$. For $A = k$ a field, the closed subscheme $C \subseteq \mathbb{P}^3$ given by the triple embedding has codimension 2, but it is easy to see that I cannot be generated by two elements. We say that C is not a complete intersection in $\mathbb{P}^3$.

  For example, for $J = (x^2 - wy, y^2 - xz)$, $V_+(J)$ is not irreducible, as it contains the line $V_+(x, y)$. For $I' = (x^2 - wy, y^3 - wz^2) \subsetneq I$, we have $\sqrt{I'} = \sqrt{I}$, so that $V_+(I') = V_+(I)$ as sets: C is a set-theoretic complete intersection.

- $n = 1$, $d = 4$, $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$. This is a twisted quartic curve. Consider

$$R' = A[u^4, u^3v, uv^3, v^4] \subseteq R^{(4)} = A[u^4, u^3v, u^2v^2, uv^3, v^4]$$

  We have $R'_n = R^{(4)}_n$ for all $n \geqslant 2$. Thus $\mathrm{Proj}(R') \xrightarrow{\sim} \mathrm{Proj}(R^{(4)})$. We have $\mathrm{Proj}(R') \hookrightarrow \mathbb{P}^3$. For A a domain, R' is not integrally closed, since $(u^2v^2)^2 \in R'$ but $u^2v^2 \notin R'$.

- $n = 2$, $d = 2$, $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. This is called the **Veronese embedding**. $R = A[u, v, w]$,

$$S = A[y_0, y_1, y_2, y_3, y_4, y_5] \twoheadrightarrow R^{(2)} = A[u^2, v^2, w^2, uv, uw, vw]$$

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Example 1.10.27. $\mathbb{P}_A(1,2,3) = \mathrm{Proj}(A[u,v,w])$, $\deg(u) = 1$, $\deg(v) = 2$, $\deg(w) = 3$. It is easy to see that $R^{(6)} = A[u^6, v^3, w^2, u^4v, u^3w, u^2v^2, uvw]$ is generated by $R_1^{(6)}$ over A. We have thus obtained a closed immersion $\mathbb{P}_A(1,2,3) \hookrightarrow \mathbb{P}^6_A$. This is a del Pezzo surface (for $A = k$ a field).

The same argument works for any finitely generated graded ring.

Lemma 1.10.28. *Let R be a graded ring, finitely generated over R_0. Then there exists $d \geqslant 1$ such that $R^{(d)}$ is generated by finitely many elements in $R_1^{(d)}$ over R_0. Moreover, if $R = R_0[f_1, \dots, f_r]$ with f_i homogeneous of degree $d_i \geqslant 1$ and $m = \mathrm{lcm}(d_1, \dots, d_r)$, then we can take $d = sm$ where s is any integer $\geq \max\{r-1, 1\}$.*

Proof. Consider $P = f_1^{e_i} \cdots f_r^{e_r}$ of total degree Nm, $N \geqslant r$. In other words, $\sum_i d_i e_i \geqslant Nm$. Then there exists i such that $e_i \geqslant \frac{m}{d_i}$ and we have $P = P_1 Q$, where $P_1 = f_i^{m/d_i}$ has degree m and Q is homogeneous of degree $(N-1)m$. Thus if $N = nsm$, we obtain by induction a decomposition $P = P_1 \cdots P_{(n-1)sm} Q$, where $P_j \in R_m$ and $Q \in R_{sm}$. Thus P can be generated by $R_1^{(d)}$ over R_0. The finiteness is clear. ☐

Corollary 1.10.29. *Let R be a graded ring, finitely generated over R_0. Then there exists a closed immersion $\mathrm{Proj}(R) \hookrightarrow \mathbb{P}^n_{R_0}$ for some n.*

Proof. Let d be as in the previous lemma. Since $R^{(d)}$ is generated by finitely many elements of $R_1^{(d)}$ over R_0, it is the quotient of the polynomial ring $R_0[X_0, \dots, X_n]$. This gives a closed immersion $\mathrm{Proj}(R) \simeq \mathrm{Proj}(R^{(d)}) \hookrightarrow \mathbb{P}^n_{R_0}$. ☐

Definition 1.10.30. We say that a morphism $f\colon X \to Y$ is **projective** if it factors as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_S \arrow[r, equal] \arrow[d, "p"] & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S \\ & S & \end{tikzcd}$$

where i is a closed immersion and p is the projection.

Remark 1.10.31. Projective morphisms are proper.

Base change

Let $\phi\colon R \to S$ be a graded ring homomorphism. We have for any $a \in R_+$ homogenous, a commutative diagram

$$\begin{tikzcd} D_+(\phi(a)) \arrow[r] \arrow[d, hook] & D_+(a) \arrow[d, hook] \\ U(\phi) \arrow[r, "r"] \arrow[d, hook] & \mathrm{Proj}(R) \arrow[dd] \\ \mathrm{Proj}(S) \arrow[d] & \\ \mathrm{Spec}(S_0) \arrow[r] & \mathrm{Spec}(R_0) \end{tikzcd}$$

The top most square is Cartesian and thus r is an affine morphism.

Proposition 1.10.32. *Let R be a graded ring and $R_0 \to S_0$ a ring homomorphism. Let $S = R \otimes_{R_0} S_0$ and $\phi\colon R \to S$. Then $U(\phi) = \mathrm{Proj}(S)$ and we have a Cartesian square*

$$\begin{tikzcd} \mathrm{Proj}(S) \arrow[r] \arrow[d] & \mathrm{Proj}(R) \arrow[d] \\ \mathrm{Spec}(S_0) \arrow[r] & \mathrm{Spec}(R_0) \end{tikzcd}$$

Proof. Since $R_+S = S_+$, we have $U(\phi) = \mathrm{Proj}(S) \backslash V(S_+) = \mathrm{Proj}(S)$. Take $a \in R_+$ homogeneous. We need to check that

$$\begin{tikzcd} D_+(\phi(a)) \arrow[r] \arrow[d] & D_+(a) \arrow[d] \\ \mathrm{Spec}(S_0) \arrow[r] & \mathrm{Spec}(R_0) \end{tikzcd}$$

is Cartesian. This follows from the fact that tensor product commutes with localization: the diagram of rings

$$\begin{tikzcd} S_{(\phi(a))} & R_{(a)} \arrow[l] \\ S_0 \arrow[u] & R_0 \arrow[l] \arrow[u] \end{tikzcd}$$

is coCartesian. □

Now let R and S be graded rings satisfying $R_0 = S_0 = A$. We will determine the fiber product of the diagram.

$$\begin{tikzcd} & \mathrm{Proj}(R) \arrow[d] \\ \mathrm{Proj}(S) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

A first attempt is to consider $R\otimes_A S$ with grading given by $(R\otimes_A S)_d = \bigoplus_{i+j=d} R_i\otimes S_j$. But for homogeneous elements $a\in R_+$ and $b\in S_+$, the map $R_{(a)}\otimes S_{(b)}\to (R\otimes_A S)_{(a\otimes b)}$ is typically not surjective.

Instead we consider the subring $R\obslash_A S = \bigoplus_{d\geq 0} R_d\otimes_A S_d \subseteq R\otimes_A S$, with grading given by $(R\obslash_A S)_d = R_d\otimes_A S_d$. We have a Cartesian square

$$\begin{tikzcd}
\mathrm{Proj}(R\obslash_A S) \arrow[r] \arrow[d] & \mathrm{Proj}(R) \arrow[d] \\
\mathrm{Proj}(S) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

Indeed, for a and b as above, we have $R_{(a)}\otimes S_{(b)}\simeq (R\obslash_A S)_{(a\otimes b)}$.

The subring can be more complicated than the tensor product, as shown by the following example.

Example 1.10.33. Let $R=A[x_0,\dots,x_r]$, $S=A[y_0,\dots,y_s]$. Then $R\otimes_A S = A[x_0,\dots,x_r,y_0,\dots y_s]$, but $R\obslash_A S = A[x_iy_j]_{\substack{0\leq i\leq r\\ 0\leq j\leq s}}$. We have a surjection

$$\begin{aligned} T = A[z_{ij}] &\to S \\ z_{ij} &\mapsto x_iy_j \end{aligned}$$

with kernel $I=(z_{ij}z_{i'j'}-z_{ij'}z_{i'j})_{i,j,i',j'}$. This gives a closed immersion $\mathbb{P}^r_A\times_{\mathrm{Spec}(A)}\mathbb{P}^r_A\simeq \mathrm{Proj}(R\obslash_A S)\hookrightarrow \mathrm{Proj}(T)=\mathbb{P}^N_A$, where $N=(r+1)(s+1)-1 = rs+r+s$. This is called the **Segre embedding**.

In the case $r=s=1$, we have $N=3$ and the image of $\mathbb{P}^1\times\mathbb{P}^1\hookrightarrow\mathbb{P}^3$ is the quadric surface defined by $xw-yz=0$.

Proposition 1.10.34. *Projective morphisms are stable under base change and composition.*

Proof. The stability under base change follows from the fact that closed immersions are stable under base change: if $X\to S$ is a projective morphism and $S'\to S$ an arbitrary morphism, then we have a diagram with Cartesian squares

$$\begin{tikzcd}
X\times_S S' \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{S'} \arrow[r] \arrow[d] & S' \arrow[d] \\
X \arrow[r, hook] & \mathbb{P}^n_S \arrow[r] & S
\end{tikzcd}$$

For the stability under composition, let $X\to Y$ and $Y\to S$ be projective morphisms and consider the following commutative diagram with Cartesian squares:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_Y \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{\mathbb{P}^m_S} \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}}\times\mathbb{P}^m_{\mathbb{Z}} \arrow[r, hook, "\text{Segre}"] \arrow[dd, "p"] & \mathbb{P}^N_{\mathbb{Z}} \arrow[ddl] \\
& Y \arrow[r, hook] \arrow[dr] & \mathbb{P}^m_S \arrow[d] & & \\
& & S \arrow[r] & \mathrm{Spec}(\mathbb{Z}) &
\end{tikzcd}$$

The square in the middle is Cartesian because $\mathbb{P}^n_{\mathbb{P}^m_S} = \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_S = \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} \times S$ by definition. Since p is projective by the Segre embedding, its base change $\mathbb{P}^n_{\mathbb{P}^m_S} \to S$ is also projective. Thus $X \to S$ is projective. $\square$

Definition 1.10.35. Let $\mathcal{P}$ be a class of schemes. $\mathcal{P}$ is called **local** if

- $X \in \mathcal{P}$, $U \subseteq X$ open $\implies U \in \mathcal{P}$.
- $X = \bigcup_i U_i$, U_i open and $U_i \in \mathcal{P}$ for all $i \implies X \in \mathcal{P}$.

Here are some local properties of schemes: reduced, normal, locally Noetherian, empty.

Non-local properties: affine, quasi-compact, separated, quasi-separated, irreducible, connected, integral, Noetherian.

Definition 1.10.36. Let $\mathcal{P}$ be a class of morphisms. We say $\mathcal{P}$ is **local on the source** if

- $(X \xrightarrow{f} Y) \in \mathcal{P}$, $U \xrightarrow{j} X$ open immersion $\implies f \circ j \in \mathcal{P}$.
- Given $f\colon X \to Y$, $X = \bigcup_i U_i$, U_i open and $\forall i$, $(f|_{U_i}\colon U_i \to X) \in \mathcal{P} \implies f \in \mathcal{P}$.

We say $\mathcal{P}$ is **local on the target** if

- $(X \xrightarrow{f} Y) \in \mathcal{P}$, $V \subseteq Y$ open $\implies (f^{-1}(V) \xrightarrow{f_V} V) \in \mathcal{P}$.
- Given $f\colon X \to Y$, $Y = \bigcup_i V_i$, $V_i \subseteq Y$ open and $\forall i$, $(f^{-1}(V_i) \xrightarrow{f_{V_i}} V_i) \in \mathcal{P} \implies f \in \mathcal{P}$.

Local on the source and the target: locally of finite type, flat, open, generizing.

Local on the target: quasi-compact, affine, closed, specializing, integral, finite, quasi-separated, separated, proper, immersion, surjective, injective.

Not local on the target: projective.

An example of Hironaka shows that projectiveness is **not** local on the target. See [H, Example B.3.4.2].

Quasi-coherent sheaves on Proj(R)

For every graded R-module M, we will construct a quasi-coherent sheaf $\widetilde{M}$ on Proj(R). Recall that a **graded R-module** is an R-module M equipped with a $\mathbb{Z}$-grading as abelian group $M = \bigoplus_{d\in\mathbb{Z}} M_d$ such that $R_d M_e \subseteq M_{d+e}$.

Given a graded R-module M and $n \in \mathbb{Z}$, we define a graded R-module $M(n)$, called the twisted module, by $M(n)_d = M_{n+d}$. If we visualize a graded R-module by writing down its pieces sequentially, then $M(1)$ corresponds to a shift to the left.

Given graded R-modules M and N, the tensor product $M \otimes_R N$ is a graded R-module as follows. The R_0-module $M \otimes_{R_0} N$ clearly admits a grading: $(M \otimes_{R_0} N)_d = \bigoplus_{i+j=d} M_i \otimes_{R_0} N_j$. Then $M \otimes_R N$ can be identified with the quotient of $M \otimes_{R_0} N$ by the graded submodule generated by $am \otimes n - m \otimes an$ with homogeneous elements $m \in M$, $n \in N$, $a \in R$.

Homomorphisms of graded modules are required to preserve degrees. We let $\mathrm{GrHom}_R(M,N)_0$ denote the R_0-module of graded homomorphisms $M \to N$. (One can define a graded R-module $\mathrm{GrHom}_R(M,N) = \bigoplus_n \mathrm{GrHom}(M,N(n))$ but this will not be used in the sequel.)

For $f \in R_+$ homogeneous, we let $M_{(f)}$ denote the degree 0 piece of M_f. The proof of the following is similar to Propositions 1.10.12 and 1.10.13.

Proposition 1.10.37. *The functor*

$$\begin{aligned} \{\textit{standard open subsets of } \mathrm{Proj}(R)\} &\to \{\textit{abelian groups}\} \\ D_+(f) &\mapsto M_{(f)} \end{aligned}$$

extends to a quasi-coherent sheaf $\widetilde{M}$ on $\mathrm{Proj}(R) = (X, \mathcal{O}_X)$. We have $\widetilde{M}|_{D_+(f)} \simeq \widetilde{M_{(f)}}$ for all $f \in R_+$ homogeneous and $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$ for all $\mathfrak{p} \in \mathrm{Proj}(R)$. Here $M_{(\mathfrak{p})}$ is the degree 0 piece of $T^{-1}M$, $T = \bigcup_{d\geqslant 0} R_d \backslash \mathfrak{p}$.

We obtain a functor

$$\begin{aligned} \mathrm{GrMod}(R) &\to \mathrm{Shv}(X, \mathcal{O}_X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

It is easy to see that this functor is exact and commutes with colimits. The canonical morphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_R N)^\sim$, given locally by $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$, is not an isomorphism in general.

We have $M_{(f)} = \mathrm{colim}(\ M_0 \xrightarrow{f} M_d \xrightarrow{f} M_{2d} \xrightarrow{f} \cdots\ )$, where $d = \deg(f)$. Thus if $Z \subseteq \mathbb{Z}$ with $\sup Z = +\infty$, then $M_{(f)}$ depends only on M_{nd}, $n \in Z$. This motivates the following.

Notation 1.10.38. For $d \geq 1$, let $U_d = \bigcup_{f \in R_d} D_+(f) \subseteq \mathrm{Proj}(R)$.

We have $\mathrm{Proj}(R) = \bigcup_{d \geqslant 1} U_d$ and $U_d \subseteq U_{dn}$ for all $n \geq 1$.

- If $\mathrm{Proj}(R)$ is quasi-compact, then $\mathrm{Proj}(R) = U_d$ for some d.
- If R is generated by R_1 over R_0, then $X = U_1$.

Definition 1.10.39. We define the quasi-coherent sheaf $\mathcal{O}_X(n)$ to be $\widetilde{R(n)}$. We call $\mathcal{O}_X(1)$ the **twisting sheaf**.

Proposition 1.10.40. *Let $X = \mathrm{Proj}(R)$. Let M and N be graded R-modules and let $n \in \mathbb{Z}$.*

- *On U_d, $\mathcal{O}_X(nd)$ is an invertible sheaf and the map*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nd) \to \widetilde{M(nd)}$$

  is an isomorphism when restricted to U_d. In particular, $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(nd)|_{U_d} \xrightarrow{\sim} \mathcal{O}_X(m+nd)|_{U_d}$, $\mathcal{O}_X(nd)|_{U_d}^{\vee} \simeq \mathcal{O}_X(-nd)|_{U_d}$.

- *The restriction of $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \xrightarrow{\sim} (M \otimes_R N)^\sim$ to U_1 is an isomorphism.*

This boils down to the following lemmas.

Lemma 1.10.41. *For $f \in R_d$, $d > 0$, we have an isomorphism $\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(nd)|_{D_+(f)}$ given by*

$$R_{(f)} \xrightarrow{\sim} R(nd)_{(f)}$$
$$a \mapsto f^n a.$$

Lemma 1.10.42. *For $f \in R_1$, the canonical map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$ is an isomorphism.*

Proof. We have $R_f = R_{(f)}[f, f^{-1}] \simeq R_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[X, X^{-1}]$. Thus $(M \otimes_R N)_f \simeq M_f \otimes_{R_f} N_f \simeq (M_{(f)} \otimes_{R_{(f)}} N_{(f)})[f, f^{-1}]$. Thus $M_{(f)} \otimes_{R_{(f)}} N_{(f)}$ is the degree 0 piece of $(M \otimes_R N)_{(f)}$. $\square$

Example 1.10.43. Consider $X = \mathbb{P}^n(d, \dots, d) = \mathrm{Proj}(R)$, where $R = A[x_0, \dots, x_n]$ with $\deg(x_i) = d \geqslant 2$. For $d \nmid n$, $R(n)_{(f)} = 0$, $f \in R_d$, since the non-zero degrees of $R(n)$ are $\equiv -n \pmod d$. Thus $\mathcal{O}(n) = 0$ for $d \nmid n$ and $0 = \mathcal{O}(1) \otimes_{\mathcal{O}} \mathcal{O}(-1) \not\simeq \mathcal{O}$.

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Recall we have defined for each graded module M over a graded ring R, a quasi-coherent sheaf $\widetilde{M}$ over $X = \mathrm{Proj}(R)$ satisfying $\widetilde{M}(D_+(f)) = M_{(f)}$. We want to study the behavior of $\widetilde{M}$ under change of R.

Let $\phi\colon R \to S$ be a graded ring homomorphism. Let $X = \mathrm{Proj}(R)$, $Y = \mathrm{Proj}(S)$, and $r\colon U(\phi) \to X$ the morphism induced by ϕ.

$$\begin{tikzcd} \mathrm{Proj}(S) & \mathrm{Proj}(R) \\ U(\phi) \arrow[u, hook] \arrow[ur, "r"] \end{tikzcd}$$

- Let N be a graded S-module. Then $r_*(\widetilde{N}|_{U(\phi)}) = \widetilde{{}_R N}$. Indeed, for each $a \in R_+$ homogeneous, $r_*(\widetilde{N}|_{U(\phi)})(D_+(a)) = \widetilde{N}(D_+(\phi(a))) = N_{(\phi(a))}$.
- Let M be a graded R-module. Then $M \otimes_R S$ is a graded S-module and we have a natural morphism $r^*(\widetilde{M}) \to \widetilde{M \otimes_R S}|_{U(\phi)}$, locally defined by $M_{(a)} \otimes_{R_{(a)}} S_{(\phi(a))} \to (M \otimes_R S)_{(\phi(a))}$ on $D_+(\phi(a))$ for $a \in R_+$ homogeneous. This is not an isomorphism in general. However, for $d \geqslant 1$ and $n \in \mathbb{Z}$, $r^*(\mathcal{O}_X(nd))|_{r^{-1}(U_d)} \xrightarrow{\sim} \mathcal{O}_Y(nd)|_{r^{-1}(U_d)}$ and $r^*(\widetilde{M})|_{r^{-1}(U_1)} \xrightarrow{\sim} \widetilde{M \otimes_R S}|_{r^{-1}(U_1)}$.

Next consider $i\colon \mathrm{Proj}(R) \cong \mathrm{Proj}(R^{(d)})$. Let M be a graded R-module. Then $i^*(\widetilde{M^{(d)}}) \xrightarrow{\sim} \widetilde{M}$. Here $M^{(d)}$ is the graded $R^{(d)}$-module defined by $(M^{(d)})_n = M_{dn}$. In particular, we have $i^*\mathcal{O}(n) = \mathcal{O}(dn)$.

The functor Γ_*

Since $M \to \widetilde{M}$ commutes with colimits, it admits a right adjoint by the adjoint functor theorem. We can describe the adjoint explicitly.

Notation 1.10.44. Given $X = \mathrm{Proj}(R)$ and an $\mathcal{O}_X$-module $\mathcal{F}$ (not necessarily quasi-coherent), we let

- $\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$;
- $\Upsilon_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-n), \mathcal{F})$.

We denote the degree n pieces of $\Gamma_*(\mathcal{F})$ and $\Upsilon_*(\mathcal{F}))$ by $\Gamma_n(\mathcal{F})$ and $\Upsilon_n(\mathcal{F})$, respectively.

Each $a \in R_d$ induces a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(n) \to \mathcal{O}_X(n+d)$. This makes $\Gamma_*(\mathcal{F})$ and $\Upsilon_*(\mathcal{F})$ into graded R-modules. The natural pairing

$$(\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(-n) \to \mathcal{F}$$

induces a homomorphism

$$\nu\colon \Gamma_*(\mathcal{F}) \to \Upsilon_*(\mathcal{F}).$$

If $X = U_d$, then ν_{dn} is an isomorphism for all $n \in \mathbb{Z}$.

We have defined functors Γ_* and Υ_* from $\mathrm{Shv}(X, \mathcal{O}_X)$ to $\mathrm{GrMod}(R)$.

Proposition 1.10.45. $\widetilde{\ } \dashv \Upsilon_*$.

Proof. We define the unit and counit

$$\phi\colon M \longrightarrow \Upsilon_*(\widetilde{M})$$
$$\psi\colon \Upsilon_*(\mathcal{F})^\sim \longrightarrow \mathcal{F}$$

as follows.

For $m \in M_d$,

$$\phi(m)\colon \mathcal{O}_X(-d) \to \widetilde{M}$$

is defined by

$$R(-d)_{(a)} \xrightarrow{\times m} M_{(a)}$$

on $D_+(a)$, $a \in R_+$ homogeneous. Here $\times m$ denotes multiplication by m.

For $a \in R_d$, $d > 0$, we define

$$\Gamma(D_+(a), \psi)\colon \Upsilon_*(\mathcal{F})_{(a)} \to \Gamma(D_+(a), \mathcal{F})$$
$$g/a^n \mapsto g(a^{-n})$$

where $g \in \Upsilon_{dn}(\mathcal{F}) = \mathrm{Hom}(\mathcal{O}_X(-dn), \mathcal{F})$, $a^{-n} \in R(-nd)_{(a)} = \Gamma(D_+(a), \mathcal{O}_X(-nd))$.

One verifies that this gives the expected adjunction. □

Proposition 1.10.46. *Assume that $X = \mathrm{Proj}(R)$ is quasi-compact. For any quasi-coherent sheaf $\mathcal{F}$ on X, $\Gamma_*(\mathcal{F})^\sim \underset{\sim}{\xrightarrow{\tilde{\nu}}} \Upsilon_*(\mathcal{F})^\sim \underset{\sim}{\xrightarrow{\psi}} \mathcal{F}$.*

Thus, for $\mathrm{Proj}(R)$ quasi-compact, Υ_* induces a fully faithful functor from $\mathrm{QCoh}(X)$ to the category of graded R-modules.

Proof. Since X is quasi-compact, we have $X = U_d$ for some $d > 0$. Then ν_{dn} is an isomorphism for all $n \in \mathbb{Z}$. It follows that $\tilde{\nu}$ is an isomorphism. Thus it suffices to show that for all $a \in R_+$ homogeneous, $\Gamma(D_+(a), \psi\tilde{\nu})\colon \Gamma_*(\mathcal{F})_{(a)} \to \Gamma(D_+(a), \mathcal{F})$ is an isomorphism. Up to replacing a by a^d, we may assume that $d \mid \deg(a) = m$. Note that X is quasi-compact and quasi-separated. It suffices to apply the lemma below to the invertible sheaf $\mathcal{O}_X(m)$ and the section defined by a. □

Let X be a scheme, $\mathcal{L}$ an invertible sheaf on X, and $\mathcal{F}$ a quasi-coherent sheaf on X.

- For every $f \in \Gamma(X, \mathcal{L})$, define $X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x\mathcal{L}_x\}$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. This is an open subset of X.
- Define $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, $\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}\otimes\mathcal{L}^{\otimes n})$. Then $\Gamma_*(X, \mathcal{L})$ is a graded ring and $\Gamma_*(X, \mathcal{L}, \mathcal{F})$ is a graded $\Gamma_*(X, \mathcal{L})$-module.

Here, for $n < 0$, $\mathcal{L}^{\otimes n}$ denotes $(\mathcal{L}^\vee)^{\otimes n}$.

Lemma 1.10.47. *Assume that X is quasi-compact. Let $f \in \Gamma(X, \mathcal{L})$. Then the canonical map*

$$\alpha\colon \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(f)} \to \Gamma(X_f, \mathcal{F})$$

is injective. Moreover, if X is quasi-separated, then α is an isomorphism.

Here $\Gamma_*(X,\mathcal{L},\mathcal{F})_{(f)}$ is simply

$$\operatorname{colim}(\Gamma(X,\mathcal{F}) \xrightarrow{f} \Gamma(X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}) \xrightarrow{f} \Gamma(X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes 2}) \to \dots).$$

For $\mathcal{L}=\mathcal{O}_X$, we recover Lemma 1.7.12.

Proof. Cover X by a finite number of open affine subsets $U_1,\dots,U_n$ such that $\mathcal{L}|_{U_i}$ is trivial, i.e. $\mathcal{L}|_{U_i}\cong\mathcal{O}_X|_{U_i}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_*(X,\mathcal{L},\mathcal{F})_{(f)} \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{i=1}^n \Gamma_*(U_i,\mathcal{L},\mathcal{F})_{(f)} \arrow[r] \arrow[d, "\theta"] & \bigoplus_{i,j=1}^n \Gamma_*(U_i\cap U_j,\mathcal{L},\mathcal{F})_{(f)} \arrow[d, "\beta"] \\
0 \arrow[r] & \Gamma(X_f,\mathcal{F}) \arrow[r] & \bigoplus_{i=1}^n \Gamma((U_i)_f,\mathcal{F}) \arrow[r] & \bigoplus_{i,j=1}^n \Gamma((U_i\cap U_j)_f,\mathcal{F})
\end{tikzcd}$$

Since each U_i is affine and $\mathcal{L}$ is trivial on U_i, θ is an isomorphism. This implies that α is injective. In the case where X is quasi-separated, $U_i\cap U_j$ is quasi-compact and β is injective by the previous case. It follows that α is an isomorphism in this case. □

The bijectivity of ϕ is more complicated. We will limit our attention to the case $M=R$. In this case, we have a commutative diagram

$$\begin{tikzcd}
 & \Gamma_*(\mathcal{O}_X) \arrow[d, "\nu"] \\
R \arrow[ur, "\varphi"] \arrow[r, "\phi"] & \Upsilon_*(\mathcal{O}_X)
\end{tikzcd}$$

Note that $\Gamma_*(\mathcal{O}_X)$ is a $\mathbb{Z}$-graded ring (in the notation above, $\Gamma_*(X,\mathcal{O}_X(1))$ is the degree ≥ 0 part of $\Gamma_*(\mathcal{O}_X)$) and φ is a homomorphism of $\mathbb{Z}$-graded rings. By contrast, there is no natural ring structure on $\Upsilon_*(\mathcal{O}_X)$ in general.

Proposition 1.10.48. *We have:*

(1) ν is an isomorphism if $X=U_1$.

(2) φ is an isomorphism if

- $R=A[x_0,\dots,x_n]$, $n\geqslant 1$*; or*
- *R is a Noetherian normal ring and* $\operatorname{ht}(R_+)\geqslant 2$.

Part (1) of the proposition is clear since ν_n is an isomorphism for all $n\in\mathbb{Z}$ in the case $X=U_1$. To prove part (2), we will give an interpretation of $\Gamma_*(\mathcal{O}_X)$.

Lemma 1.10.49. *Let X be a quasi-compact scheme and $\{\mathcal{F}_i\}_{i\in I}$ a family of quasi-coherent sheaves. Then the natural map*

$$\epsilon\colon \bigoplus_{i\in I}\Gamma(X,\mathcal{F}_i)\to\Gamma(X,\bigoplus_{i\in I}\mathcal{F}_i)$$

is injective. Moreover, if X is quasi-separated, then ϵ is an isomorphism.

Proof. The map ϵ is an isomorphism for X affine. In general, proceed as in Lemma 1.10.47. $\square$

Remark 1.10.50. The first (resp. second) statement of the lemma holds in fact for any quasi-compact (resp. quasi-compact, quasi-separated, admitting a quasi-compact open basis) topological space X and any abelian sheaf $\mathcal{F}$ on X.

In the case $X = \mathrm{Proj}(R)$, consider $\epsilon\colon \Gamma_*(\mathcal{O}_X) \to \Gamma(X, \mathcal{A})$, where $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$. Note that $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-algebra. Consider $f\colon \underline{\mathrm{Spec}}(\mathcal{A}) \to \mathrm{Proj}(R)$. The restriction of f to $D_+(a) \subseteq \mathrm{Proj}(R)$ can be identified with $\mathrm{Spec}(R_a) \to \mathrm{Spec}(R_{(a)})$. Thus $\underline{\mathrm{Spec}}(\mathcal{A})$ can be identified with the open subscheme $U = \bigcup_{a\in R_{+,\mathrm{homog}}} D(a) = \mathrm{Spec}(R)\backslash V(R_+)$ of $\mathrm{Spec}(R)$. The following is easy to check.

Lemma 1.10.51. $\epsilon\varphi\colon R \to \Gamma(X, \mathcal{A})$ *can be identified with the restriction map* $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$*, where* $Y = \mathrm{Spec}(R)$.

Proof of Proposition 1.10.48(2). Note that in both cases $\mathrm{Proj}(R)$ is quasi-compact, so that ϵ is an isomorphism. Thus it suffices to show that the restriction map $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$ is an isomorphism.

Case $R = A[x_0, \dots, x_n]$, $n \geqslant 1$. We have $U = \bigcup_{i=0}^n D(x_i)$, $D(x_i) = \mathrm{Spec}(R_{x_i})$, $D(x_i) \cap D(x_j) = \mathrm{Spec}(R_{x_i x_j})$. The relevant rings can be compatibly regarded as subrings $R_{x_0\cdots x_n}$ and $\mathcal{O}_Y(U) = \bigcap_{i=0}^n R_{x_i} = R$.

Case R Noetherian normal ring and $\mathrm{ht}(R_+) \geqslant 2$. A Noetherian normal ring is finite product of Noetherian normal domains. The Proposition then follows immediately from the following Lemma. $\square$

Lemma 1.10.52. *Let R be a Noetherian normal domain. Then* $R = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} R_{\mathfrak{p}}$.

This is a consequence of Krull's principal ideal theorem. See [M2, Theorem 11.5].

Example 1.10.53.
- $R = A[x]$, $\Gamma_*(\mathcal{O}_X) = A[x, x^{-1}]$. In this case $R \hookrightarrow \Gamma_*(\mathcal{O}_X)$ is not an isomorphism unless $A = 0$.
- $R = k[u^4, u^3v, uv^3, v^4]$, $\mathrm{Proj}(R) \cong \mathbb{P}^1_k$. We have remarked that R is not integrally closed. The map $R \hookrightarrow \Gamma_*(\mathcal{O}_X)$ identifies $\Gamma_*(\mathcal{O}_X)$ with the integral closure of R (exercise).

The morphism $f\colon \mathrm{Spec}(R)\backslash V(R_+) \to \mathrm{Proj}(R)$ gives an interpretation of $\mathrm{Proj}(R)$ as a quotient. We now give some indications towards this direction.

The affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$ is equipped with a multiplication $m\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ and a unit morphism $e\colon \mathrm{Spec}(\mathbb{Z}) \to \mathbb{A}^1$, making $\mathbb{A}^1$ a monoid scheme. The morphisms m and e are given by the following ring homomorphisms, called comultiplication and counit:

$$\begin{array}{cc} \mathbb{Z}[x] \to \mathbb{Z}[y] \otimes \mathbb{Z}[z] & \mathbb{Z}[x] \to \mathbb{Z} \\ x \mapsto y \otimes z & x \mapsto 1 \end{array}$$

Equipped with these homomorphisms, $\mathbb{Z}[x]$ is a bialgebra. The open subscheme $\mathbb{G}_{\mathrm{m}} = \mathbb{A}^1\backslash V(x) = \mathrm{Spec}(\mathbb{Z}[x, 1/x])$ is a group scheme, called the **multiplicative**

group. It is equipped with the inverse morphism $i\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, which is defined by the antipode

$$\begin{aligned} \mathbb{Z}[x, x^{-1}] &\to \mathbb{Z}[x, x^{-1}] \\ x &\mapsto x^{-1} \end{aligned}$$

This makes $\mathbb{Z}[x, x^{-1}]$ into a Hopf algebra.

An action $\mathbb{A}^1 \curvearrowright X$ is a morphism $a\colon \mathbb{A}^1 \times X \to X$ compatible with m and e. If $X = \mathrm{Spec}(R)$ is affine, then a is defined by a ring homomorphism

$$\begin{aligned} R &\to \mathbb{Z}[x] \otimes R = R[x] \\ r &\mapsto \sum_{d \geq 0} r_d x^d \end{aligned}$$

One checks that an action of $\mathbb{A}^1$ on $\mathrm{Spec}(R)$ is equivalent to a grading on R. Similarly, an action of $\mathbb{G}_{\mathrm{m}}$ on $\mathrm{Spec}(R)$ is equivalent to a $\mathbb{Z}$-grading on R. One can interpret $V(R_+)$ as the fixed point locus by the action of $\mathbb{A}^1$ on $\mathrm{Spec}(R)$, and $\mathrm{Proj}(R)$ as the quotient of $\mathrm{Spec}(R) \backslash V(R_+)$ by the action of $\mathbb{G}_{\mathrm{m}}$.

Proposition 1.10.54. *Let R be a graded ring such that $X = \mathrm{Proj}(R)$ is quasi-compact and $\varphi\colon R \to \Gamma_*(\mathcal{O}_X)$ is an isomorphism. Then any closed subscheme of X is defined by a homogeneous ideal of R.*

Proof. Let $Z \subseteq X$ be a closed subscheme defined by a quasi-coherent ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$. Then $\Gamma_*(\mathcal{I}_Z) \hookrightarrow \Gamma_*(\mathcal{O}_X) \simeq R$ can be identified with a homogeneous ideal $\mathfrak{a}$ of R. Thus $\tilde{\mathfrak{a}} \simeq \Gamma_*(\mathcal{I}_Z)^\sim \xrightarrow{\sim} \mathcal{I}_Z$. Since the ideal sheaf of the closed subscheme $\mathrm{Proj}(R/\mathfrak{a}) \subseteq X$ is $\tilde{\mathfrak{a}}$, we have $Z = \mathrm{Proj}(R/\mathfrak{a})$ as subscheme of X. □

Corollary 1.10.55. *A morphism of schemes $f\colon X \to \mathrm{Spec}(A)$ is projective if and only if there exists a graded ring R finitely generated over $R_0 = A$ such that $X = \mathrm{Proj}(R)$ and f is the canonical morphism.*

Proof. $\Longleftarrow$. This is Corollary 1.10.29.

$\Longrightarrow$. Let $X \hookrightarrow \mathbb{P}^n_A \to \mathrm{Spec}(A)$ be a factorization. Since X is a closed subscheme of $\mathbb{P}^n_A$, it is defined by a homogeneous ideal $I \subseteq R = A[x_0, \ldots, x_n]$. In other words, $X = \mathrm{Proj}(R/I)$. □

Functor represented by $\mathrm{Proj}(R)$

We are mainly interested in the functor represented by the open subscheme U_1 of $\mathrm{Proj}(R)$. Let $\varphi\colon R \to \Gamma_*(U_1, \mathcal{O}(1)) = \bigoplus_{n \geq 0} \Gamma(U_1, \mathcal{O}(n))$ be the canonical homomorphism of graded rings.

Definition 1.10.56. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module and $\Sigma \subseteq \Gamma(X, \mathcal{F})$ a subset. We say that $\mathcal{F}$ is **generated** by Σ if

$$\bigoplus_{s \in \Sigma} \mathcal{O}_X \xrightarrow{(s)} \mathcal{F}$$

is an epimorphism. We say that $\mathcal{F}$ is **globally generated** if $\mathcal{F}$ is generated by $\Gamma(X, \mathcal{F})$.

Note that if X is a scheme (or a locally ringed space), an invertible sheaf $\mathcal{L}$ is generated by Σ if and only if $\bigcup_{s\in\Sigma} X_s = X$. In particular, $\mathcal{O}_{\mathrm{Proj}(R)}(dn)|_{U_d}$ is generated by $\varphi(R_{dn})$ for $d, n \geq 1$.

Example 1.10.57. On $\mathbb{P}^n_A$, $\Gamma(\mathbb{P}^n_A, \mathcal{O}(d)) = A[x_0, \dots, x_n]_d$ for $d \geqslant 0$ and $\Gamma(\mathbb{P}^n_A, \mathcal{O}(-d)) = 0$ for $d \geq 1$. In particular, $\mathcal{O}(-d)$ is not globally generated.

Proposition 1.10.58. *Let Y be a scheme and $X = \mathrm{Proj}(R)$. Then there is a bijection*

$$\mathrm{Hom}_{\mathrm{Sch}}(Y, U_1) \longrightarrow \left\{ (\mathcal{L}, \gamma) \mid \begin{array}{r} \mathcal{L} \textit{ invertible sheaf on } Y, \\ \gamma\colon R \to \Gamma_*(Y, \mathcal{L}) \textit{ homomorphism of graded rings} \\ \textit{such that } \mathcal{L} \textit{ is generated by } \gamma(R_1) \end{array} \right\} / \cong$$
$$(f\colon Y \to U_1) \mapsto [(f^*(\mathcal{O}(1)|_{U_1}), R \xrightarrow{\varphi} \Gamma_*(U_1, \mathcal{O}(1)) \to \Gamma_*(Y, f^*(\mathcal{O}(1))))],$$

where $(\mathcal{L}, \gamma) \cong (\mathcal{L}', \gamma')$ if there exists $c\colon \mathcal{L} \cong \mathcal{L}'$ rendering

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & \Gamma_*(X, \mathcal{L}) \arrow[d, "c"] \\ & \Gamma_*(X, \mathcal{L}') \end{tikzcd}$$

commutative.

Proof. We construct the inverse $[(\mathcal{L}, \gamma)] \mapsto f$ as follows. For $a \in R_d$, $d > 0$ satisfying $D_+(a) \subseteq U_1$, we have a ring homomorphism

$$R_{(a)} \xrightarrow{\gamma} \Gamma_*(Y, \mathcal{L})_{\gamma(a)} \to \Gamma(Y_{\gamma(a)}, \mathcal{O}_Y).$$

This gives a morphism $Y_{\gamma(a)} \to D_+(a)$. Since $\mathcal{L}$ is generated by $\gamma(R_1)$, we have $\bigcup_{a\in R_1} Y_{\gamma(a)} = Y$. Thus these morphisms glue to a morphism $f\colon Y \to U_1$. □

Corollary 1.10.59. *For $X = \mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, we have a bijection*

$$\mathrm{Hom}_{\mathrm{Sch}}(Y, \mathbb{P}^n_{\mathbb{Z}}) \longrightarrow \left\{ (\mathcal{L}, s_0, \dots, s_n) \mid \begin{array}{r} \mathcal{L} \textit{ invertible sheaf on } Y, s_i \in \Gamma(Y, \mathcal{L}), \\ \mathcal{L} \textit{ is generated by } s_0, \dots, s_n \end{array} \right\} / \cong$$
$$f \mapsto (f^*(\mathcal{O}(1)), f^*x_0, \dots, f^*x_n)$$

The functor represented by U_d can be described with the help of the isomorphism $\mathrm{Proj}(R) \simeq \mathrm{Proj}(R^{(d)})$. Indeed, this isomorphism restricts to $U_{d,R} \simeq U_{1,R^{(d)}}$.

Remark 1.10.60. Given a scheme Y and $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is a line bundle on Y and $\gamma\colon R \to \Gamma_*(X, \mathcal{L})$ is a homomorphism of graded rings (without assumptions on generation by global sections), the construction in the proof above produces a morphism of schemes $f\colon Y_\gamma \to \mathrm{Proj}(R)$, where $Y_\gamma = \bigcup_{a\in R_{+,\mathrm{homog}}} Y_{\gamma(a)}$.

1.11 Ample invertible sheaves

Given a graded ring R, the opens $D_+(f) \cap U_1$ form a basis for the topology on $U_1 \subseteq \mathrm{Proj}(R)$. Each $f \in R_d$, $d > 0$ gives rise to an element $\varphi(f) \in \Gamma(U_1, \mathcal{O}(d))$ and $D_+(f) \cap U_1 = (U_1)_{\varphi(f)}$. Thus the open subsets $(U_1)_s$, $s \in \bigcup_{d \geq 1} \Gamma(U_1, \mathcal{O}(d))$ form a basis for the topology on U_1. We generalize this property to arbitrary invertible sheaves on schemes as follows.

Definition 1.11.1. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. We way that $\mathcal{L}$ is **ample** if

- X is quasi-compact and
- $\{X_s \mid s \in \Gamma(X, \mathcal{L}^{\otimes d}), d \geqslant 1\}$ forms a basis for the topology on X.

Remark 1.11.2. Let $U = \mathrm{Spec}(A) \subseteq X$ be open affine such that $\mathcal{L}$ is trivial on U. Then for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, $X_s \cap U = \mathrm{Spec}(A_s)$ is affine. In particular, if $X_s \subseteq U$, then X_s is affine. (Exercise: Show that assumption that $\mathcal{L}$ is trivial can be removed.)

Lemma 1.11.3. *Let X be an affine scheme. Then any invertible sheaf $\mathcal{L}$ on X is ample.*

Proof. Let $X = \mathrm{Spec}(A)$. Then $\mathcal{L} \simeq \widetilde{M}$ for some A-module M. The opens X_{am}, $a \in A$, $m \in M$ form a basis for the topology on X. Indeed, $X_{am} \subseteq D(a)$ and $\bigcup_{m \in M} X_m = X$. □

Lemma 1.11.4. *Given $d \geqslant 1$, $\mathcal{L}$ is ample $\iff$ $\mathcal{L}^{\otimes d}$ is ample.*

Proof. We have $X_s = X_{s^{\otimes d}}$. □

Lemma 1.11.5. *Let $i \colon Y \to X$ be a quasi-compact immersion. For any ample invertible sheaf $\mathcal{L}$ on X, $i^*\mathcal{L}$ is ample on Y.*

Proof. We have $Y_{i^*s} = Y \cap X_s$. □

Theorem 1.11.6. *Let X be a quasi-compact scheme and let $\mathcal{L}$ be an invertible sheaf on X. Let $S = \Gamma_*(X, \mathcal{L})$. Then the following conditions are equivalent:*

(a) $\mathcal{L}$ *is ample.*

(b) $\{X_s \text{ affine} \mid s \in S_{+,\mathrm{homog}}\}$ *is a basis for X.*

(c) $\{X_s \text{ affine} \mid s \in S_{+,\mathrm{homog}}\}$ *covers X.*

(d) The morphism $X \hookrightarrow \mathrm{Proj}(S)$ defined by $(\mathcal{L}, id \colon S \to S)$ is an open immersion.

(e) There exists a graded ring R, an immersion $i \colon X \hookrightarrow U_1 \subseteq \mathrm{Proj}(R)$ and $d \geqslant 1$ such that $\mathcal{L}^{\otimes d} \simeq i^*\mathcal{O}(1)$.

(f) $\forall$ $\mathcal{F}$ quasi-coherent sheaf on X, $\bigcup_{n \geqslant 1} \mathrm{Im}(\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathcal{L}^{\otimes -n}) \to \mathcal{F}) = \mathcal{F}$.

(g) $\forall$ $\mathcal{F}$ quasi-coherent ideal sheaf on X, the condition in (f) holds.

Proof. (a) $\Longrightarrow$ (b). This follows from Remark 1.11.2.

(b) $\Longrightarrow$ (c). Trivial.

(c) $\Longrightarrow$ (d). We first prove that X is quasi-separated. By assumption $X = \bigcup_{i=1}^n X_{s_i}$ with X_{s_i} affine. There exists an affine open covering $X = \bigcup_{k=1}^m U_k$ such that $\mathcal{L}$ is trivial on each U_k. Then $X_{s_i} \cap X_{s_j} = X_{s_i \otimes s_j} = \bigcup_{k=1}^m (X_{s_i \otimes s_j} \cap U_k)$ is quasi-compact, since $X_{s_i \otimes s_j} \cap U_k$ is affine. (In fact $X_{s_i} \cap X_{s_j}$ is affine by the exercise mentioned in Remark 1.11.2.)

We can now apply Lemma 1.10.47 to see $S_{(s)} \xrightarrow{\sim} \Gamma(X_s, \mathcal{O}_X)$. For X_s affine, this implies $X_s \xrightarrow{\sim} D_+(s)$. Thus $X \hookrightarrow \mathrm{Proj}(S)$ is an open immersion.

(d) $\Longrightarrow$ (e). Since X is quasi-compact, the image of the open immersion $j\colon X \hookrightarrow \mathrm{Proj}(S)$ in (d) is contained in U_d for some d. We take $R = S^{(d)}$ and let $i\colon X \hookrightarrow U_{d,S} \simeq U_{1,R}$. Then $i^*(\mathcal{O}_{U_{1,R}}(1)) = j^*(\mathcal{O}_{U_{d,S}}(d)) = \mathcal{L}^{\otimes d}$.

(e) $\Longrightarrow$ (a). By the discussion at the beginning of the section, $\mathcal{O}(1)|_{U_1}$ is ample. By Lemma 1.11.4, $\mathcal{L}^{\otimes d} \simeq i^*(\mathcal{O}(1)|_{U_1})$ is ample, which implies that $\mathcal{L}$ ample by Lemma 1.11.3.

(a) $\Longrightarrow$ (f). We have shown that (a) implies that X is quasi-separated. Thus, by Lemma 1.10.47, $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \xrightarrow{\sim} \Gamma(X_s, \mathcal{F})$ for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, $d \geqslant 1$. Elements in $\Gamma(X_s, \mathcal{F})$ can be written as $a = b|_{X_s} \otimes s^{-n}$, $b \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes nd})$, $s^{-n} \in \Gamma(X_s, \mathcal{L}^{\otimes -nd})$. Thus a is in the image of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes nd}) \otimes_{\mathbb{Z}} \mathcal{L}^{\otimes -nd} \to \mathcal{F}$. Since X_s forms an open basis, $\mathcal{F}$ equals the union as shown in (f).

(f) $\Longrightarrow$ (g). Trivial.

(g) $\Longrightarrow$ (a). Let $x \in U \subseteq X$ be an open neighborhood of x. It suffices to show that there exists $s \in S_+$ homogeneous such that $x \in X_s \subseteq U$. Let $Z = X \backslash U$ and equip it with the induced reduced closed subscheme structure. Let $\mathcal{I}_Z$ be the corresponding ideal sheaf. Then $\mathcal{I}_Z|_U = \mathcal{O}_X|_U$. The assumption in (g) implies

$$\bigcup_{n \geqslant 1} \mathrm{Im}(\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes -n}) \to \mathcal{I}_Z) = \mathcal{I}_Z.$$

In particular, there exists $n \geqslant 1$, $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n})$ such that $s_x \notin \mathfrak{m}_x(\mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})_x = (\mathcal{L}^{\otimes n})_x$, where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. Let $i\colon Z \to X$ be the closed immersion. The exact sequence $0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$ induces an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(Z, i^*\mathcal{L}^{\otimes n}|_Z).$$

Regarding s as an element of $\Gamma(X, \mathcal{L}^{\otimes n})$, we have $x \in X_s$. The image of s in $\Gamma(Z, i^*\mathcal{L}^{\otimes n}|_Z)$ is zero, which implies that $X_s \cap Z = \emptyset$ and $X_s \subseteq U$. □

Corollary 1.11.7. *Any scheme admitting an ample invertible sheaf is separated.*

Indeed, $\mathrm{Proj}(R)$ is separated.

Date: 11.3

(Additional equivalent conditions have been inserted into Theorem 1.11.6.)

Definition 1.11.8. We say that a scheme X is **quasi-affine** if X is a quasi-compact open subset of an affine scheme.

Corollary 1.11.9. *A scheme X is quasi-affine $\iff$ $\mathcal{O}_X$ is ample.*

Proof. $\Longrightarrow$. If $j\colon X \hookrightarrow \operatorname{Spec}(A)$ is a quasi-compact open immersion, then $\mathcal{O}_X = j^*\mathcal{O}_{\operatorname{Spec}(A)}$ is ample.

$\Longleftarrow$. We apply the theorem above with $S = \bigoplus_{d\geqslant 0}\Gamma(X,\mathcal{O}_X) = A[x]$, where $A = \Gamma(X,\mathcal{O}_X)$. Then $j\colon X \to \operatorname{Proj}(S) = \operatorname{Spec}(A)$ is an open immersion. By assumption, X is quasi-compact. It follows that j is quasi-compact. □

Definition 1.11.10. Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is **of finite type** if there exists an open cover $\{U_i\}$ of X, integers $n_i \geqslant 0$ and epimorphisms $\mathcal{O}_{U_i}^{n_i} \twoheadrightarrow \mathcal{F}|_{U_i}$.

Remark 1.11.11. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.

- Every quotient of $\mathcal{F}$ is of finite type.
- If X is a locally Noetherian scheme and $\mathcal{F}$ is quasi-coherent, then every quasi-coherent subsheaf of $\mathcal{F}$ is also of finite type.

Corollary 1.11.12. *Let X be a scheme, $\mathcal{L}$ an ample invertible sheaf on X, and $\mathcal{F}$ a quasi-coherent sheaf of finite type on X. Then there exists an integer n_0 such that for all $n \geqslant n_0$, $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ is globally generated.*

Remark 1.11.13. The tensor product of two globally generated $\mathcal{O}_X$-modules is globally generated.

Proof. By Theorem 1.11.6 (a)⇒(f) and the lemma below, for any quasi-coherent sheaf $\mathcal{G}$ of finite type on X, there exists $e = e(\mathcal{G},\mathcal{L}) \geqslant 1$ such that $\mathcal{G}\otimes\mathcal{L}^{\otimes e}$ is globally generated. Let $d = e(\mathcal{O}_X,\mathcal{L})$, so that $\mathcal{L}^{\otimes d}$ is globally generated. For $0 \leq i < d$, let $e_i = e(\mathcal{F}\otimes\mathcal{L}^{\otimes i},\mathcal{L}^{\otimes d})$, so that $\mathcal{F}\otimes\mathcal{L}^{\otimes de_i+i}$ is globally generated. It follows that $\mathcal{F}\otimes\mathcal{L}^{\otimes de+i}$ is globally generated for $e \geq e_i$. Take $n_0 = \max_{0\leqslant i<d}\{de_i+i\}$. Then for all $n \geqslant n_0$, $\mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is globally generated. □

Lemma 1.11.14. *Let $(X,\mathcal{O}_X)$ be a ringed space with X quasi-compact. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

(1) Assume $\mathcal{F} = \operatorname{colim}_{i\in I}\mathcal{F}_i$ with I filtered. Then there exists i such that the canonical morphism $\mathcal{F}_i \to \mathcal{F}$ is an epimorphism.

(2) If $\mathcal{F}$ is globally generated, then $\mathcal{F}$ is generated by finitely many global sections.

Proof. (1) For any $x \in X$, there exist an open neighborhood U and an epimorphism $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$. Shrinking U if necessary, we can find i such that $\mathcal{O}_U^n \to \mathcal{F}_i|_U \to \mathcal{F}|_U$. Then $\mathcal{F}_i|_U \twoheadrightarrow \mathcal{F}|_U$. Since X is quasi-compact, we can find an $i \in I$ such that $\mathcal{F}_i|_U \twoheadrightarrow \mathcal{F}|_U$ holds for U running through an open cover of X. This shows $\mathcal{F}_i \twoheadrightarrow \mathcal{F}$.

(2) We have

$$\mathcal{F} = \bigcup_{\Sigma\subseteq\Gamma(X,\mathcal{F})\ \text{finite}} \operatorname{Im}(\mathcal{O}_X^\Sigma \to \mathcal{F})$$

By (1), there exists $\Sigma \subseteq \Gamma(X,\mathcal{F})$ finite such that $\mathcal{O}_X^\Sigma$ surjects onto $\mathcal{F}$. □

Corollary 1.11.15. *Let X be a scheme, $\mathcal{L}$ an ample invertible sheaf on X. Then for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X, there exists $n \geqslant 1$, $m \geqslant 0$ such that $\mathcal{F}$ is a quotient of $(\mathcal{L}^{\otimes -n})^m$.*

Proof. There exists n such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. By the lemma above, there exist m and an epimorphism $\mathcal{O}_X^m \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$. Thus $(\mathcal{L}^{\otimes -n})^m \twoheadrightarrow \mathcal{F}$. $\square$

Remark 1.11.16. If X is Noetherian, then the condition in the corollary is equivalent to the ampleness of $\mathcal{L}$. Indeed, in this case, every ideal sheaf is finitely generated. In fact, the equivalence holds as long as X is quasi-compact and quasi-separated, because in this case every quasi-coherent $\mathcal{O}_X$-module is the union of its submodules of finite type [SP, 01PG].

Relative ampleness

Definition 1.11.17. Let $f\colon X \to S$ be a morphism of schemes and let $\mathcal{L}$ be an invertible sheaf on X.

- We say that $\mathcal{L}$ is f-**ample** if f is quasi-compact and for every affine open $V \subseteq S$, $\mathcal{L}|_{f^{-1}(V)}$ is ample.
- We say that $\mathcal{L}$ is f-**very ample** if there exists a decomposition

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

  where i is an immersion such that $\mathcal{L} \simeq i^*\mathcal{O}_{\mathbb{P}^n_S}(1)$. Here $\mathcal{O}_{\mathbb{P}^n_S}(1) := p^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$, where $p\colon \mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times S \to \mathbb{P}^n_{\mathbb{Z}}$ is the projection.

Lemma 1.11.18. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes and let $\mathcal{L}$ be an invertible sheaf on X.*

(1) If $\mathcal{L}$ is ample, then $\mathcal{L}$ is f-ample.

(2) If $\mathcal{L}$ is f-very ample, then $\mathcal{L}$ is f-ample.

Theorem 1.11.19. *Let $f\colon X \to S$ be a morphism locally of finite type and let $\mathcal{L}$ be an ample invertible sheaf on X. Then there exists $d \geqslant 1$ such that $\mathcal{L}^{\otimes d}$ is f-very ample.*

Proof. Let $R = \bigoplus_{d \geqslant 0} \Gamma(X, \mathcal{L}^{\otimes d})$. Since $\{X_s \text{ affine} \mid s \in R_d, d \geqslant 1\}$ forms a basis for the topology of X and X is quasi-compact, there exists a finite cover $X = \bigcup_{i=1}^n X_{s_i}$ with $X_{s_i} = \operatorname{Spec}(B_i)$ such that $f(X_{s_i}) \subseteq V_i = \operatorname{Spec}(A_i)$, where V_i is an affine open of S. Since f is locally of finite type, B_i is a finitely generated A_i-algebra, say $B_i = A_i[b_{i,1}, \dots, b_{i,n_i}]$. By Lemma 1.10.47, $R_{(s_i)} \simeq \Gamma(X_{s_i}, \mathcal{O}_X)$. Thus $b_{ij} = f_{ij}/s_i^{e_{ij}}$, with f_{ij} homogeneous of degree $e_{ij} \deg(s_i)$.

Take d such that $\deg(s_i) \mid d$ and $d \geqslant \deg(f_{ij})$ for all i, j. Let

$$\Sigma = \{s_i^{d/\deg(s_i)}, f_{ij}s_i^{d/\deg(s_i) - e_{ij}}\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n_i}} \subseteq R_d.$$

Then Σ generates $\mathcal{L}^{\otimes d}$ since $\bigcup_{s\in\Sigma} X_s \supseteq \bigcup_i X_{s_i} = X$. Let $T = \mathbb{Z}[x_i, x_{ij}]_{i,j}$ and consider the ring homomorphism

$$\begin{aligned} T &\to R \\ x_i &\mapsto s_i^{d/\deg(s_i)} \\ x_{ij} &\mapsto f_{ij} s_i^{d/\deg(s_i)-e_{ij}}. \end{aligned}$$

This gives a morphism of schemes $X \to \operatorname{Proj}(T) = \mathbb{P}^N_{\mathbb{Z}}$, $N = \#\Sigma - 1$. This induces a commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "f"'] & \mathbb{P}^N_S \arrow[r] \arrow[d] & \mathbb{P}^N_{\mathbb{Z}} \arrow[d] \\ & S \arrow[r] & \operatorname{Spec}(\mathbb{Z}). \end{tikzcd}$$

We have $r^{-1}(D_+(x_i) \times S) = X_{s_i}$ and the restriction of r is the composition

$$X_{s_i} \xrightarrow{v} D_+(x_i) \times V_i \xrightarrow{u} D_+(x_i) \times S,$$

where u is an open immersion and v is a morphism of affine schemes. The ring homomorphism corresponding to v

$$\begin{aligned} T_{(x_i)} \otimes_{\mathbb{Z}} A_i &\to B_i \\ x_{ij}/x_i &\mapsto b_{ij} \end{aligned}$$

is surjective, which implies that v is a closed immersion. Therefore, r is an immersion. By construction, $r^*\mathcal{O}(1) \simeq \mathcal{L}^{\otimes d}$. □

Remark 1.11.20. The conclusion of the theorem can be strengthened to the existence of an integer d_0 such that for all $d \geqslant d_0$, $\mathcal{L}^{\otimes d}$ is f-very ample (exercise).

Corollary 1.11.21. *Let S be an affine scheme, $f\colon X \to S$ a morphism of finite type, $\mathcal{L}$ an invertible sheaf on X. Then the following conditions are equivalent:*

(a) $\mathcal{L}$ is ample.

(b) $\mathcal{L}$ is f-ample.

(c) there exists $d \geqslant 1$ such that $\mathcal{L}^{\otimes d}$ is f-very ample.

Definition 1.11.22. We say that a morphism of schemes $f\colon X \to S$ is **quasi-projective** if there exists a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

where i is a quasi-compact immersion.

Warning 1.11.23. Our definitions of f-very ampleness and quasi-projectiveness differ from the EGA. We will see later that being f-very ample in our sense is not local on S.

Example 1.11.24. $X = \mathbb{P}^n_A$, $A \neq 0$, $n > 1$. For $d > 0$, $\mathcal{O}(d)$ is very ample over A. Indeed, if $i_d\colon \mathbb{P}^n_A \hookrightarrow \mathbb{P}^N_Z$ denotes the d-uple embedding, then $i_d^*\mathcal{O}(1) \simeq \mathcal{O}(d)$. For $d < 0$, $\mathcal{O}(d)$ has no nonzero global sections. It follows that for $d \leq 0$, $\mathcal{O}_X(d)$ is not ample, because $\mathcal{O}_X(d)^{\otimes n} \otimes \mathcal{O}(-1) = \mathcal{O}(dn-1)$ is not globally generated for any $n \geq 0$. In summary,

$$\mathcal{O}_X(d) \text{ is } \begin{cases} \text{very ample over } A & d > 0 \\ \text{not ample} & d \leqslant 0. \end{cases}$$

Example 1.11.25. Let

$$\begin{tikzcd} X = \mathbb{P}^m_A \times_{\mathrm{Spec}(A)} \mathbb{P}^n_A \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{P}^m_A \\ \mathbb{P}^n_A \end{tikzcd}$$

with $A \neq 0$, $m, n \geqslant 1$. Let $\mathcal{L}_{a,b} = \mathcal{O}(a) \boxtimes_A \mathcal{O}(b) = p_1^*\mathcal{O}(a) \otimes_{\mathcal{O}_X} p_2^*\mathcal{O}(b)$. We have

$$\mathcal{L}_{a,b} \text{ is } \begin{cases} \text{very ample over } A & a, b > 0 \\ \text{not ample} & a \leqslant 0 \text{ or } b \leqslant 0. \end{cases}$$

For $a, b > 0$, let $i_a\colon \mathbb{P}^m_A \to \mathbb{P}^M_A$ and $i_b\colon \mathbb{P}^n_A \to \mathbb{P}^N_A$ be the a-uple and b-uple embeddings, respectively. Let $i\colon \mathbb{P}^M \times \mathbb{P}^N \to \mathbb{P}^r$ be the Segre embedding. Then

$$\begin{tikzcd} \mathbb{P}^m \times \mathbb{P}^n \arrow[r, "i_a \times i_b"] \arrow[rr, bend left, "j"] & \mathbb{P}^M \times \mathbb{P}^N \arrow[r, "i"] & \mathbb{P}^r \end{tikzcd}$$

and $j^*\mathcal{O}(1) \simeq (i_a \times i_b)^*(\mathcal{O}(1) \boxtimes_A \mathcal{O}(1)) \simeq \mathcal{O}(a) \boxtimes_A \mathcal{O}(b)$. In fact, on $\mathrm{Proj}(R \otimes_A S) \simeq \mathrm{Proj}(R) \times_A \mathrm{Proj}(S)$, we have $\widetilde{M \otimes_A N} \simeq \widetilde{M} \boxtimes_A \widetilde{N}$, where $(M \otimes N)_d = M_d \otimes_A N_d$.

For $a \leqslant 0$, we choose a section s of $\mathbb{P}^n_A \to \mathrm{Spec}(A)$ satisfying $s^*\mathcal{O}(1) = \mathcal{O}$ and consider the pullback

$$\begin{tikzcd} \mathbb{P}^m \arrow[r, hook, "t"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{P}^m \times \mathbb{P}^n \arrow[d, "p_2"] \\ \mathrm{Spec}(A) \arrow[r, hook, "s"] & \mathbb{P}^n \end{tikzcd}$$

Then $t^*(\mathcal{O}(a) \boxtimes \mathcal{O}(b)) \simeq \mathcal{O}(a)$, which is not ample on $\mathbb{P}^m$. Thus $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$ is not ample. The case $b \leq 0$ is similar.

Example 1.11.26. Let k be an algebraically closed field, C an integral normal k-scheme of dimension 1 and proper over k. Assume $C \not\simeq \mathbb{P}^1_k$. We will show later that the properness of C implies $\dim_k(\Gamma(C, \mathcal{O}_C)) < \infty$. Since $\Gamma(C, \mathcal{O}_C)$ is an integral finite-dimensional k-algebra, it is k itself. Let $P \in C$ be a closed point, corresponding to the ideal sheaf $\mathcal{I}_P$. Then $\mathcal{I}_P$ is an invertible sheaf. Let $\mathcal{L}(P) := \mathcal{I}_P^\vee$. We will show later that $\mathcal{L}(P)$ is ample. Let us show that $\mathcal{L}(P)$ is not very ample over k.

Let $i\colon P \to C$ be the inclusion. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_C \longrightarrow i_*k \longrightarrow 0$$

Tensoring the above sequence with $\mathcal{L}(P)$, we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{L}(P) \longrightarrow i_*k \longrightarrow 0$$

Taking global sections, we see that $\dim_k(\Gamma(X, \mathcal{L}(P))) \leq 2$. Suppose there exists an immersion $j\colon C \to \mathbb{P}(V) := \operatorname{Proj}(\operatorname{Sym}_k(V))$, where V is a finite-dimensional k-vector space, such that $i^*\mathcal{O}(1) \simeq \mathcal{L}(P)$. Then i corresponds to a k-linear map $\phi\colon V \to \Gamma(X, \mathcal{L}(P))$ whose image generates $\mathcal{L}(P)$. Let $W = \operatorname{im}(\phi)$. Then $\dim_k(W) \leq 2$ and i factorizes through $i\colon C \to \mathbb{P}(W)$. Then i is a closed immersion. It follows that $C \simeq \mathbb{P}(W) \simeq \mathbb{P}^1$. Contradiction.

1.12 Relative homogeneous spectrum

Let S be a scheme and let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. A **graded $\mathcal{O}_S$-algebra** $\mathcal{R}$ is and $\mathcal{O}_S$-algebra $\mathcal{R}$ equipped with a grading $\mathcal{R} = \bigoplus_{d\geqslant 0} \mathcal{R}_d$ such that $\mathcal{R}_d\mathcal{R}_e \subseteq \mathcal{R}_{d+e}$.

We define a scheme $\underline{\operatorname{Proj}}(\mathcal{R})$ and a morphism $\pi\colon \underline{\operatorname{Proj}}(\mathcal{R}) \to S$ by gluing. If $V' \subseteq V \subseteq S$ are affine open subsets, we have Cartesian squares

$$\begin{tikzcd}
\operatorname{Proj}(\mathcal{R}(V')) \arrow[r] \arrow[d] & \operatorname{Proj}(\mathcal{R}(V)) \arrow[r] \arrow[d] & \underline{\operatorname{Proj}}(\mathcal{R}) \arrow[d, "\pi"] \\
V' \arrow[r, hook] & V \arrow[r, hook] & S
\end{tikzcd}$$

Remark 1.12.1. π is separated.

Example 1.12.2. $S = \operatorname{Spec}(A)$, $\mathcal{R} = \widetilde{R}$, where R is a graded A-algebra. Then $\underline{\operatorname{Proj}}(\widetilde{R}) = \operatorname{Proj}(R)$.

Example 1.12.3. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Then $\underline{\operatorname{Proj}}(\mathcal{A}[x]) = \underline{\operatorname{Spec}}(\mathcal{A})$.

Example 1.12.4. $\underline{\operatorname{Proj}}(\mathcal{O}_S[x_0, \ldots, x_n]) \cong \mathbb{P}^n_S$.

Example 1.12.5. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. Then $\mathbb{P}(\mathcal{E}) := \underline{\operatorname{Proj}}(\operatorname{Sym}(\mathcal{E}))$ is called the **projective bundle** over S associated to $\mathcal{E}$. (In the literature, $\mathcal{E}$ is sometimes assumed to be locally free.)

Date: 11.5

The following treatment of $\mathcal{O}$-modules is based on a method of Berthelot [SGA6, VI 2].

Quasi-coherent sheaves on $\underline{\mathrm{Spec}}(\mathcal{A})$

Let S be a scheme, $\mathcal{A}$ a quasi-coherent $\mathcal{O}_S$-algebra, and $\pi\colon X = \underline{\mathrm{Spec}}(\mathcal{A}) \to S$. We have $\pi_*(\mathcal{O}_X) = \mathcal{A}$, which gives by adjunction a morphism of sheaves of rings $(\pi_\natural)^\sharp\colon \pi^{-1}\mathcal{A} \to \mathcal{O}_X$ on X. The morphism $(\pi_\natural)^\sharp$ is flat. To see this, let $V = \mathrm{Spec}(B)$ be an affine open of S and let $\mathcal{A}|_V = \widetilde{A}$, where A is a B-algebra. Then, the stalk of $(\pi_\natural)^\sharp$ at $\mathfrak{p} \in \mathrm{Spec}(A)$ is the localization $(\pi^{-1}\mathcal{A})_\mathfrak{p} \simeq A_\mathfrak{q} \to A_\mathfrak{p} \simeq \mathcal{O}_{X,\mathfrak{p}}$. Here $\mathfrak{q} = \mathfrak{p} \cap B$.

The morphism π, when regarded as a morphism of ringed spaces, can be decomposed into $(X, \mathcal{O}_X) \xrightarrow{\pi_\natural} (S, \mathcal{A}) \to (S, \mathcal{O}_S)$. The morphism of ringed spaces $\pi_\natural$ induces a pair of functors $\mathrm{Shv}(X, \mathcal{O}_X) \underset{\pi_\natural^*}{\overset{\pi_*}{\rightleftarrows}} \mathrm{Shv}(S, \mathcal{A})$, where $\pi_\natural^*\mathcal{M} = \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{A}} \mathcal{O}_X$.

Proposition 1.12.6. *(1) $\pi_\natural^* \dashv \pi_*$ and $\pi_\natural^*$ is exact.*

(2) The functors induce equivalences of categories

$$\mathrm{QCoh}(X, \mathcal{O}_X) \underset{\pi_\natural^*}{\overset{\pi_*}{\rightleftarrows}} \mathrm{QCoh}(S, \mathcal{A})$$

quasi-inverse to each other. Moreover, for $\mathcal{M} \in \mathrm{QCoh}(S, \mathcal{A})$ and $V \subseteq S$ an affine open, $\pi_\natural^*(\mathcal{M})|_{\pi^{-1}V} = \widetilde{\mathcal{M}(V)}$.

Proof. (1) This holds for any flat morphism of ringed spaces.

(2) That π_* carries quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{A}$-modules follows from the lemma below. The proof of the other statements is similar, by choosing a presentation locally. □

Lemma 1.12.7. *An $\mathcal{A}$-module $\mathcal{M}$ is quasi-coherent as $\mathcal{A}$-module $\iff$ $\mathcal{M}$ is quasi-coherent as $\mathcal{O}_S$-module.*

Proof. $\Longrightarrow$. Locally, $\mathcal{M} \simeq \mathrm{Coker}(\mathcal{A}^{\oplus I} \to \mathcal{A}^{\oplus J})$. Since $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-module, so is $\mathcal{M}$.

$\Longleftarrow$. We may assume $S = \mathrm{Spec}(B)$. Then $\mathcal{M} = \widetilde{M}$, where M is a B-module, and $\mathcal{A} = \widetilde{A}$, where A is a B-algebra. The $\mathcal{A}$-module structure on $\mathcal{M}$ induces an A-module structure on M. Choose a presentation

$$A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0.$$

This induces an exact sequence

$$\mathcal{A}^{\oplus I} \longrightarrow \mathcal{A}^{\oplus J} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Thus $\mathcal{M}$ is quasi-coherent as $\mathcal{A}$-module. □

Quasi-coherent sheaves on $\underline{\mathrm{Proj}}(\mathcal{R})$

Let S be a scheme and let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\pi\colon X = \underline{\mathrm{Proj}}(\mathcal{R}) \to S$ be the canonical morphism.

Definition 1.12.8. A **graded $\mathcal{R}$-module** is an $\mathcal{R}$-module $\mathcal{M}$ equipped with a $\mathbb{Z}$-grading $\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n$ such that $\mathcal{R}_d\mathcal{M}_e \subseteq M_{d+e}$.

Let $\mathrm{QCohGr}(S,\mathcal{R})$ denote the category of quasi-coherent graded $\mathcal{R}$-modules. Consider the functor

$$\begin{aligned} \mathrm{QCohGr}(S,\mathcal{R}) &\to \mathrm{QCoh}(X) \\ \mathcal{M} &\mapsto \widetilde{\mathcal{M}} \end{aligned}$$

where $\widetilde{\mathcal{M}}$ is constructed by gluing: for every open affine subset $V \subseteq S$, $\widetilde{M}|_{\pi^{-1}(V)} \simeq \widetilde{\mathcal{M}(V)}$. Note that $\mathcal{M}(V) = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n(V)$.

Definition 1.12.9. $\mathcal{O}_X(n) = \widetilde{\mathcal{R}(n)}$.

We now proceed to extend the functor $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ to graded modules that are not necessarily quasi-coherent. We have a morphism of $\mathbb{Z}$-graded $\mathcal{O}_S$-algebras

$$\mathcal{R} \to \bigoplus_{n\in\mathbb{Z}} \pi_*\mathcal{O}_X(n)$$

given locally on an affine open $V \subseteq S$ by $\varphi_n\colon \mathcal{R}_n(V) \to \Gamma(\pi^{-1}(V), \mathcal{O}(n))$. Recall that $\pi^{-1}(V) \simeq \mathrm{Proj}(\mathcal{R}(V))$. By adjunction, we obtain a morphism of $\mathbb{Z}$-graded sheaf of rings

$$\pi^{-1}\mathcal{R} \to \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n),$$

which is flat as in the case of $\underline{\mathrm{Spec}}(\mathcal{A})$.

We consider the following categories and functors:

$$\begin{tikzcd} \mathrm{Shv}(X,\mathcal{O}_X) \arrow[r, bend left, "(\bullet)_l"] \arrow[r, bend right, "(\bullet)_r"'] & \mathrm{GrShv}(X, \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)) \arrow[l, "(\ )_0"'] \arrow[r, bend right, "\pi_*^{\oplus}"'] & \mathrm{GrShv}(S,\mathcal{R}) \arrow[l, bend right, "\pi_\natural^*"'] \end{tikzcd}$$

The functor $(\ )_0$ are obvious. The functor $\pi_*^{\oplus}$ is defined by $\pi_*^{\oplus}(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n) := \bigoplus_{n\in\mathbb{Z}} \pi_*\mathcal{F}_n$. For $\mathcal{M} \in \mathrm{GrShv}(S,\mathcal{R})$,

$$\pi_\natural^*\mathcal{M} := \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{R}} (\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)).$$

For $\mathcal{F}$ an $\mathcal{O}_X$-module,

$$\begin{aligned} \mathcal{F}(\bullet)_l &:= \bigoplus_{n\in\mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n), \\ \mathcal{F}(\bullet)_r &:= \bigoplus_{n\in\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-n), \mathcal{F}). \end{aligned}$$

We have adjunctions

$$(\bullet)_l \dashv (\;)_0 \dashv (\bullet)_r \qquad \pi_\natural^* \dashv \pi_*$$

In particular, we have a canonical natural transformation $\nu\colon \mathcal{F}(\bullet)_l \to \mathcal{F}(\bullet)_r$. For $d \geq 1$, let $U_d := \bigcup_{V,s} D_+(V,s)$ where V runs through affine opens of S and s runs through elements of $\mathcal{R}_d(V)$. Then ν_{dn} is an isomorphism on U_{dn} for all $n \in \mathbb{Z}$.

By composition, we obtain functors

$$\begin{tikzcd} \mathrm{Shv}(X,\mathcal{O}_X) \arrow[r, bend left, "\underline{\Gamma}_*"] \arrow[r, bend right, "\underline{\Upsilon}_*"'] & \mathrm{GrShv}(S,\mathcal{R}) \arrow[l, "\sim"'] \end{tikzcd}$$

For $\mathcal{M} \in \mathrm{GrShv}(S,\mathcal{R})$,

$$\widetilde{\mathcal{M}} := (\pi_\natural^* \mathcal{M})_0.$$

This extends the definition of $^\sim$ on $\mathrm{QCohGr}(S,\mathcal{R})$. For $\mathcal{F} \in \mathrm{Shv}(X,\mathcal{O}_X)$,

$$\begin{aligned} \underline{\Gamma}_*(\mathcal{F}) &:= \bigoplus_{n\in\mathbb{Z}} \pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)), \\ \underline{\Upsilon}_*(\mathcal{F}) &:= \bigoplus_{n\in\mathbb{Z}} \pi_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-n),\mathcal{F}). \end{aligned}$$

We denote by $\upsilon\colon \underline{\Gamma}_* \to \underline{\Upsilon}_*$ the natural transformation induced by $\nu\colon (\bullet)_l \to (\bullet)_r$.

Proposition 1.12.10. *(1) $^\sim \dashv \underline{\Upsilon}_*$ and $^\sim$ is exact.*

(2) Suppose that π is quasi-compact. For $\mathcal{F} \in \mathrm{QCoh}(X,\mathcal{O}_X)$, we have $\underline{\Gamma}_*(\mathcal{F}) \in \mathrm{QCoh}(S,\mathcal{R})$ and

$$\underline{\Gamma}_*(\mathcal{F})^\sim \xrightarrow[\sim]{\widetilde{\upsilon}} \underline{\Upsilon}_*(\mathcal{F})^\sim \xrightarrow{\sim} \mathcal{F}$$

Proof. (1) is clear. For (2), the quasi-coherence is clear. The last statement follows from the corresponding result for Proj (Proposition 1.10.12). □

Corollary 1.12.11. *The functor $\underline{\Upsilon}_*\colon \mathrm{QCoh}(X) \to \mathrm{GrShv}(S,\mathcal{R})$ is fully faithful.*

Proposition 1.12.12. *Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of rank ≥ 2 and let $\pi\colon \mathbb{P}(\mathcal{E}) \to S$. Then the morphism $\mathcal{R} \to \underline{\Gamma}_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})})$ is an isomorphism, where $\mathcal{R} = \mathrm{Sym}(\mathcal{E})$. In other words, the morphism $\mathrm{Sym}(\mathcal{E}) \to \bigoplus_{n\in\mathbb{Z}} \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n))$ is an isomorphism.*

In particular, $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) = 0$ for $n < 0$, $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{O}_S$, and $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{E}$.

Proof. We reduce to the case where S is affine and $\mathcal{E}$ is a free module. In this case, $\mathbb{P}(\mathcal{E})$ is a projective space and Proposition 1.10.48(2) applies. □

Functor represented by $\mathrm{Proj}(\mathcal{R})$

Let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra and let $U_1 \subseteq \underline{\mathrm{Proj}}(\mathcal{R})$ be the open subscheme as above.

Proposition 1.12.13. *Let $f\colon Y \to S$ be a morphism of schemes. Then we have a bijection*

$$\mathrm{Hom}_S(Y, U_1) \xleftrightarrow{1:1} \left\{ \begin{aligned} &(\mathcal{L}, \gamma) \mid \mathcal{L} \text{ invertible sheaf on } Y \\ &\qquad \gamma\colon f^*\mathcal{R} \to \bigoplus_{n\geqslant 0} \mathcal{L}^{\otimes n} \text{ homomorphism of graded } \mathcal{O}_Y\text{-algebras} \\ &\qquad \gamma_1\colon f^*\mathcal{R}_1 \twoheadrightarrow \mathcal{L} \end{aligned} \right\} / \simeq$$

$$g \mapsto (g^*\mathcal{O}(1), f^*\mathcal{R} \xrightarrow{g^*(\varphi|_{U_1})} \bigoplus_{n\in\mathbb{Z}} g^*\mathcal{O}(n))$$

where $(\mathcal{L}, \gamma) \simeq (\mathcal{L}', \gamma')$ if there exists $c\colon \mathcal{L} \simeq \mathcal{L}'$ rendering

$$\begin{tikzcd} f^*\mathcal{R} \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n} \arrow[d, "c"] \\ & \bigoplus_{n\geq 0} \mathcal{L}'^{\otimes n} \end{tikzcd}$$

commutative. Here $\varphi\colon \pi^*\mathcal{R} \to \bigoplus_{n\geq 0} \mathcal{O}_X(n)$ denotes the canonical morphism.

Corollary 1.12.14. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. Then we have a bijection*

$$\mathrm{Hom}_S(Y, \mathbb{P}(\mathcal{E})) \xleftrightarrow{1:1} \left\{ \begin{aligned} &(\mathcal{L}, \gamma_1) \mid \mathcal{L} \text{ invertible sheaf on } Y \\ &\qquad \gamma_1\colon f^*\mathcal{E} \twoheadrightarrow \mathcal{L} \text{ homomorphism of } \mathcal{O}_Y\text{-modules} \end{aligned} \right\} / \simeq$$

Example 1.12.15. Let k be a field and let V be a k-vector space. Then we have bijections

$$\begin{aligned} \mathbb{P}(V)(k) &\xleftrightarrow{1:1} \{\text{quotients of } V \text{ of dimension } 1\} \\ &\xleftrightarrow{1:1} \{\text{hyperplanes of } V\} \end{aligned}$$

The functor represented by the projective bundle $\mathbb{P}(\mathcal{E})$ should be compared to the functor represented by the vector bundle $\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{E}))$. For any morphism $f\colon Y \to S$, we have

$$\mathrm{Hom}_S(Y, \mathbb{V}(\mathcal{E})) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Sym}(\mathcal{E}), f_*\mathcal{O}_Y) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, \mathcal{O}_Y).$$

In particular, for $Y = S$, we have

$$\mathrm{Hom}_S(S, \mathbb{V}(\mathcal{E})) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y).$$

If $\mathcal{E}$ is locally free, then $\mathrm{Hom}_S(S, \mathbb{V}(\mathcal{E})) \simeq \Gamma(S, \mathcal{E}^\vee)$. In words, sections of the morphism $\pi\colon \mathbb{V}(\mathcal{E}) \to S$ correspond to sections of the sheaf $\mathcal{E}^\vee$.

The S-scheme $\mathbb{P}(\mathcal{E})$ classifies quotients of $\mathcal{E}$ locally free of rank 1. More generally, we have the following.

Theorem 1.12.16. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module and let $r \geq 0$ be an integer. Then there exists an S-scheme $\mathbb{G}\mathrm{rass}_r(\mathcal{E})$, called the **Grassmannian**, equipped with a functorial bijection*

$$\mathrm{Hom}(Y, \mathbb{G}\mathrm{rass}_r(\mathcal{E})) \simeq \{\text{quotients of } f^*\mathcal{E} \text{ that are locally free of rank } r\}.$$

Moreover, the morphism

$$\begin{aligned} \mathbb{G}\mathrm{rass}_r(\mathcal{E}) &\to \mathbb{P}(\textstyle\bigwedge^r \mathcal{E}) \\ (f^*\mathcal{E} \twoheadrightarrow \mathcal{F}) &\mapsto (f^*(\textstyle\bigwedge^r \mathcal{E}) \twoheadrightarrow \textstyle\bigwedge^r \mathcal{F}) \end{aligned}$$

*is a closed immersion, called the **Plücker embedding**.*

The proof is a good exercise. See [GD, 9.7, 9.8].

Remark 1.12.17. $\mathbb{G}\mathrm{rass}_1(\mathcal{E}) = \mathbb{P}(\mathcal{E})$.

Functoriality

Let $\phi\colon \mathcal{R} \to \mathcal{R}'$ be a morphism of quasi-coherent graded $\mathcal{O}_S$-algebra. Let $U(\phi) := \bigcup D_+(V, \phi(s))$, where the union is taken over all $V \subseteq S$ affine open and $s \in \mathcal{R}_+(V)$ homogeneous. We have a commutative diagram

$$\begin{tikzcd} U(\phi) \arrow[r, "r"] \arrow[d, hook] & \underline{\mathrm{Proj}}(\mathcal{R}) \arrow[d] \\ \underline{\mathrm{Proj}}(\mathcal{R}') \arrow[r] & S \end{tikzcd}$$

where r is an affine morphism.

Base change

Let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra and let $f\colon S' \to S$ be a morphism of schemes. Then we have a Cartesian square

$$\begin{tikzcd} \underline{\mathrm{Proj}}(f^*\mathcal{R}) \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathrm{Proj}}(\mathcal{R}) \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

For any quasi-coherent graded $\mathcal{R}$-module $\mathcal{M}$, we have $f'^*\widetilde{\mathcal{M}} = \widetilde{f^*\mathcal{M}}$.

Let $\mathcal{R}$ and $\mathcal{R}'$ be quasi-coherent graded $\mathcal{O}_S$-algebras. Then we have a Cartesian square

$$\begin{tikzcd} \underline{\mathrm{Proj}}(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}') \arrow[r, "p"] \arrow[d, "p'"] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathrm{Proj}}(\mathcal{R}) \arrow[d] \\ \underline{\mathrm{Proj}}(\mathcal{R}') \arrow[r] & S \end{tikzcd}$$

Here $\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}' := \bigoplus_{d\geq 0} \mathcal{R}_d \otimes_{\mathcal{O}_S} \mathcal{R}'_d$. For a quasi-coherent graded $\mathcal{R}$-module $\mathcal{M}$ and a quasi-coherent graded $\mathcal{R}'$-module $\mathcal{M}'$, we have

$$(\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{M}')^{\sim} \simeq \widetilde{\mathcal{M}} \boxtimes_S \widetilde{\mathcal{M}'} := p^*\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{X''}} p'^*\widetilde{\mathcal{M}'}.$$

Here $\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{M}' := \bigoplus_{d\in\mathbb{Z}} \mathcal{M}_d \otimes_{\mathcal{O}_S} \mathcal{M}'_d$ and $X'' = \underline{\mathrm{Proj}}(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}')$.

Example 1.12.18. Let $\mathcal{E}$ and $\mathcal{E}'$ be quasi-coherent $\mathcal{O}_S$-modules. Then we have $\mathrm{Sym}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}') \twoheadrightarrow \mathrm{Sym}(\mathcal{E}) \otimes_{\mathcal{O}_S} \mathrm{Sym}(\mathcal{E}')$, which induces a closed immersion $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}')$. This is called the **Segre embedding** and generalizes the Segre embedding for the product of two projective spaces.

Example 1.12.19. Let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S and let $\mathcal{R}' = \mathrm{Sym}(\mathcal{L}) = \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d}$. Then $\mathcal{R}'' = \mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}' = \bigoplus_{d\geqslant 0} \mathcal{R}_d \otimes \mathcal{L}^{\otimes d}$. We have a Cartesian square

$$\begin{tikzcd}
\underline{\mathrm{Proj}}(\mathcal{R}'') \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] \arrow[dr, "\pi''"] & \underline{\mathrm{Proj}}(\mathcal{R}) \arrow[d, "\pi"] \\
\mathbb{P}(\mathcal{L}) \arrow[r, "\pi'"] & S
\end{tikzcd}$$

Note that π' is an isomorphism, because it is so locally. Thus $p\colon \underline{\mathrm{Proj}}(\mathcal{R}'') \to \underline{\mathrm{Proj}}(\mathcal{R})$ is an isomorphism. We have $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(d) = \pi'^*\mathcal{L}^{\otimes d}$ and $\mathcal{O}_{X''}(d) = p^*\mathcal{O}_X(d) \otimes \pi''^*\mathcal{L}^{\otimes d}$, where $X = \underline{\mathrm{Proj}}(\mathcal{R})$, $X'' = \underline{\mathrm{Proj}}(\mathcal{R}'')$.

Proposition 1.12.20. *Let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{R}_1$ over $\mathcal{O}_S$ and let $\pi\colon X = \mathrm{Proj}(\mathcal{R}) \to S$. Suppose that $\mathcal{R}_1$ is a $\mathcal{O}_S$-module of finite type. Then*

(1) π is proper.

(2) If S is quasi-compact and $\mathcal{L}$ is an invertible sheaf on S such that $\mathcal{R}_1 \otimes_{\mathcal{O}_S} \mathcal{L}$ is generated by global sections, then $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \pi^*\mathcal{L}$ is π-very ample. In particular, if S admits an ample invertible sheaf, then π is projective.

Note that the condition on generation by $\mathcal{R}_1$ implies that the morphism $\mathcal{O}_S \to \mathcal{R}_0$ is an epimorphism of sheaves of sets.

Proof. (1) The problem being local on S, we may assume S affine. Then π is projective and thus proper.

(2) Let $\mathcal{L}$ be an invertible sheaf on S such that $\mathcal{R}_1 \otimes \mathcal{L}$ is generated by globally sections. In the case where S admits an ample invertible sheaf $\mathcal{M}$, we can take $\mathcal{L}$ to be $\mathcal{M}^{\otimes d}$ for some d. Since S is quasi-compact, there exists an epimorphism $\mathcal{O}_S^{n+1} \twoheadrightarrow \mathcal{R}_1 \otimes \mathcal{L}$. The morphisms of $\mathcal{O}_S$-algebras

$$\mathrm{Sym}(\mathcal{O}_S^{n+1}) \to \mathrm{Sym}(\mathcal{R}_1 \otimes \mathcal{L}) \to \mathcal{R} \otimes \mathrm{Sym}(\mathcal{L})$$

are epimorphisms of $\mathcal{O}_S$-modules. The composition induces a closed embedding $i\colon \underline{\mathrm{Proj}}(\mathcal{R}) \hookrightarrow \underline{\mathrm{Proj}}(\mathcal{R}')$, where $\mathcal{R}' = \mathcal{R} \otimes \mathrm{Sym}(\mathcal{L})$, that fits in the commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Proj}}(R) \arrow[dr, "\pi"'] & \underline{\mathrm{Proj}}(\mathcal{R}') \arrow[l, "p"', "\sim"] \arrow[r, "i"] \arrow[d, "\pi'"] & \mathbb{P}^n_S \arrow[dl] \\ & S. & \end{tikzcd}$$

We have $i^*\mathcal{O}_{\mathbb{P}^n_S}(1) \simeq \mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{R}')}(1) \simeq p^*\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{R})}(1) \otimes \pi'^*\mathcal{L}$. Let $f = i \circ p^{-1} \colon \underline{\mathrm{Proj}}(\mathcal{R}) \to \mathbb{P}^n_S$. Then $f^*\mathcal{O}_{\mathbb{P}^n_S}(1) \simeq \mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{R})}(1) \otimes \pi^*\mathcal{L}$. □

Remark 1.12.21. Let $\pi \colon X \to S$ be a morphism of schemes. If $\mathcal{L}$ is π-very ample, then $\mathcal{L}$ is globally generated. Indeed, $\mathcal{O}_{\mathbb{P}^n_S}(1)$ is globally generated. Conversely, if π is an isomorphism, S is quasi-compact and $\mathcal{L}$ is globally generated, then $\mathcal{L}$ is π-very ample by the preceding proposition. Thus very ampleness is **not** local on S.

Remark 1.12.22. A morphism of schemes $f \colon X \to S$ is said to be **EGA projective** if there exists a decomposition

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & S \end{tikzcd}$$

where i is a closed immersion and $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_S$-module of finite type.

Blowing up

Definition 1.12.23. Let S be a scheme, $\mathcal{I} \subseteq \mathcal{O}_S$ a quasi-coherent ideal sheaf which defines a closed subscheme Z. Consider the quasi-coherent graded $\mathcal{O}_S$-module $\mathcal{R} = \bigoplus_{n\geqslant 0} \mathcal{I}^n$, where $\mathcal{I}^0 = \mathcal{O}_S$. Then

$$X = \underline{\mathrm{Proj}}(\mathcal{R}) \xrightarrow{\pi} S$$

is called the **blowing up** of S along Z (or with **center** Z, or in $\mathcal{I}$). The closed subscheme $\pi^{-1}(Z) \subseteq X$ is called the **exceptional divisor**.

On an affine open $\mathrm{Spec}(B) = V \subseteq S$, we have $\mathcal{I}|_V = \widetilde{I}$ where $I \subseteq B$ an ideal, and $\pi^{-1}(V) = \mathrm{Proj}(\bigoplus_{n\geqslant 0} I^n)$, where $I^0 = B$. We have $\pi^{-1}(V) = \bigcup_{a\in I} D_+(V, a^{(1)})$, where for $a \in I$, $a^{(1)}$ denotes a viewed as an element of $\mathcal{R}_1(V) = I$. We have $D_+(V, a^{(1)}) = \mathrm{Spec}(B[\frac{I}{a}])$, where $B[\frac{I}{a}] := (\bigoplus_{n\geq 0} I^n)_{(a^{(1)})}$ is called the affine blow up algebra. Elements of $B[\frac{I}{a}]$ are of the form x/a^n, $x \in I^n$ and $x/a^n = y/a^m$ if and only if there exists k such that $a^k(a^m x - a^n y) = 0$.

We will discuss divisors more thoroughly later in the course. Here we limit our attention to effective Cartier divisors.

Definition 1.12.24. An **effective Cartier divisor** on a scheme X is a closed subscheme $D \subseteq X$ whose sheaf of ideals $\mathcal{I}_D$ is invertible.

Lemma 1.12.25. *Let $D \subseteq X$ be a closed subscheme. Then D is an effective Cartier divisor if and only if every $x \in X$ admits an affine open neighborhood $x \in U = \mathrm{Spec}(A)$ such that $U \cap D = \mathrm{Spec}(A/(f))$, where $f \in A$ is a non zero-divisor.*

Proof. An ideal I of A is free of rank 1 if and only if I is generated by a non zero-divisor of A. $\square$

Lemma 1.12.26. *Let $D \subseteq X$ be an effective Cartier divisor. Then $j\colon X\backslash D \hookrightarrow X$ is schematically dense. Namely, $j^\flat\colon \mathcal{O}_X \to j_*(\mathcal{O}_{X\backslash D})$ is a monomorphism.*

Proof. Locally, j corresponds to the ring homomorphism $A \to A_f$, where $f \in A$ is a non zero-divisor. The ring homomorphism is clearly injective. $\square$

Remark 1.12.27. Let D_1 and D_2 be two effective Cartier divisors on X with ideal sheaves $\mathcal{I}_{D_1}$, $\mathcal{I}_{D_2}$, respectively. Then the natural morphism $\mathcal{I}_{D_1} \otimes_{\mathcal{O}_X} \mathcal{I}_{D_2} \to \mathcal{I}_{D_1}\mathcal{I}_{D_2}$ is an isomorphism. (Indeed, it is by definition an epimorphism of sheaves of $\mathcal{O}_X$-modules and the morphism $\mathcal{I}_{D_1} \otimes \mathcal{I}_{D_2} \to \mathcal{O}_X$ is a monomorphism by flatness.) We define the sum of the two divisors $D_1 + D_2$ as the subscheme defined by $\mathcal{I}_{D_1}\mathcal{I}_{D_2}$. Then $\mathrm{CaDiv}_+(X) = (\{\text{effective Cartier divisors on } S\}, +)$ is a commutative monoid.

Proposition 1.12.28. *Let $Z \subseteq S$ be a closed subscheme and let X be the blowing up of S along Z. Then*

(1) $\pi|_{\pi^{-1}(S\backslash Z)}\colon \pi^{-1}(S\backslash Z) \to S\backslash Z$ is an isomorphism.

(2) $E = \pi^{-1}(Z)$ is an effective Cartier divisor with $\mathcal{I}_E = \mathcal{O}_X(1)$.

Proof. (1) The construction being compatible with restriction to open subschemes, we may assume $Z = \emptyset$. Then $\mathcal{I}_Z = \mathcal{O}_X$ and $X = \underline{\mathrm{Proj}}(\bigoplus_{n\geqslant 0}\mathcal{O}_S) = \mathbb{P}^0_S \simeq S$.

(2) Let $\mathcal{R} = \bigoplus_{n\geqslant 0}\mathcal{I}^n$. We have

$$\mathcal{I}_E = \widetilde{\mathcal{I}\mathcal{R}} = (\bigoplus_{n\geqslant 0}\mathcal{I}^{n+1})^\sim,$$
$$\mathcal{O}(1) = \widetilde{\mathcal{R}(1)} = (\bigoplus_{n\geqslant -1}\mathcal{I}^{n+1})^\sim.$$

They are isomorphic as sheaves. $\square$

Proposition 1.12.29 (Universal property of blowing up)**.** *Let $Z \subseteq S$ be a closed subschema and let X be the blowing up of S along Z. Let $f\colon Y \to S$ be a morphism of schemes such that $f^{-1}(Z)$ is an effective Cartier divisor on Y. Then there exists a unique $g\colon Y \to X$ such that $f = \pi g$.*

$$\begin{tikzcd} Y \arrow[r, dashed, "g"] \arrow[dr, "f"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, hook] & S \end{tikzcd}$$

Proof. Existence. Let $\mathcal{I}$ be the ideal sheaf of Z and let $D = f^{-1}(Z)$. Then $\mathcal{I}_D = f^{-1}(\mathcal{I})\mathcal{O}_Y$. We have epimorphisms $\gamma_n\colon f^*\mathcal{I}^n \twoheadrightarrow \mathcal{I}_D^n = \mathcal{I}_D^{\otimes n}$, which induces a morphism of graded $\mathcal{O}_Y$-algebras $\gamma\colon f^*(\bigoplus_{n\geqslant 0}\mathcal{I}^n) \twoheadrightarrow \bigoplus_{n\geqslant 0}\mathcal{I}_D^{\otimes n}$. This corresponds to an S-morphism $g\colon Y \to X$.

Uniqueness. Suppose there are two S-morphisms $g, g' \colon Y \rightrightarrows X$. Let E be their equalizer:

$$\begin{tikzcd} E \arrow[r, hook] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] \arrow[dr, "f"'] & X \arrow[d, "\pi"] \\ & & S \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "{(g,g')}"] \\ X \arrow[r, hook, "\Delta_S"] & X \times_S X \end{tikzcd}$$

Since π is separated, E is a closed subscheme of Y. Moreover, since π is an isomorphism on $\pi^{-1}(S\backslash Z)$, we have $E \supseteq f^{-1}(S\backslash Z)$ as subschemes. Since $Y\backslash f^{-1}(Z) \hookrightarrow Y$ is schematically dense in Y by Lemma 1.12.26, so is E. Therefore, $E = Y$ and $g = g'$. □

Corollary 1.12.30. *Let $Z \subseteq S$ be a closed subscheme, $f \colon S' \to S$ a morphism of schemes, $\pi \colon X \to S$ the blowing up of S along Z, and $\pi' \colon X' \to S'$ the blowing up of S' along $f^{-1}(Z)$. Then there exists a unique $g \colon X' \to X$ such that $\pi g = f\pi'$:*

$$\begin{tikzcd} X' \arrow[r, dashed, "g"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

Moreover,

(a) If f is a closed immersion, so is g.

(b) If f is flat, then the square is Cartesian.

Proof. The existence and uniqueness of g is follows from the universal property of blowing up.

For (a), we may assume $S' = \underline{\mathrm{Spec}}(\mathcal{O}_S/\mathcal{J})$, where $\mathcal{J} \subseteq \mathcal{O}_S$ is an ideal sheaf. Let $\mathcal{I} \subseteq \mathcal{O}_S$ be the ideal sheaf of Z. Then the ideal sheaf of $f^{-1}(Z)$ is $\mathcal{I}\mathcal{J}/\mathcal{J}$. Then the canonical morphism $\bigoplus_{n\geqslant 0} \mathcal{I}^n \to \bigoplus_{n\geqslant 0} \mathcal{I}^n\mathcal{J}/\mathcal{J}$ is an epimorphism as sheaves of $\mathcal{O}_S$-modules and the corresponding morphism $X' \to X$ is a closed immersion.

For (b), we need to show that the morphism $X' \to X \times_S S'$ is an isomorphism. Since f is flat, we have $f^*(\mathcal{I}^n) \simeq f^{-1}(\mathcal{I}^n)\mathcal{O}_{S'}$. Thus $X' =\simeq \mathrm{Proj}(\bigoplus_{n\geqslant 0} f^{-1}(\mathcal{I}^n)\mathcal{O}_{S'}) \simeq \mathrm{Proj}(\bigoplus_{n\geqslant 0} f^*(\mathcal{I}^n)) \simeq X \times_S S'$. □

Definition 1.12.31. In case (a), X' is called the **strict transform** of S'.

Remark 1.12.32. The exceptional divisor of the blowing up of a scheme S along a closed subscheme Z defined by the ideal sheaf $\mathcal{I}$ is $E = \mathrm{Proj}(\mathcal{R}/\mathcal{I}\mathcal{R}) \simeq \mathrm{Proj}(\bigoplus_{n\geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1})$. This is a closed subscheme of $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$ and is sometimes called the **projective normal cone** of $Z \subseteq S$. Here $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^n$.

Example 1.12.33. Let $S = \mathbb{A}^n_A = \mathrm{Spec}(B)$, where $B = A[x_1, \dots, x_n]$. Let Z be the closed subscheme defined by $I = (x_1, \dots, x_n)$. Let $X = \mathrm{Bl}_Z(S) = \mathrm{Proj}(R)$, where $R = \bigoplus_{n\geqslant 0} I^n$. We have the surjective homomorphism

$$\begin{aligned} B[y_1, \dots, y_n] &\to R \\ y_i &\mapsto x_i^{(1)}, \end{aligned}$$

which gives a closed immersion $X \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{A}^n_A}$. We have $R \simeq B[y_1, \dots, y_n]/(x_i y_j - x_j y_i)$. The exceptional divisor $E \simeq \mathbb{P}^{n-1}_A$.

For $S = \mathbb{A}^2$, the strict transforms of lines ℓ through the origin are disjoint. The intersection of the strict transform of ℓ with the exceptional divisor is given by the slope of ℓ.

Let $B' = A[x, y]/(y^2 - x^2(x + 1))$. The blowing up of B' in (x, y) is $B[\frac{I}{x}] = A[x, z]/(z^2 - (x + 1))$, where $z = y/x$. By contrast, the blowing up of B' in (x) is B', because $x \in B'$ is a non zero-divisor. We see that blowing up depends on the closed subscheme and not only on the closed subset.

Chapter 2

Cohomology of Quasi-coherent Sheaves

Date: 11.17

2.1 Homological algebra

We will give a brief introduction to derived categories and derived functors and refer to [GM] and [Z, Chapter 2] for more complete treatments.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. For any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, we have, by the left exactness of F, an exact sequence

$$0 \to FX \to FY \to FZ$$

in $\mathcal{B}$. Under suitable conditions, we can define additive functors $R^nF\colon \mathcal{A} \to \mathcal{B}$, $i \geq 1$, called the **right derived functors** of F, such that the exact sequence in $\mathcal{B}$ extends to a long exact sequence

$$\begin{aligned} 0 \to FX \to FY \to FZ \to R^1FX \to R^1FY \to R^1FZ \to \cdots \\ \to R^nFX \to R^nFY \to R^nFZ \to \cdots . \end{aligned}$$

Roughly speaking, the right derived functors measure the lack of right exactness of F. The functors can be assembled into one single functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ between **derived categories**.

Recall that an object I of $\mathcal{A}$ is said to be **injective** if $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ is exact. Assume that $\mathcal{A}$ admits enough injectives (namely, every object of $\mathcal{A}$ can be embedded into an injective object of $\mathcal{A}$). Then every object X of $\mathcal{A}$ admits an **injective resolution** of X, namely an exact sequence

$$0 \to X \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

with I^i injective. Then RFX is computed by the complex

$$FI\colon \cdots \to 0 \to FI^0 \xrightarrow{Fd^0} FI^1 \xrightarrow{Fd^1} \cdots$$

and R^iFX is computed by the i-th cohomology of RFX: $\ker(Fd^i)/\mathrm{im}(Fd^{i-1})$.

Definition 2.1.1. Let $\mathcal{A}$ be an additive category. A **(cochain) complex** in $\mathcal{A}$ consists of $X = (X^n, d^n)_{n\in\mathbb{Z}}$, where X^n is an object of $\mathcal{A}$, $d_X^n \colon X^n \to X^{n+1}$ is a morphism of $\mathcal{A}$ (called **differential**) such that for any n, $d_X^{n+1}d_X^n = 0$. The index n in X^n is called the **degree**. A **(cochain) morphism** of complexes $X \to Y$ is a collection of morphisms $(f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n \colon X^n \to Y^n$ in $\mathcal{A}$ such that $d_Y^n f^n = f^{n+1}d_X^n$. We let $C(\mathcal{A})$ denote the category of complexes in $\mathcal{A}$.

Note that $C(\mathcal{A})$ is an additive category. We have $(X\oplus Y)^n = X^n \oplus Y^n$ and the zero complex 0 with $0^n = 0$ is a zero object of $C(\mathcal{A})$.

Let $\mathcal{A}$ be an abelian category. Then $C(\mathcal{A})$ is an abelian category as well, with $\mathrm{Ker}(f)^n = \mathrm{Ker}(f^n)$ and $\mathrm{coker}(f)^n = \mathrm{coker}(f^n)$.

Definition 2.1.2. Let X be a complex in $\mathcal{A}$. We define

$$\begin{aligned} Z^nX &= \mathrm{Ker}(d_X^n \colon X^n \to X^{n+1}),\\ B^nX &= \mathrm{im}(d_X^{n-1} \colon X^{n-1} \to X^n),\\ H^nX &= \mathrm{coker}(B^nX \hookrightarrow Z^nX), \end{aligned}$$

and call them the **cocycle**, **coboundary**, **cohomology** objects, of degree n.

The letter Z stands for German **Zyklus**, which means cycle. We get additive functors

$$Z^n, B^n, H^n \colon C(\mathcal{A}) \to \mathcal{A},$$

with Z^n left exact.

Definition 2.1.3. A complex X is said to be **acyclic** if $H^nX = 0$ for all n. A morphism of complexes $X \to Y$ is called a **quasi-isomorphism** if $H^nf \colon H^nX \to H^nY$ is an isomorphism for all n.

We will soon define the derived category $D(\mathcal{A})$ of $\mathcal{A}$. Roughly speaking, $D(\mathcal{A})$ is $C(\mathcal{A})$ modulo quasi-isomorphisms. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F induces $C(F) \colon C(\mathcal{A}) \to C(\mathcal{B})$ (also denoted by F). If F is exact, then $C(F)$ preserves quasi-isomorphisms and induces a functor $D(\mathcal{A}) \to D(\mathcal{B})$. For the general case, it is convenient to introduce an intermediary between $C(\mathcal{A})$ and $D(\mathcal{A})$.

Let $\mathcal{A}$ be an additive category. Let X and Y be complexes in $\mathcal{A}$. We let

$$\mathrm{Ht}(X,Y) = \prod_n \mathrm{Hom}_{\mathcal{A}}(X^n, Y^{n-1})$$

denote the abelian group of families of morphisms $h = (h^n \colon X^n \to Y^{n-1})_{n\in\mathbb{Z}}$. Given h, consider $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n \colon X^n \to Y^n$. We have

$$d_Y^n f^n = d_Y^{n-1}d_Y^n h^n + d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n + h^{n+2}d_X^{n+1}d_X^n = f^{n+1}d_X^n.$$

Thus we get a morphism of complexes $f \colon X \to Y$. We get a homomorphism of abelian groups

$$\mathrm{Ht}(X,Y) \to \mathrm{Hom}_{C(\mathcal{A})}(X,Y). \tag{2.1.1}$$

Definition 2.1.4. We say that a morphism of complexes $f\colon X \to Y$ is **null-homotopic** if there exists $h \in \mathrm{Ht}(X,Y)$ such that $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n$. We say that two morphisms of complexes $f, g\colon X \to Y$ are **homotopic** if $f - g$ is null-homotopic.

Lemma 2.1.5. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of complexes in $\mathcal{A}$. If f or g is null-homotopic, then gf is null-homotopic.*

Proof. If $f = dh + hd$ for $h \in \mathrm{Ht}(X,Y)$, then $gf = gdh + ghd = d(gh) + (gh)d$, where $gh \in \mathrm{Ht}(X,Z)$. The other case is similar. □

Definition 2.1.6. We define the **homotopy category of complexes in** $\mathcal{A}$, $K(\mathcal{A})$, as follows. The objects of $K(\mathcal{A})$ are objects of $C(\mathcal{A})$, that is, complexes in $\mathcal{A}$. For complexes X and Y, we put

$$\mathrm{Hom}_{K(\mathcal{A})}(X,Y) = \mathrm{coker}(\mathrm{Ht}(X,Y) \xrightarrow{(2.1.1)} \mathrm{Hom}_{C(\mathcal{A})}(X,Y)).$$

In other words, morphisms in $K(\mathcal{A})$ are homotopy classes of morphisms of complexes.

Remark 2.1.7. The category $K(\mathcal{A})$ is an additive category and the functor $C(\mathcal{A}) \to K(\mathcal{A})$ carrying a complex to itself and a morphism of complexes to its homotopy class is an additive functor.

Definition 2.1.8. Let $\mathcal{A}$ be an abelian category. We call $D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$ the **derived category** of $\mathcal{A}$, where S is the collection of quasi-isomorphisms in $K(\mathcal{A})$.

By definition, objects of $D(\mathcal{A})$ are complexes in $\mathcal{A}$ and morphisms are equivalence classes of zigzags of morphisms of $K(\mathcal{A})$

$$\to \cdots \to \leftarrow \cdots \leftarrow \to \cdots \to \cdots \leftarrow \cdots \leftarrow,$$

where each $\leftarrow$ represents an element of S. One advantage of defining $D(\mathcal{A})$ as a localization of $K(\mathcal{A})$ instead of $C(\mathcal{A})$ is that left and right calculus of fractions holds:

$$\mathrm{Hom}_{D(\mathcal{A})}(X,Y) \simeq \operatorname*{colim}_{(Y',s)\in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X,Y') \simeq \operatorname*{colim}_{(X',s)\in S_{/X}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(X',Y).$$

In general, $D(\mathcal{A})$ does not have small Hom sets, even if $\mathcal{A}$ has small Hom sets. See however Remark 2.1.35 below.

The categories $K(\mathcal{A})$ and $D(\mathcal{A})$ admit an additional structure, making them **triangulated categories**. To introduce this structure, we need a couple of constructions.

Let $\mathcal{A}$ be an additive category.

Definition 2.1.9. Let X be a complex and let k be an integer. We define a complex $X[k]$ by $X[k]^n = X^{n+k}$ and $d^n_{X[k]} = (-1)^k d_X^{n+k}$. For a morphism of complexes $f\colon X \to Y$, we define $f[k]\colon X[k] \to Y[k]$ by $f[k]^n = f^{n+k}$. The functor $[k]\colon C(\mathcal{A}) \to C(\mathcal{A})$ is called the **translation** (or shift) functor of degree k.

The sign in the definition of $X[k]$ will be explained after the following definition.

Definition 2.1.10. Let $f\colon X \to Y$ be a morphism of complexes in $\mathcal{A}$. We define the **mapping cone** of f to be the complex $\mathrm{Cone}(f)^n = X[1]^n \oplus Y^n = X^{n+1} \oplus Y^n$ with differential

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f[1]^n & d^n_Y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}.$$

Intuitively, for $\begin{pmatrix} x \\ y \end{pmatrix} \in X^{n+1} \oplus Y^n$, $d^n_{\mathrm{Cone}(f)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X x \\ f^{n+1}x + d^n_Y y \end{pmatrix}$.

Note that the sign in the definition of the differential of $X[1]$ makes $\mathrm{Cone}(f)$ a complex:

$$d^n_{\mathrm{Cone}(f)} d^{n-1}_{\mathrm{Cone}(f)} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \begin{pmatrix} -d^n_X & 0 \\ f^n & d^{n-1}_Y \end{pmatrix} = \begin{pmatrix} d^{n+1}_X d^n_X & 0 \\ d^n_Y f^n - f^{n+1} d^n_X & d^n_Y d^{n-1}_Y \end{pmatrix} = 0.$$

Example 2.1.11. If X and Y are concentrated in degree 0, then $\mathrm{Cone}(f)$ can be identified with the complex $X^0 \xrightarrow{f^0} Y^0$ concentrated on degrees -1 and 0.

Triangulated categories

Given a category $\mathcal{D}$ equipped with a functor $X \mapsto X[1]$, diagrams of the form $X \to Y \to Z \to X[1]$ are called **triangles**. It is sometimes useful to visualize such diagrams as

$$\begin{tikzcd} & Z \arrow[dl, "+1"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

Such a morphism is an isomorphism if and only if f, g, h are isomorphisms.

Definition 2.1.12 (Verdier)**.** A **triangulated category** consists of the following data:

(1) An additive category $\mathcal{D}$.

(2) A **translation functor** $\mathcal{D} \to \mathcal{D}$ which is an equivalence of categories. We denote the functor by $X \mapsto X[1]$.

(3) A collection of **distinguished triangles** $X \to Y \to Z \to X[1]$.

These data are subject to the following axioms:
(TR1)

- The collection of distinguished triangles is stable under isomorphism.

- Every morphism $f\colon X \to Y$ in $\mathcal{D}$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.
- For every object X of $\mathcal{D}$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle.

(TR2) A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if the (clockwise) rotated diagram $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(TR4) Given three distinguished triangles

$$\begin{aligned} &X \xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} X[1], \\ &Y \xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} Y[1], \\ &X \xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} X[1], \end{aligned}$$

with $h = gf$, there exists a distinguished triangle $U \xrightarrow{i} V \xrightarrow{i'} W \xrightarrow{i''} U[1]$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"] & & Z \arrow[rr, "g'"] \arrow[dr, "h'"] & & W \arrow[rr, "i''"] \arrow[dr, "g''"] & & U[1] \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"] & & V \arrow[ur, "i'"] \arrow[dr, "h''"] & & Y[1] \arrow[ur, "f'[1]"] & \\
& & U \arrow[ur, "i"] \arrow[rr, "f''"] & & X[1] \arrow[ur, "f[1]"] & &
\end{tikzcd}$$

This notion was introduced by Verdier (see his 1967 thesis of **doctorat d'État** [V]). Some authors call the translation functor the suspension functor and denote it by Σ. (TR4) is sometimes known as the octahedron axiom, as the four distinguished triangles and the four commutative triangles can be visualized as the faces of an octahedron.

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Remark 2.1.13. The original definition included an axiom (TR3): Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "j"] \arrow[d, "g"] & Z \arrow[r, "k"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "j'"] & Z' \arrow[r, "k'"] & X'[1] \end{tikzcd}$$

in which both rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative. Note that we do **not** require the dotted arrow to be unique.

May [M3, Section 2] observed that this axiom can be deduced from (TR1) and (TR4). Indeed, by (TR1), we may extend $gi = i'f$ to a distinguished triangle

$$X \xrightarrow{gi} Y' \xrightarrow{j''} Z'' \xrightarrow{k''} X[1].$$

Applying (TR1) to g and (TR4) to the distinguished triangles with bases g, i, and gi, we get a morphism $Z \xrightarrow{h'} Z''$ such that $h'j = j''g$ and $k = k''h'$. Similarly, applying (TR1) to f and (TR4) to the distinguished triangles with bases f, i', and gi, we get $Z'' \xrightarrow{h''} Z$ such that $j' = h''j''$ and $f[1]k'' = k'h''$. It suffices to take $h = h''h'$.

Corollary 2.1.14. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle. Then $gf = 0$.*

Proof. By (TR1), $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle. By (TR3), there exists a morphism $0 \to Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "\mathrm{id}_{X[1]}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd} \tag{2.1.2}$$

commutes. The commutativity of the square in the middle implies $gf = 0$. □

Proposition 2.1.15. *Let $\mathcal{D}$ be a triangulated category. Let W be an object of $\mathcal{D}$ and let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle. Then the sequences*

$$\mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z),$$
$$\mathrm{Hom}_{\mathcal{D}}(Z, W) \to \mathrm{Hom}_{\mathcal{D}}(Y, W) \to \mathrm{Hom}_{\mathcal{D}}(X, W)$$

are exact.

If $\mathcal{D}$ has small Hom sets, then the proposition means that the functors

$$\mathrm{Hom}_{\mathcal{D}}(W, -)\colon \mathcal{D} \to \mathrm{Ab}, \quad \mathrm{Hom}_{\mathcal{D}}(-, W)\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$$

are cohomological functors.

Proof. Let us show that the first sequence is exact, the other case being similar. Since $gf = 0$, the composition is zero. Thus it suffices to show that for $j\colon W \to Y$ satisfying $gj = 0$, there exists $i\colon W \to X$ such that $j = fi$. Applying (TR1), (TR2), (TR3), we get the following commutative diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "j"] & 0 \arrow[r] \arrow[d] & W[1] \arrow[r, "-\mathrm{id}_{W[1]}"] \arrow[d, dashed, "i[1]"] & W[1] \arrow[d, "j[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r] & X[1] \arrow[r, "-f[1]"] & Y[1].
\end{tikzcd}$$

□

Corollary 2.1.16. *Let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

be a morphism of distinguished triangles. If f and g are isomorphisms, so is the third one.

Thus triangles extending a morphism $X \to Y$ are unique up to **non-unique** isomorphisms.

Proof. Let W be any object of the triangulated category. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(W,X) \arrow[r] \arrow[d, "\mathrm{Hom}(W,f)"] & \mathrm{Hom}(W,Y) \arrow[r] \arrow[d, "\mathrm{Hom}(W,g)"] & \mathrm{Hom}(W,Z) \arrow[r] \arrow[d, "\mathrm{Hom}(W,h)"] & \mathrm{Hom}(W,X[1]) \arrow[r] \arrow[d, "\mathrm{Hom}(W,f[1])"] & \mathrm{Hom}(W,Y[1]) \arrow[d, "\mathrm{Hom}(W,g[1])"] \\
\mathrm{Hom}(W,X') \arrow[r] & \mathrm{Hom}(W,Y') \arrow[r] & \mathrm{Hom}(W,Z') \arrow[r] & \mathrm{Hom}(W,X'[1]) \arrow[r] & \mathrm{Hom}(W,Y'[1])
\end{tikzcd}$$

with exact rows. By the five lemma, $\mathrm{Hom}(W,h)$ is an isomorphism. Therefore h is an isomorphism by Yoneda's lemma. □

Corollary 2.1.17. *In a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, f is an isomorphism if and only if Z is a zero object.*

Proof. Applying Corollary 2.1.16 to the diagram (2.1.2), we see that f is an isomorphism if and only if h is an isomorphism. □

Definition 2.1.18. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories. A **triangulated functor** consists of the following data:

(1) An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$.

(2) A natural isomorphism $\phi_X\colon F(X[1]) \simeq (FX)[1]$ of functors $\mathcal{D} \to \mathcal{D}'$.

These data are subject to the condition that F carries distinguished triangles in $\mathcal{D}$ to distinguished triangles in $\mathcal{D}'$. That is, for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}$, $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\phi_Z \circ Fh} (FX)[1]$ is a distinguished triangle in $\mathcal{D}'$.

Let $(F, \phi), (F', \phi') \colon \mathcal{D} \to \mathcal{D}'$ be triangulated functors. A **natural transformation of triangulated functors** is a natural transformation $\alpha \colon F \to F'$ such that the following diagram commutes for all X:

$$\begin{tikzcd} F(X[1]) \arrow[r, "\phi_X"] \arrow[d, "\alpha(X[1])"'] & (FX)[1] \arrow[d, "\alpha(X)[1]"] \\ F'(X[1]) \arrow[r, "\phi'_X"] & (F'X)[1]. \end{tikzcd}$$

Derived categories

Let $\mathcal{A}$ be an additive category. We equip $K(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.9. We say that a triangle in $K(\mathcal{A})$ is **distinguished** if it is isomorphic to a standard triangle, namely a triangle of the form $X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1]$, where i and p are the canonical morphisms. If $\mathcal{A}$ is abelian, we equip $D(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.9 and we say that a triangle in $D(\mathcal{A})$ is **distinguished** if it is isomorphic to a standard triangle.

Theorem 2.1.19. *Let $\mathcal{A}$ be an additive category.*

(1) $K(\mathcal{A})$ is a triangulated category.

(2) If $\mathcal{A}$ is abelian, then $D(\mathcal{A})$ is a triangulated category and the functor $Q \colon K(\mathcal{A}) \to D(\mathcal{A})$ (equipped with the trivial natural isomorphism $Q(X[1]) = (QX)[1]$) is a triangulated functor.

For a proof, see for example [Z, Chapter 2].

We define **naive truncation** functors

$$\sigma^{\le n} \colon C(\mathcal{A}) \to C(\mathcal{A}), \quad \sigma^{\ge n} \colon C(\mathcal{A}) \to C(\mathcal{A})$$

by $(\sigma^{\le n} X)^m = X^m$ for $m \le n$, $(\sigma^{\le n} X)^m = 0$ for $m > n$ and $(\sigma^{\ge n} X)^m = X^m$ for $m \ge n$, $(\sigma^{\ge n} X)^m = 0$ for $m < n$.

Let $\mathcal{A}$ be an abelian category. The morphisms $H^n X \to H^n \sigma^{\le n} X$, $H^n \sigma^{\ge n} X \to H^n X$ are not isomorphisms in general. Moreover, if $f \colon X \to Y$ is a quasi-isomorphism, $\sigma^{\le n} f \colon \sigma^{\le n} X \to \sigma^{\le n} Y$ and $\sigma^{\ge n} f \colon \sigma^{\ge n} X \to \sigma^{\ge n} Y$ are not quasi-isomorphisms in general. To remedy this problem, we introduce the following **truncation** functors.

Definition 2.1.20. Let X be a complex. We define

$$\tau^{\le n} X = (\cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} Z^n X \to 0 \to \cdots),$$
$$\tau^{\ge n} X = (\cdots \to 0 \to X^n / B^n X \xrightarrow{d_X^n} X^{n+1} \to \cdots).$$

Here $X^n / B^n X$ denotes $\mathrm{coker}(d_X^{n-1})$.

We obtain functors
$$\tau^{\leq n}, \tau^{\geq n} \colon C(\mathcal{A}) \to C(\mathcal{A}),$$
with $\tau^{\leq n}$ left exact and $\tau^{\geq n}$ right exact.

Remark 2.1.21. The morphism $\tau^{\leq n}X \to X$ induces an isomorphism $H^m\tau^{\leq n}X \to H^mX$ for $m \leq n$ and $H^m\tau^{\leq n}X = 0$ for $m > n$. The morphism $X \to \tau^{\geq n}X$ induces an isomorphism $H^mX \to H^m\tau^{\geq n}X$ for $m \geq n$ and $H^m\tau^{\geq n}X = 0$ for $m < n$. The functors $\tau^{\leq n}$ and $\tau^{\geq n}$ preserve quasi-isomorphisms.

Remark 2.1.22. For $a \leq b$, we have $\tau^{\leq a}\tau^{\geq b}X \simeq \tau^{\geq b}\tau^{\leq a}X$ and we write $\tau^{[a,b]}X$ for either of them. We have $\tau^{[n,n]}X \simeq (H^nX)[-n]$.

The functor H^n is neither left exact nor right exact in general. However, it has the following important property.

Proposition 2.1.23. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of complexes. Then we have a* ***long exact sequence***

$$\cdots \to H^nL \xrightarrow{H^nf} H^nM \xrightarrow{H^ng} H^nN \xrightarrow{\delta} H^{n+1}L \xrightarrow{H^{n+1}f} H^{n+1}M \xrightarrow{H^{n+1}g} H^{n+1}N \to \cdots,$$

which is functorial with respect to the short exact sequence.

The morphism δ is called the **connecting** morphism.

Proof. The sequence $\tau^{[n,n+1]}L \to \tau^{[n,n+1]}M \to \tau^{[n,n+1]}N$ provides a commutative diagram

$$\begin{tikzcd}
 & L^n/B^nL \arrow[r] \arrow[d] & M^n/B^nM \arrow[r] \arrow[d] & N^n/B^nN \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}L \arrow[r] & Z^{n+1}M \arrow[r] & Z^{n+1}N &
\end{tikzcd}$$

with exact rows. Applying the snake lemma, we obtain the desired exact sequence. □

Corollary 2.1.24. *For every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $D(\mathcal{A})$, we have a long exact sequence $H^nX \xrightarrow{H^nf} H^nY \xrightarrow{H^ng} H^nZ \xrightarrow{H^nh} H^{n+1}X$.*

Proof. We may assume that the triangle is standard: $X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1]$. The short exact sequence of complexes

$$0 \to Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0.$$

induces a long exact sequence

$$\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta} H^nY \xrightarrow{H^ni} H^n(\mathrm{Cone}(f)) \xrightarrow{H^np} H^n(X[1]) \to \cdots.$$

It suffices to check that, via the isomorphism $H^{n-1}(X[1]) \simeq H^nX$, the connecting morphism can be identified with H^nf. The connecting morphism is constructed using the snake lemma applied to the commutative diagram

$$\begin{tikzcd}
 & Y^{n-1}/B^{n-1}Y \arrow[r] \arrow[d] & C^{n-1}/B^{n-1}C \arrow[r] \arrow[d] & X^n/B^nX \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^nY \arrow[r] & Z^nC \arrow[r] & Z^{n+1}X, &
\end{tikzcd}$$

where $C = \mathrm{Cone}(f)$. We reduce by the Freyd-Mitchell Theorem to the case of modules. Let $x \in Z^n X$. Then $\begin{pmatrix} x \\ 0 \end{pmatrix} + B^{n-1}C$ is a lifting of $x + B^n X$. We conclude by $d_C^{n-1}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f^n(x) \end{pmatrix}$. □

Corollary 2.1.25. *Consider a short exact sequence of complexes* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. *Then the map* $\phi = (0, g)\colon \mathrm{Cone}(f) \to Z$ *is a quasi-isomorphism.*

In this case, we get a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{p\phi^{-1}} X[1]$ in $D(\mathcal{A})$.

Proof. We have a commutative diagram of long exact sequences

$$\begin{tikzcd} H^n X \arrow[r, "H^n f"] \arrow[d, equal] & H^n Y \arrow[r, "H^n i"] \arrow[d, equal] & H^n(\mathrm{Cone}(f)) \arrow[r, "H^n p"] \arrow[d, "H^n \phi"] \arrow[dr, phantom, "(*)"] & H^{n+1} X \arrow[r, "H^{n+1} f"] \arrow[d, equal] & H^{n+1} Y \arrow[d, equal] \\ H^n X \arrow[r, "H^n f"] & H^n Y \arrow[r, "H^n g"] & H^n Z \arrow[r, "-\delta"] & H^{n+1} X \arrow[r, "H^{n+1} f"] & H^{n+1} Y. \end{tikzcd}$$

Indeed, for the commutativity of the square $(*)$ we reduce by the Freyd-Mitchell Theorem to the case of modules, and it suffices to note that for $\begin{pmatrix} x \\ y \end{pmatrix} \in Z^n\mathrm{Cone}(f)$, we have $f^n(x) + d^n y = 0$. By the five lemma, $H^n\phi$ is an isomorphism. □

Definition 2.1.26. Let $\mathcal{A}$ be an additive category. We say that a complex X is **bounded below** (resp. **bounded above**) if $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$). We say that X is **bounded** if it is bounded below and bounded above. For an interval $I \subseteq \mathbb{Z}$, we say that X is concentrated in degrees in I if $X^n = 0$ for $n \notin I$. We let $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^I(\mathcal{A})$ denote the full subcategories of $C(\mathcal{A})$ consisting of complexes bounded below, bounded above, bounded, concentrated in I, respectively. We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$, $K^I(\mathcal{A})$ denote their respective images in $K(\mathcal{A})$.

For $\mathcal{A}$ abelian, we let $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, resp. $D^b(\mathcal{A})$, resp. $D^I(\mathcal{A})$) denote the full subcategories of $D(\mathcal{A})$ consisting of complexes satisfying $H^n = 0$ for all $n \ll 0$ (resp. $n \gg 0$, resp. $|n| \gg 0$, resp. $n \notin I$).

Proposition 2.1.27. *The functor* $H^0\colon D^{[0,0]}(\mathcal{A}) \to \mathcal{A}$ *is an equivalence of categories.*

Proof. Consider the functor $F\colon \mathcal{A} \to D^{[0,0]}(\mathcal{A})$ carrying A to a complex X concentrated in degree 0 with $X^0 = A$. We have $H^0FA \simeq A$. For any complex X concentrated in degree 0, $X \simeq \tau^{[0,0]}X \simeq FH^0X$. □

Derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. We have remarked that F extends to an additive functor $C(F)\colon C(\mathcal{A}) \to C(\mathcal{B})$, which induces a triangulated $K(F)\colon K(\mathcal{A}) \to K(\mathcal{B})$. We have a commutative diagram

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r] \arrow[d, "C(F)"'] & K(\mathcal{A}) \arrow[d, "K(F)"] \\ C(\mathcal{B}) \arrow[r] & K(\mathcal{B}). \end{tikzcd}$$

Definition 2.1.28. Let $Q_{\mathcal{A}}\colon K^+(\mathcal{A}) \to D^+(\mathcal{A})$ and $Q_{\mathcal{B}}\colon K^+(\mathcal{B}) \to D^+(\mathcal{B})$ be the localization functors. A **right derived functor** of F is a pair (RF, ϵ), where $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is a triangulated functor, and $\epsilon\colon Q_{\mathcal{B}}(K^+F) \to (RF)Q_{\mathcal{A}}$ is a natural transformation of triangulated functors, such that for every such pair (G, η), there exists a unique natural transformation of triangulated functors $\alpha\colon RF \to G$ such that $\eta = (\alpha Q_{\mathcal{A}})\epsilon$.

If RF exists, we put $R^nFK = H^nRFK \in \mathcal{B}$ for $K \in D^+(\mathcal{A})$ (sometimes called the **hypercohomology** of K with respect to RF). The functor $R^nF\colon \mathcal{A} \to \mathcal{B}$ is called the n-th right derived functor of F.

In the sequel, we will often abbreviate $C(F)$ and $K(F)$ to F. For F exact, we also let F denote the functor $D(\mathcal{A}) \to D(\mathcal{B})$ given by F.

To show the existence of the right derived functor, we need resolutions.

Theorem 2.1.29. *Let $\mathcal{J} \subseteq \mathcal{A}$ be a full additive subcategory. Assume that for every object X of $\mathcal{A}$, there exists a monomorphism $X \to Y$ with Y in $\mathcal{J}$.*

(1) For every $K \in C^{\geq n}(\mathcal{A})$, there exist $L \in C^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $f\colon K \to L$ such that $\tau^{\geq m}f$ is a monomorphism of complexes for each m.

(2) The functor $K^+(\mathcal{J}) \to D^+(\mathcal{A})$ induces an equivalence of triangulated categories
$$K^+(\mathcal{J})[S^{-1}] \to D^+(\mathcal{A}),$$
where S is the collection of quasi-isomorphisms in $K^+(\mathcal{J})$.

Part (2) follows from part (1) and a general result on localization of triangulated categories (omitted).

Proof of (1). It suffices to construct $L_m = (\cdots \to L^m \to 0 \to \cdots) \in C^{[n,m]}(\mathcal{J})$ and a morphism $f_m\colon K \to L_m$ of complexes for each m such that f^i_m and $K^i/B^iK \to L^i/B^iL$ are monomorphisms for each $i \leq m$, H^if_m is an isomorphism for each $i < m$, $L_m = \sigma^{\leq m}L_{m+1}$ and f_m equals the composite $K \xrightarrow{f_{m+1}} L_{m+1} \to L_m$. We proceed by induction on m. For $m < n$, we take $L_m = 0$. Given L_m, we construct L_{m+1} as follows. Form the pushout square

$$\begin{tikzcd} K^m/B^mK \arrow[r] \arrow[d] & L^m/B^mL \arrow[d] \\ K^{m+1} \arrow[r] & X. \end{tikzcd}$$

By induction hypothesis, the upper horizontal arrow is a monomorphism. It follows that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K^m/B^mK \arrow[r] \arrow[d] & L^m/B^mL \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K^{m+1} \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

with exact rows. By assumption, there exists a monomorphism $X \to L^{m+1}$ with L^{m+1} in $\mathcal{J}$. We define $f^{m+1}\colon K^{m+1} \to L^{m+1}$ and $d^m_L\colon L^m \to L^{m+1}$ by the obvious compositions. Then f_{m+1} is a morphism of complexes. It is clear that f^{m+1} is a monomorphism. Applying the snake lemma to the above diagram, we see that $K^{m+1}/B^{m+1}K \to L^{m+1}/B^{m+1}L$ is a monomorphism and $H^m f$ is an isomorphism. □

Definition 2.1.30. Let $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. A full additive subcategory $\mathcal{J} \subseteq \mathcal{A}$ is said to be ***F*-injective** if it satisfies the following conditions:

(a) For every $X \in \mathcal{A}$, there exists a monomorphism $X \to Y$ with $Y \in \mathcal{J}$.

(b) For every $L \in K^+(\mathcal{J})$ acyclic, FL is acyclic.

The terminology is not completely standard. Our definition here follows [KS, Definitions 10.3.2, 13.3.4]. Some authors replace (b) by the stronger condition (b′) below.

Proposition 2.1.31. *Condition (b′) below implies (b).*

(b′) Every monomorphism $X' \to X$ in $\mathcal{A}$ with $X', X \in \mathcal{J}$ can be completed into a short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$ with $X'' \in \mathcal{J}$ such that the sequence

$$0 \to FX' \to FX \to FX'' \to 0$$

is exact.

Proof. Let $L \in K^+(\mathcal{J})$ be an acyclic complex. Then L breaks into short exact sequences

$$0 \to Z^n L \to L^n \to Z^{n+1} L \to 0.$$

By (b), one shows by induction on n that $Z^n L$ is isomorphic to an object in $\mathcal{J}$ and we have short exact sequences

$$0 \to F(Z^n L) \to F(L^n) \to F(Z^{n+1}L) \to 0,$$

so that $K^+(F)(L)$ is acyclic. □

Corollary 2.1.32. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and let $\mathcal{J} \subseteq \mathcal{A}$ be an F-injective subcategory.*

(1) The right derived functor $(RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B}), \epsilon)$ of F exists and for $L \in K^+(\mathcal{J})$, $\epsilon_L\colon FL \xrightarrow{\sim} RFL$ is an isomorphism. Moreover, RF carries $D^{\geq n}(\mathcal{A})$ into $D^{\geq n}(\mathcal{B})$.

(2) If F is left exact, then the morphism $FX \to R^0FX$ is an isomorphism for all $X \in \mathcal{A}$.

Part (1) follows from the theorem.

Proof of (2). Choose a quasi-isomorphism $X \to L$ with $L \in K^{\geq 0}(\mathcal{J})$, corresponding to an exact sequence
$$0 \to X \to L^0 \to L^1 \to \cdots .$$
Applying F, we obtain an exact sequence
$$0 \to FX \to FL^0 \to FL^1.$$
Thus $R^0FX \simeq H^0FL \simeq FX$. □

Corollary 2.1.33. *Let $\mathcal{A}$ be an abelian category with enough injectives. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects.*

(1) For every $K \in C^{\geq n}(\mathcal{A})$, there exist $L \in C^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $f\colon K \to L$.

(2) The triangulated functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence of triangulated categories.

(3) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then $\mathcal{I}$ is F-injective. In particular, the right derived functor $(RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B}), \epsilon)$ of F exists and for $L \in K^+(\mathcal{I})$, $\epsilon_L\colon FL \xrightarrow{\sim} RFL$ is an isomorphism.

Proof. This follows from the theorem and Corollary 2.1.32(1). For (2), we need the following lemma. For (3), note that $\mathcal{I}$ satisfies conditions (a) and (b′). Indeed, any short exact sequence of injective objects splits. □

Lemma 2.1.34. *Let $\mathcal{A}$ be an abelian category. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects. Then any acyclic complex in $K^+(\mathcal{I})$ is isomorphic to zero in $K^+(\mathcal{I})$.*

Proof. Let $L \in K^+(\mathcal{I})$ be an acyclic complex. Then L breaks into short exact sequences
$$0 \to Z^nL \to L^n \to Z^{n+1}L \to 0.$$
One shows by induction on i that Z^nL is injective and the sequence splits. Thus L^n can be identified with $Z^n \oplus Z^{n+1}$. Then $h^n\colon Z^n \oplus Z^{n+1} \to Z^n \to Z^{n-1} \oplus Z^n$ satisfies $hd + dh = \mathrm{id}_X$. □

Remark 2.1.35. By the preceding corollary, if $\mathcal{A}$ has small Hom sets and admits enough injectives, then $D^+(\mathcal{A})$ has small Hom sets.

Proposition 2.1.36. *Let $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories. Let $\mathcal{J} \subseteq \mathcal{B}$ be a G-injective subcategory. Assume that $\mathcal{A}$ admits enough injectives and $FI \in \mathcal{J}$ for every injective object I of $\mathcal{A}$. Then the natural transformation $\eta_L\colon R(GF) \to (RG)(RF)$ given by the universal property of right derived functors is a natural isomorphism.*

This applies in particular to the case where $\mathcal{B}$ admits enough injectives and F preserves injectives.

Proof. Let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects. For $L \in K^+(\mathcal{I})$, the composite $(GF)L \xrightarrow{\epsilon_L} R(GF)L \xrightarrow{\eta_L} (RG)(RF)L$ and ϵ_L are both isomorphisms in $D^+(\mathcal{C})$, and hence so is η_L. □

2.2 Derived direct image

Proposition 2.2.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then* $\mathrm{Shv}(X, \mathcal{O}_X)$ *admits enough injectives.*

Proof. Let $\mathcal{F} \in \mathrm{Shv}(X, \mathcal{O}_X)$. For $x \in X$, let $i_x\colon \{x\} \to X$ be the inclusion. Let us show that the canonical morphism $\mathcal{F} \to \prod_{x\in X} i_{x*}i_x^{-1}\mathcal{F}$ is a monomorphism. For every $y \in X$, we have a commutative diagram

$$\begin{tikzcd} i_y^{-1}\mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dr] & i_y^{-1}\prod_{x\in X} i_{x*}i_x^{-1}\mathcal{F} \arrow[d] \\ i_y^{-1}\mathcal{F} & i_y^{-1}i_{y*}i_y^{-1}\mathcal{F} \arrow[l] \end{tikzcd}$$

It follows that the top horizontal arrow is injective at every stalk, and hence a monomorphism.

Each $i_x^{-1}\mathcal{F}$ is an $\mathcal{O}_{X,x}$-module and can be embedded into an injective $\mathcal{O}_{X,x}$-module $i_x^{-1}\mathcal{F} \hookrightarrow \mathcal{I}_x$. Then $\mathcal{F} \hookrightarrow \prod_{x\in X} i_{x*}i_x^{-1}\mathcal{F} \hookrightarrow \prod_{x\in X} i_{x*}I_x$. Note that $i_{x*}\mathcal{I}_x$ is injective by the next lemma applied to the adjoint functors $i_x^{-1} \vdash i_{x*}$ with i_x^{-1} exact. We conclude by the fact that a product of injective sheaves is injective. □

Lemma 2.2.2. *Let* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *be functors between abelian categories with $\mathcal{F} \dashv G$ and F exact. For $X \in \mathrm{Ob}(\mathcal{B})$ injective, $G(X)$ is injective.*

Proof. In fact, $\mathrm{Hom}_{\mathcal{A}}(-, G(X)) \simeq \mathrm{Hom}_{\mathcal{B}}(F-, X)$ is exact. □

Example 2.2.3. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a flat morphism of ringed spaces. Then f^{-1} is exact. It follows that f_* sends injective sheaves to injective sheaves.

Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor $f_*\colon \mathrm{Shv}(X, \mathcal{O}_X) \to \mathrm{Shv}(Y, \mathcal{O}_Y)$ is left exact. It follows from the proposition that f_* admits a right derived functor Rf_*.

Example 2.2.4. $Y = \mathrm{pt}$, $\mathcal{O}_Y = \mathbb{Z}$. Then

$$f_* = \Gamma(X, -)\colon \mathrm{Shv}(X, \mathcal{O}_X) \to \mathrm{Ab}$$
$$Rf_* = \mathrm{R}\Gamma(X, -)\colon D^+(X, \mathcal{O}_X) \to D^+(\mathrm{Ab}).$$

For $L \in D^+(X, \mathcal{O}_X)$, we call $\mathrm{H}^n(X, L) := \mathrm{R}^n\Gamma(X, -)$ the i-th (hyper)cohomology of L.

If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a flat morphism of ringed spaces, then for $M \in D^+(Y, \mathcal{O}_Y)$, there is a natural restriction morphism $\mathrm{R}\Gamma(Y, M) \to \mathrm{R}\Gamma(X, f^*M)$.

Proposition 2.2.5. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $L \in D^+(X, \mathcal{O}_X)$. Then there is a canonical isomorphism*

$$\mathrm{R}^i f_* L \simeq a(V \mapsto \mathrm{H}^i(f^{-1}(V), L|_{f^{-1}(V)})$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Shv}(X,\mathcal{O}_X) \arrow[r, "\iota"] \arrow[d, "f_*"] & \mathrm{PShv}(X,\mathcal{O}_X) \arrow[d, "f_*^{\mathrm{psh}}"] \\ \mathrm{Shv}(Y,\mathcal{O}_Y) & \mathrm{PShv}(Y,\mathcal{O}_Y) \arrow[l, "a"] \end{tikzcd}$$

Since a and f_*^{psh} are exact, we have $\mathrm{R}f_* = af_*^{\mathrm{psh}}\mathrm{R}\iota$. We conclude by the next lemma, which computes $\mathrm{R}\iota$. ☐

Lemma 2.2.6. *For $L \in D^+(X,\mathcal{O}_X)$, $\mathrm{R}^i\iota L\colon U \mapsto \mathrm{H}^i(U, L|_U)$,*

Proof. Let $L \to I$ be a quasi-isomorphism with $I \in K^+(X,\mathcal{O}_X)$ and I^i injective for all i. Then $R^i\iota L \simeq \mathrm{H}^n(\iota I)$. Since $\Gamma(U,-)$ is exact on presheaves, we have $(\mathrm{H}^i\iota I)(U) \simeq \mathrm{H}^i(I(U)) \simeq \mathrm{H}^i(U, L|_U)$. In the last isomorphism we used the fact that $L|_U \to I|_U$ is a quasi-isomorphism and $I|_U^i$ is injective for all i. To see this last point, let $j\colon U \to X$ be the inclusion. Then j^* preserves injectives, because it has an exact left adjoint, $j_!$, and Lemma 2.2.2 applies. ☐

Flabby sheaves

Definition 2.2.7. A sheaf $\mathcal{F}$ on X is said to be **flabby** if for all $U \subseteq X$ open, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Remark 2.2.8. If $\mathcal{F}$ is flabby, then for any inclusion $V \subseteq U$ of opens, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. In other words, if $\mathcal{F}$ is flabby, then $\mathcal{F}|_U$ is flabby for every open $U \subseteq X$.

Proposition 2.2.9. *Consider a short exact sequence in* $\mathrm{Shv}(X,\mathcal{O}_X)$*:*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0.$$

(1) If $\mathcal{F}'$ is flabby, then $0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$ *is exact.*

(2) If $\mathcal{F}'$ and $\mathcal{F}$ are flabby, then so is $\mathcal{F}''$.

(3) Injective $\mathcal{O}_X$-modules are flabby.

Proof. (1) Take $s \in \mathcal{F}''(X)$. Consider

$$\Omega = \left\{ (U,t) \;\middle|\; \begin{array}{l} U \subseteq X \text{ open} \\ t \in \mathcal{F}(U), \quad \psi(t) = s|_U \end{array} \right\}$$

Define a partial order on Ω by $(U,t) \le (U',t')$ if $U \subseteq U'$ and $t'|_U = t$. By Zorn's lemma, there exists a maximal element (U,t) of Ω. If $U = X$, we are done. Assume $U \subsetneq X$. Take $x \in X \backslash U$. Since $\mathcal{F} \to \mathcal{F}''$ is surjective, there exists an open neighborhood $V \ni x$ and $r \in \mathcal{F}(V)$ such that $\psi(r) = s|_V$. Since $\psi(t|_{U\cap V}) = \psi(r|_{U\cap V}) = s|_{U\cap V}$, there exists $v \in \mathcal{F}'(U \cap V)$ such that $t|_{U\cap V} - r|_{U\cap V} = \phi(v)$. Since $\mathcal{F}'$ is flabby, there exists $\bar{v} \in \mathcal{F}'(V)$ such that $\bar{v}|_{U\cap V} = v$. By construction, $t \in \mathcal{F}(U)$ and $r + \phi(\bar{v}) \in \mathcal{F}(V)$ agree on $U \cap V$. Thus they define $\bar{t} \in \mathcal{F}(U \cup V)$ such that $\bar{t}|_U = t$, $\bar{t}|_V = r + \phi(\bar{v})$. Clearly $\psi(\bar{t}) = s|_{U\cup V}$. This shows $(U \cup V, \bar{t}) \in \Omega$ and contradicts the maximality of (U, t).

(2) Consider the commutative diagram:

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d, two heads] & \mathcal{F}''(X) \arrow[d] \\ \mathcal{F}(U) \arrow[r, two heads] & \mathcal{F}''(U) \end{tikzcd}$$

The left vertical arrow is surjective since $\mathcal{F}$ is flabby. The bottom horizontal arrow is surjective by (1). It follows that the right vertical arrow is surjective.

(3) Let $U \subseteq X$ be an open subset and let $j \colon U \to X$ be the inclusion. Since $j_!\mathcal{O}_U \hookrightarrow \mathcal{O}_X$ is a monomorphism and $\mathcal{F}$ is injective, $\mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) \to \mathrm{Hom}(j_!\mathcal{O}_U, \mathcal{F})$ is surjective. This can be identified with the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ via the adjunction $\mathrm{Hom}(j_!\mathcal{O}_U, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{O}_U, j^{-1}\mathcal{F}) \simeq \mathcal{F}(U)$. □

Corollary 2.2.10. *Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then the full subcategory consisting of the flabby $\mathcal{O}_X$-modules is f_*-injective (Definition 2.1.30).*

Proof. Condition (b′) of Proposition 2.1.31 follows from (1) and (2). Condition (a) Definition 2.1.30 follows from (3) and the existence of enough injectives. One can give a more direct proof of (a). For any $\mathcal{O}_X$-module $\mathcal{F}$, we have $\mathcal{F} \hookrightarrow \mathcal{G} = \prod_{x\in X} i_{x*}i_x^{-1}\mathcal{F}$, where $\mathcal{G}$ is flabby because $\mathcal{G}(U) = \prod_{x\in U} i_{x*}i_x^{-1}\mathcal{F}$. □

Remark 2.2.11. It is clear that f_* sends flabby sheaves to flabby sheaves.

Corollary 2.2.12. *Let $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then $R(gf)_*L \simeq Rg_*Rf_*L$ for all $L \in D^+(X, \mathcal{O}_X)$.*

Example 2.2.13. Consider the commutative diagram

$$\begin{tikzcd} (X, \mathbb{Z}_X) \arrow[d, "f_0"] & (X, \mathcal{O}_X) \arrow[l, "\mu_X"'] \arrow[d, "f"] \\ (Y, \mathbb{Z}_X) & (Y, \mathcal{O}_Y) \arrow[l, "\mu_Y"'] \end{tikzcd}$$

Then $\mu_{Y*}Rf_* \simeq Rf_{0*}\mu_{X*}$. Here μ_{X*} is the functor forgetting the $\mathcal{O}_X$-module structure. In other words, the functor Rf_* does not depend on sheaf of rings.

Theorem 2.2.14 (Grothendieck)**.** *Let X be a Noetherian topological space of finite dimension d. Then for any abelian sheaf $\mathcal{F}$ on X, $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > d$.*

Remark 2.2.15. By a result of Spaltenstein [S], the derived functor $Rf_* \colon D(X, \mathcal{O}_X) \to D(Y, \mathcal{O}_Y)$ between unbounded derived categories exists. We refer to [KS, Chapters 14, 18] for more details.

2.3 Čech cohomology

Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X and let $\{U_i\}_I = \mathfrak{U}$ be an open cover. The sheaf condition is the exactness of the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \prod_i \mathcal{F}(U_i) \arrow[r] & \prod_{i,j} \mathcal{F}(U_{ij}) \\ & & (s_i) \arrow[r, mapsto] & s_i|_{U_{ij}} - s_j|_{U_{ij}} \end{tikzcd}$$

where $U_{ij} = U_i \cap U_j$. In Čech cohomology, we extend this sequence to the right.

Definition 2.3.1 (Čech complex)**.** Let $\mathcal{F}$ be a presheaf. The **Čech complex** $C^\bullet(\mathcal{U}, \mathcal{F}) \in C^{\geq 0}(\mathrm{Ab})$ is defined by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0,\ldots,i_p}), \quad U_{i_0,\ldots,i_p} = \bigcap_{k=0}^{p} U_{i_k}$$

with the differential given by

$$(d^p s)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_0,\ldots,\widehat{i_k},\ldots,i_p}|_{U_{i_0,\ldots,i_{p+1}}}$$

for $s \in C^p((U), \mathcal{F})$. One can check $dd = 0$. We call $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = \mathrm{H}^p C^\bullet(\mathcal{U}, \mathcal{F})$ the **Čech cohomology**.

Remark 2.3.2. The global section functor factors as

$$\begin{tikzcd} \mathrm{Shv}(X) \arrow[r, "\iota"] \arrow[rr, bend right, "\Gamma(X,-)"'] & \mathrm{PShv}(X) \arrow[r, "\check{\mathrm{H}}^0(\mathfrak{U},-)"] & \mathrm{Ab} \end{tikzcd}$$

We will show that $\check{\mathrm{H}}^p(\mathfrak{U}, -)$ are the right derived functors of $\check{\mathrm{H}}^0(\mathfrak{U}, -)$.

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Let X be a topological space. Recall for an open cover $\mathcal{U} = \{U_i\}$ of X and an abelian presheaf $\mathcal{F}$, we have defined the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$. We can extend it to a Čech complex of presheaves $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \in C^{\geqslant 0}(\mathrm{PShv}(X))$, $\Gamma(V, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})) = C^\bullet(\mathcal{U} \cap V, \mathcal{F})$, where $\mathcal{U} \cap V = \{U_i \cap V\}$. In other words,

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}} j_{i_0,\dots i_p *} j^{-1}_{i_0,\dots i_p} \mathcal{F}, \qquad j_{i_0,\dots i_p} \colon U_{i_0,\dots i_p} \hookrightarrow X.$$

Let $f \colon \coprod_i U_i \to X$. Then $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \mathcal{C}^0(\mathcal{U}, \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})) = \underbrace{f_* f^{-1} \cdots f_* f^{-1}}_{p+1} \mathcal{F}$, where $f_* f^{-1}$ appears $p+1$ in the expression.

Proposition 2.3.3. *Let $\mathcal{F}$ be a sheaf on X. Then*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

is exact in $\mathrm{Shv}(X)$.

We will prove a more general form of the proposition.

Lemma 2.3.4. *Let $F \colon \mathcal{A} \to \mathcal{B}$ a functor between abelian categories admitting a right adjoint G. Let $A \in \mathcal{A}$. Consider the complex $L \in C^{\geq -1}(\mathcal{A})$:*

$$0 \longrightarrow A \xrightarrow{d^{-1}} GFA \xrightarrow{d^0} GFGFA \xrightarrow{d^1} \cdots$$

where

$$L^p = \underbrace{GF \cdots GF}_{p+1} A$$
$$d^p = \sum_{k=0}^{p+1} (-1)^k \underbrace{GF \cdots GF}_{k} \epsilon \underbrace{GF \cdots GF}_{p+1-k} A$$

where $\epsilon \colon \mathrm{id} \to GF$ is the unit. Then $FL = 0$ in $K(\mathcal{B})$.

Proof. Define $h \in \mathrm{Ht}(FL, FL)$ as follows. Let $\eta \colon FG \to \mathrm{id}$ be the counit. We take

$$h^p = \eta F \underbrace{GF \cdots GF}_{p} A \colon FL^p \to FL^{p-1}$$

One checks that $dh + hd = \mathrm{id}$. $\square$

Proposition 2.3.5. *Let $f \colon Y \to X$ be a surjective continuous map and let $\mathcal{F}$ be a sheaf on X. Define $\mathcal{C}^p(f, \mathcal{F}) := \underbrace{f_* f^{-1} \cdots f_* f^{-1}}_{p+1} \mathcal{F}$. Then*

$$0 \to \mathcal{F} \to \mathcal{C}^0(f, \mathcal{F}) \to \mathcal{C}^1(f, \mathcal{F}) \to \dots$$

is an exact sequence.

Proof. Take $F = f^{-1}$, $G = f_*$ in the lemma. Then $f^{-1}(L)$ is acyclic. Since f is surjective, L is acyclic. $\square$

In fact, in the lemma, L is acyclic if F is conservative.

Example 2.3.6. Let $f\colon Y = \coprod_{x\in X} x \to X$. Then $f^{-1}f_*\mathcal{F} = \prod_{x\in X} i_{x*}i_x^{-1}\mathcal{F}$. This is used in the proof of the existence of enough flabby sheaves. The flabby resolution given by the proposition is called **Godement resolution**. Note that every sheaf on Y is flabby and f_* preserves flabby sheaves.

Corollary 2.3.7. *Let $\mathcal{F}$ be a flabby sheaf on X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$ and open cover $\mathcal{U}$ of X.*

Proof. Let $f\colon \coprod_i U_i \to X$. Since f_* and f^{-1} both preserve flabby sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U},\mathcal{F}) \longrightarrow \dots$$

is a flabby resolution of $\mathcal{F}$. Taking global sections, we get the exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow C^0(\mathcal{U},\mathcal{F}) \longrightarrow C^1(\mathcal{U},\mathcal{F}) \longrightarrow \dots.$$

$\square$

Let $\mathcal{F}$ be a sheaf. Then we can replace $\mathcal{F}$ by $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}))$ in $\mathcal{D}(\mathrm{Shv}(X))$. In general, $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ is not flabby. We can choose a quasi-isomorphism $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \to L^\bullet$ with $L \in K^+$ and L^p injective or flabby for all p. This gives a canonical homomorphism

$$\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F}),$$

which is an isomorphism for $p = 0$. We have the following criterion for the map to be an isomorphism.

Theorem 2.3.8 (Leray)**.** *Let $\mathcal{F}$ be a sheaf. Assume $H^n(U_{i_0,\dots,i_p},\mathcal{F}) = 0$ for all $p \geqslant 0$, $(i_0,\dots,i_p) \in I^{p+1}$, $n \geqslant 1$. Then the canonical map $\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$ is an isomorphism for all $p \geqslant 0$.*

We will give a proof based on an interpretation of the Čech cohomology as derived Hom. For this we need more homological algebra.

Double complexes

Let $\mathcal{A}$ be an additive category.

Definition 2.3.9. We define the category of **double complexes** in $\mathcal{A}$ to be $C^2(\mathcal{A}) = C(C(\mathcal{A}))$. Thus a double complex consists of objects $X^{i,j}$ for $i,j \in \mathbb{Z}$ and differentials $d_I\colon X^{i,j} \to X^{i+1,j}$, $d_{II}\colon X^{i,j} \to X^{i,j+1}$ such that $d_I^2 = 0$, $d_{II}^2 = 0$, $d_I d_{II} = d_{II} d_I$.

Definition 2.3.10. Let X be a double complex in $\mathcal{A}$. We define two complexes in $\mathcal{A}$ with $(\mathrm{tot}_\oplus X)^n = \oplus_{i+j=n} X^{i,j}$ (if the coproducts exist) and $(\mathrm{tot}_\Pi X)^n = \prod_{i+j=n} X^{i,j}$ (if the products exist), called **total complex** of X with respect to coproducts and products, respectively. The differentials are defined as follows. Let $i+j=n$. The composition $X^{i,j} \to (\mathrm{tot}_\oplus X)^n \xrightarrow{d^n} (\mathrm{tot}_\oplus X)^{n+1}$ is given by

$$d_I^{i,j} + (-1)^i d_{II}^{i,j}. \tag{2.3.1}$$

The composition $(\mathrm{tot}_\Pi X)^{n-1} \xrightarrow{d^{n-1}} (\mathrm{tot}_\Pi X)^n \to X^{i,j}$ is given by

$$d_I^{i-1,j} + (-1)^i d_{II}^{i,j-1}. \tag{2.3.2}$$

Remark 2.3.11. The sign in (2.3.1) and (2.3.2) ensures that $d^2 = 0$. If Y is the transpose of X defined by $Y^{i,j} = X^{j,i}$ and by swapping the two differentials, then we have an isomorphism $\mathrm{tot}_\oplus X \simeq \mathrm{tot}_\oplus Y$ given by $(-1)^{ij}\mathrm{id}_{X^{i,j}}$. The same holds for tot_Π.

In the literature, a variant of Definition 2.3.9 with $d_I d_{II} + d_{II} d_I = 0$ is sometimes used. If we adopt this variant, then (2.3.1) can be simplified to $d = d_I + d_{II}$. The two definitions correspond to each other by multiplying $d_{II}^{i,j}$ by the sign $(-1)^i$.

Definition 2.3.12. We say that a double complex X is **biregular** if for every n, $X^{i,j} = 0$ for all but finitely many pairs (i,j) with $i+j=n$. We let $C^2_{\mathrm{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A})$ denote full subcategory consisting of biregular double complexes. It is an additive subcategory.

If $X^{i,j} = 0$ for $i < a$ or $j < b$ (X concentrated in a (translated) first quadrant) or $X^{i,j} = 0$ for $i > a$ or $j > b$ (X concentrated in a (translated) third quadrant), then X is biregular. If $X^{i,j} = 0$ for $|i| \gg 0$ (concentrated in a vertical stripe) or $X^{i,j} = 0$ for $|j| \gg 0$ (concentrated in a horizontal stripe), then X is biregular.

Remark 2.3.13. If X is a biregular double complex, then $\mathrm{tot}_\oplus X$ and $\mathrm{tot}_\Pi X$ exist and we have $\mathrm{tot}_\oplus X \xrightarrow{\sim} \mathrm{tot}_\Pi X$. We will simply write $\mathrm{tot}X$. We get an additive functor $\mathrm{tot}\colon C^2_{\mathrm{reg}}(\mathcal{A}) \to C(\mathcal{A})$.

Example 2.3.14. Let $f\colon L \to M$ be a morphism of complexes in $\mathcal{A}$. We define a double complex X by $X^{-1,j} = L^j$, $X^{0,j} = M^j$, $X^{i,j} = 0$ for $i \neq -1, 0$, $d_I^{-1,j} = f^j$, d_{II} given by d_L and d_M. Then $\mathrm{tot}X = \mathrm{Cone}(f)$.

Let $\mathcal{A}$ be an abelian category. For a double complex X in $\mathcal{A}$, we put

$$H_I(X)^{i,j} = \mathrm{Ker}(d_I^{i,j})/\mathrm{im}(d_I^{i-1,j}), \quad H_{II}(X)^{i,j} = \mathrm{Ker}(d_{II}^{i,j})/\mathrm{im}(d_{II}^{i,j-1}).$$

The full additive subcategory $C^2_{\mathrm{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A})$ is stable under subobjects and quotients. Thus $C^2_{\mathrm{reg}}(\mathcal{A})$ is an abelian category and the inclusion functor is exact. The functor $\mathrm{tot}\colon C^2_{\mathrm{reg}}(\mathcal{A}) \to C(\mathcal{A})$ is exact.

Proposition 2.3.15. *Let X be a biregular double complex such that $H_I^{i,\bullet}(X)$ is acyclic for every i. Then* $\mathrm{tot}X$ *is acyclic.*

A similar statement holds for H_{II}, which generalizes the fact that the cone of a quasi-isomorphism is acyclic.

Proof. For each m, there exists N such that $H^m(\mathrm{tot}X) = H^m\mathrm{tot}(\tau_I^{\leq n}X)$ for all $n \geq N$. It suffices to show that $H^m\mathrm{tot}(\tau_I^{\leq n}X) = 0$ for all n. We proceed by induction on n (for a fixed m). For $n \ll 0$, $(\mathrm{tot}(\tau_I^{\leq n}X))^m = 0$. Assume that $H^m\tau_I^{\leq n-1}(X) = 0$ and consider the short exact sequence of double complexes

$$0 \to \tau_I^{\leq n-1}X \to \tau_I^{\leq n}X \to Y \to 0,$$

where $Y = (B_I^{n,\bullet}X \xrightarrow{f} Z_I^{n,\bullet}X)$ is concentrated on the columns $n-1$ and n. Applying tot, we get an exact sequence of complexes

$$0 \to \mathrm{tot}\tau_I^{\leq n-1}X \to \mathrm{tot}\tau_I^{\leq n}X \to \mathrm{tot}Y \to 0.$$

We have a quasi-isomorphism $\mathrm{tot}(Y)[n] \simeq \mathrm{Cone}((-1)^n f) \to H_I^{n,\bullet}(X)$. It follows $\mathrm{tot}Y$ is acyclic. Taking long exact sequence, we get

$$H^m\mathrm{tot}\tau_I^{\leq n}X \simeq H^m\mathrm{tot}\tau_I^{\leq n-1}X = 0.$$

□

Corollary 2.3.16. *Let X be a biregular double complex such that $X^{\bullet,j}$ is acyclic for every j (namely, every row of X is acyclic). Then* $\mathrm{tot}X$ *is acyclic.*

A similar statement holds for columns of X: if $X^{i,\bullet}$ is acyclic for every i, then $\mathrm{tot}X$ is acyclic.

Corollary 2.3.17. *Let $f\colon X \to Y$ be a morphism of biregular double complexes such that $H_I^{i,\bullet}(f)\colon H_I^{i,\bullet}(X) \to H_I^{i,\bullet}(Y)$ is a quasi-isomorphism for each i. Then* $\mathrm{tot}(f)\colon \mathrm{tot}(X) \to \mathrm{tot}(Y)$ *is a quasi-isomorphism.*

Proof. We let $W = \mathrm{Cone}_{II}(f)$ with $W^{i,j} = X^{i,j+1} \oplus Y^{i,j}$. Then $H_I^{i,\bullet}(W) \simeq \mathrm{Cone}(H_I^{i,\bullet}(f))$ is acyclic. By the proposition applied to W, $\mathrm{tot}(W) \simeq \mathrm{Cone}(\mathrm{tot}(f))$ is acyclic. □

Corollary 2.3.18. *Let $f\colon X \to Y$ be a morphism of biregular double complexes such that $f^{\bullet,j}\colon X^{\bullet,j} \to Y^{\bullet,j}$ is a quasi-isomorphism for each j. Then* $\mathrm{tot}(f)\colon \mathrm{tot}(X) \to \mathrm{tot}(Y)$ *is a quasi-isomorphism.*

Derived Hom

Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be additive categories. Let $F\colon \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ be a functor that is additive in each variable. Then F extends to a functor $C^2(F)\colon C(\mathcal{A}) \times C(\mathcal{A}') \to C^2(\mathcal{A}'')$ additive in each variable. For $X \in C(\mathcal{A})$, $Y \in C(\mathcal{A}')$, the double complex $C^2(F)(X,Y)$ is defined by $C^2(F)(X,Y)^{i,j} = F(X^i, Y^j)$, with $d_I^{i,j} = F(d_X^i, \mathrm{id}_{Y^j})$, $d_{II}^{i,j} = F(\mathrm{id}_{X^i}, d_Y^j)$.

Example 2.3.19. Let $\mathcal{A}$ be an additive category with small Hom sets. The functor $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$ is additive in each variable. We have an isomorphism $C(\mathcal{A})^{\mathrm{op}} \simeq C(\mathcal{A}^{\mathrm{op}})$, carrying (X,d) to $((X^{-n}), (-1)^n d^{-n-1})$. Thus $\mathrm{Hom}_{\mathcal{A}}$ extends to a functor

$$\mathrm{Hom}_{\mathcal{A}}^{\bullet\bullet}\colon C(\mathcal{A})^{\mathrm{op}} \times C(\mathcal{A}) \to C^2(\mathrm{Ab}),$$

additive in each variable. For $X, Y \in C(\mathcal{A})$, $\operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}(X,Y)^{i,j} = \operatorname{Hom}_{\mathcal{A}}(X^{-j}, Y^i)$, with

$$d_I^{i,j} = \operatorname{Hom}_{\mathcal{A}}(X^{-j}, d_Y^i), \quad d_{II}^{i,j} = \operatorname{Hom}_{\mathcal{A}}((-1)^j d_X^{-j-1}, Y^i).$$

We define $\operatorname{Hom}^\bullet_{\mathcal{A}}$ as the composite functor

$$C(\mathcal{A})^{\mathrm{op}} \times C(\mathcal{A}) \xrightarrow{\operatorname{Hom}^{\bullet\bullet}_{\mathcal{A}}} C^2(\mathrm{Ab}) \xrightarrow{\mathrm{tot}_\Pi} C(\mathrm{Ab}).$$

We have

$$\operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y)^n = \prod_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^j, Y^{n+j}),$$

and for $f = (f^j) \in \operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y)^n$,

$$(d^n f)^j = d_Y^{j+n} f^j + (-1)^{n+1} f^{j+1} d_X^j.$$

Proposition 2.3.20. *We have*

$$\begin{aligned} Z^0 \operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y) &\simeq \operatorname{Hom}_{C(\mathcal{A})}(X,Y), \\ B^0 \operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y) &\simeq \operatorname{im}(\operatorname{Ht}(X,Y) \to \operatorname{Hom}_{C(\mathcal{A})}(X,Y)), \\ H^0 \operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y) &\simeq \operatorname{Hom}_{K(\mathcal{A})}(X,Y). \end{aligned}$$

Proof. We have $d^0(f) = df - fd$, so that $d^0(f) = 0$ if and only if $f\colon X \to Y$ is a morphism of complexes. We have $\operatorname{Ht}(X,Y) = \operatorname{Hom}^\bullet_{\mathcal{A}}(X,Y)^{-1}$, and for $h \in \operatorname{Ht}(X,Y)$, $d^{-1}(h) = dh + hd$. □

Definition 2.3.21. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be triangulated categories. A **triangulated bifunctor** is a functor $F\colon \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ equipped with natural isomorphisms $F(X[1], Y) \simeq F(X,Y)[1]$, $F(X, Y[1]) \simeq F(X,Y)[1]$, such that the following diagram anticommutes

$$\begin{tikzcd} F(X[1],Y[1]) \arrow[r] \arrow[d] & F(X,Y[1])[1] \arrow[d] \\ F(X[1],Y)[1] \arrow[r] & F(X,Y)[2] \end{tikzcd}$$

and such that F is triangulated in each variable.

Note that $\operatorname{Hom}^\bullet$ factorizes through a triangulated bifunctor $K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathrm{Ab})$.

Proposition 2.3.22. *Assume that $\mathcal{A}$ admits enough injectives. Then the triangulated bifunctor*

$$\operatorname{Hom}^\bullet_{\mathcal{A}}\colon K(\mathcal{A})^{\mathrm{op}} \times K^+(\mathcal{A}) \to K(\mathrm{Ab})$$

admits a right derived bifunctor

$$R\operatorname{Hom}_{\mathcal{A}}\colon D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathrm{Ab})$$

such that, for $M \in K^+(\mathcal{A})$ with injective components and $L \in K(\mathcal{A})$, we have

$$\operatorname{Hom}^\bullet_{\mathcal{A}}(L,M) \xrightarrow{\sim} R\operatorname{Hom}_{\mathcal{A}}(L,M).$$

Sketch of proof. We need to show that for $L \in K(\mathcal{A})$, $M \in K^+(\mathcal{A})$, M^n injective for all n, with L or M acyclic, then $\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M)$ is acyclic. Indeed,

$$H^n \mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \simeq \mathrm{Hom}_{K(\mathcal{A})}(L, M[n]) \simeq \mathrm{Hom}_{D(\mathcal{A})}(L, M[n]) = 0.$$

□

Remark 2.3.23. Assume that $\mathcal{A}$ has enough injectives. For $L \in D(\mathcal{A})$, $M \in D^+(\mathcal{A})$, we have

$$H^n R\,\mathrm{Hom}_{\mathcal{A}}(L, M) \simeq H^n \mathrm{Hom}^\bullet_{\mathcal{A}}(L, M') \simeq \mathrm{Hom}_{K(\mathcal{A})}(L, M'[n]) \simeq \mathrm{Hom}^n(L, M[n]),$$

where we have taken a quasi-isomorphism $M \to M' \in K^+(\mathcal{A})$ such that M' has injective components. In particular, for $X \in \mathcal{A}$, $\mathrm{Hom}_{D(\mathcal{A})}(X, -[n])$ is the n-th right derived functor of $\mathrm{Hom}(X, -)$.

Dually, we have the following.

Proposition 2.3.24. *Assume that $\mathcal{A}$ admits enough projectives. Then the triangulated bifunctor*

$$\mathrm{Hom}^\bullet_{\mathcal{A}} \colon K^-(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathrm{Ab}).$$

admits a right derived bifunctor

$$R\,\mathrm{Hom}_{\mathcal{A}} \colon D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathrm{Ab})$$

such that for $L \in K^-(\mathcal{A})$ with projective components and $M \in K(\mathcal{A})$, we have

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \xrightarrow{\sim} R\,\mathrm{Hom}_{\mathcal{A}}(L, M).$$

Remark 2.3.25. In the case where $\mathcal{A}$ admits enough injectives and enough projectives, the functors $R\,\mathrm{Hom}$ defined in Propositions 2.3.22 and 2.3.24 are isomorphic when restricted to $D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A})$. Indeed, for $L \in D^-(\mathcal{A})$ and $M \in D^+(\mathcal{A})$, $R\,\mathrm{Hom}(L, M)$ can be computed by finding quasi-isomorphisms $L' \to L$ and $M \to M'$ such that L' has projective components and M' has injective components and taking $\mathrm{Hom}^\bullet(L, M)$.

Back to Čech cohomology

Let $\mathcal{F}$ be a presheaf. We have

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0,\dots i_p}) \simeq \mathrm{Hom}(\mathcal{C}_p(\mathcal{U}), \mathcal{F}),$$

where

$$\mathcal{C}_p(\mathcal{U}) = \bigoplus_{i_0,\dots,i_p\in I^{p+1}}^{\mathrm{psh}} j^{\mathrm{psh}}_{i_0,\dots i_p!}\mathbb{Z}^{\mathrm{psh}}_{U_{i_0,\dots,i_p}}, \qquad j_{i_0,\dots i_p} \colon U_{i_0,\dots i_p} \hookrightarrow X.$$

Here $\mathbb{Z}^{\mathrm{psh}}_{U_{i_0,\dots,i_p}}$ denotes the constant presheaf. There is a complex $\mathcal{C}_\bullet(\mathcal{U})$ in $C^{\le 0}$ with $\mathcal{C}_\bullet(\mathcal{U})^{-p} = \mathcal{C}_p(\mathcal{U})$ satisfying $C^\bullet(\mathcal{U}, \mathcal{F}) \simeq \mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), \mathcal{F})$. To specify the differentials and to study this complex, it is convenient to consider the functor

$$f^{\mathrm{psh}}_! \mathcal{G} = \bigoplus_{i\in I} j^{\mathrm{psh}}_{i!}(\mathcal{G}|_{U_i})$$

between categories of presheaves, where $f\colon \coprod_{i\in I} U_i \to X$. This functor is left adjoint to f^{-1}, and we have the counit $\eta_{\mathcal{U}}\colon f_!^{\mathrm{psh}} f^{-1} \to \mathrm{id}$. Note that $(f_!^{\mathrm{psh}} f^{-1}\mathcal{F})(U) \simeq \bigoplus_{U\subseteq U_i} \mathcal{F}(U_i)$. Moreover,

$$\mathcal{C}_p(\mathcal{U}) \simeq \underbrace{f_!^{\mathrm{psh}} f^{-1} \dots f_!^{\mathrm{psh}} f^{-1}}_{p+1} \mathbb{Z}_X^{\mathrm{psh}}.$$

We define the differentials of $\mathcal{C}_\bullet(\mathcal{U})$ by

$$d^{-p} = \sum_{k=0}^{p} (-1)^{k+1} \underbrace{f_!\mathrm{psh} f^{-1} \dots f_!^{\mathrm{psh}} f^{-1}}_{k} \eta_{\mathcal{U}} \underbrace{f_!\mathrm{psh} f^{-1} \dots f_!^{\mathrm{psh}} f^{-1}}_{p-k}.$$

Lemma 2.3.26. *(1) The sequence*

$$(*) \qquad \cdots \longrightarrow \mathcal{C}_1(\mathcal{U}) \longrightarrow \mathcal{C}_0(\mathcal{U}) \xrightarrow{\eta_{\mathcal{U}}} \mathbb{Z}_X^{\mathrm{psh}}$$

is exact.

(2) $\mathcal{C}_p(\mathcal{U})$ is projective for each $p \geqslant 0$.

Proof. (2) $\mathrm{Hom}(\mathbb{Z}_X^{\mathrm{psh}}, -) = \Gamma(X, -)$ is exact, which implies that $\mathbb{Z}_X^{\mathrm{psh}}$ is projective. Moreover, since $f_!^{\mathrm{psh}} \dashv f^{-1} \dashv f_*$ and $f_!$ and f_* are exact, the functors $f_!^{\mathrm{psh}}$ and f^{-1} preserve projectives.

(1) $f^{-1}(*)$ is exact by Lemma 2.3.4. Thus $(*)|_{U_i}$ is exact for every $i \in I$. Let $U \subseteq X$ be an open subset. If there exists an $i \in I$ such that $U \subseteq U_i$, then $\Gamma(U, (*))$ is exact. Otherwise, $\Gamma(U,(*)) = (\ \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}\ )$, which is exact. □

From the Lemma, we see $\mathcal{C}_\bullet(\mathcal{U})$ is a projective resolution of $\mathrm{im}(\eta_{\mathcal{U}})$, and hence

$$C^\bullet(\mathcal{U}, \mathcal{F}) \simeq \mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), \mathcal{F}) \simeq R\,\mathrm{Hom}(\mathrm{Im}(\eta_{\mathcal{U}}), \mathcal{F}) \simeq R\check{\Gamma}(\mathcal{U}, \mathcal{F}),$$

where $R\check{\Gamma}(\mathcal{U}, -)$ denotes the right derived functor of $\check{H}^0(\mathcal{U}, -)$. For the last isomorphism we note

$$\mathrm{Hom}(\mathrm{im}(\eta_{\mathcal{U}}), \mathcal{F}) \simeq \check{H}^0(\mathcal{U}, \mathcal{F}),$$

which implies that for $L \in D^+(\mathrm{PShv}(X))$, we have

$$R\,\mathrm{Hom}(\mathrm{im}(\eta_{\mathcal{U}}), L) \simeq R\check{\Gamma}(\mathcal{U}, L)$$

Consider

$$\begin{tikzcd} & \mathrm{PShv}(X) \arrow[dr, "\check{H}^0(\mathcal{U},-)"] & \\ \mathrm{Shv}(X) \arrow[ur, "\iota"] \arrow[rr, "\Gamma(X,-)"] & & \mathrm{Ab} \end{tikzcd}$$

.

Lemma 2.3.27. *$R\check{\Gamma}(\mathcal{U}, R\iota L) \simeq R\Gamma(X, L)$, $\forall L \in D^+(\mathrm{Shv}(X))$.*

Proof. Since $a \dashv \iota$ and a is exact, ι preserves injective objects. □

For a sheaf $\mathcal{F}$, the canonical morphism $\iota\mathcal{F} \to R\iota\mathcal{F}$ induces

$$R\check{\Gamma}(\mathcal{U}, \iota\mathcal{F}) \to R\check{\Gamma}(\mathcal{U}, R\iota\mathcal{F}) \simeq R\Gamma(X, \mathcal{F}),$$

which in turn induces the maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ by taking cohomology.

We are now in a position to prove Leray's theorem.

Proof of Leray's theorem. Let $L = R\iota\mathcal{F}$. For $p \geqslant 0$, consider the morphism of complexes

$$\begin{array}{ccc} \mathrm{Hom}^{p,\bullet}(\mathcal{C}_\bullet(\mathcal{U}), \iota\mathcal{F}) & \longrightarrow & \mathrm{Hom}^{p,\bullet}(\mathcal{C}(\mathcal{U}), L) \\ \| & & \| \\ \prod_{(i_0,\cdots i_p)} \Gamma(U_{i_0,\dots i_p}, \iota\mathcal{F}) & & \prod_{(i_0,\cdots i_p)} \Gamma(U_{i_0,\dots i_p}, L) \end{array}$$

By assumption, $H^n(U_{i_0,\dots i_p}, \mathcal{F}) = 0$, $n \geqslant 1$, which means

$$\Gamma(U_{i_0,\dots i_p}, \iota\mathcal{F}) \to \Gamma(U_{i_0,\dots i_p}, L)$$

is an quasi-isomorphism. Thus

$$\mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), \iota\mathcal{F}) \to \mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), L)$$

is also a quasi-isomorphism. Therefore,

$$R\check{\Gamma}(\mathcal{U}, \iota\mathcal{F}) \simeq R\check{\Gamma}(\mathcal{U}, L).$$

$\square$

Proposition 2.3.28. *Let $\mathcal{F}$ be a sheaf.*

(1) We have an exact sequence

$$0 \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, R^1\iota\mathcal{F}).$$

(2) We have

$$\operatorname*{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, R^1\iota\mathcal{F}) = 0,$$

where $\mathcal{U}$ runs through open covers of X. In particular,

$$\operatorname*{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F}).$$

Proof. (1) This follows from Lemma 2.3.29 applied to $R\Gamma(X, -)$.

(2) Let $\mathcal{F} \xrightarrow{\sim} L$ be a quasi-isomorphism with $L \in K^+$ and L^i injective. Then in $\mathrm{PShv}(X)$,

$$R^q\iota\mathcal{F} = \ker(\iota L^q \to \iota L^{q+1})/\mathrm{im}(\iota L^{q-1} \to \iota L^q).$$

For $q > 0$, $aR^q\iota\mathcal{F} \simeq \mathcal{H}^q\mathcal{F} = 0$. Here $\mathcal{H}^q$ denotes the q-th cohomology sheaf. We conclude by Lemma 2.3.30 below. $\square$

Lemma 2.3.29. *Let $F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be a triangulated functor carrying $D^{\geq 0}(\mathcal{A})$ into $D^{\geq 0}(\mathcal{B})$. Let $X \in D^{\geq 0}(\mathcal{A})$. We have an isomorphism $H^0FH^0X \simeq H^0FX$ and an exact sequence*

$$0 \to H^1FH^0X \to H^1FX \to H^0FH^1X \to H^2FH^0X \to H^2FX.$$

We leave this as an exercise.

Lemma 2.3.30. *Let $\mathcal{F}$ be a presheaf. Then the canonical map*

$$\operatorname*{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}) \to \Gamma(X, a\mathcal{F})$$

is injective.

Proof. By definition, $a\mathcal{F} = (\mathcal{F}')'$, where

$$\mathcal{F}'(U) = \operatorname*{colim}_{\mathcal{V}} \check{H}^0(\mathcal{V}, \mathcal{F}),$$

with $\mathcal{V}$ running through open covers of U. Since $\mathcal{F}'$ is separated, $\mathcal{F}' \to (\mathcal{F}')' = a\mathcal{F}$ is a monomorphism of presheaves. In particular, $\Gamma(X, \mathcal{F}') = \operatorname{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}) \to \Gamma(X, a\mathcal{F})$ is injective. □

Remark 2.3.31. The map

$$\operatorname*{colim}_{\mathcal{U}} \check{H}^2(\mathcal{U}, \mathcal{F}) \to H^2(X, \mathcal{F})$$

is injective but not bijective in general. Using hypercovers one can get isomorphisms to H^q all q.

Remark 2.3.32 (Alternating Čech complex)**.** The following variant of the Čech complex is very useful. For a presheaf $\mathcal{F}$ on X and an open cover $\mathcal{U} = \{U_i\}_{i\in I}$, we define a subcomplex $C^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}) \subseteq C^\bullet(\mathcal{U}, \mathcal{F})$, called the **alternating Čech complex**. An element $s = (s_{i_0,\dots,i_p}) \in \prod_{i_0,\dots,i_p\in I} \mathcal{F}(U_{i_0,\dots,i_p}) = C^p(\mathcal{U}, \mathcal{F})$ is said to be **alternating** if

$$\begin{cases} s_{i_0,\dots,i_p} = 0 & \text{if } i_j = i_k, \\ s_{i_{\sigma(0)}, \dots, i_{\sigma(p)}} = \operatorname{sgn}(\sigma) s_{i_0,\dots,i_p} & \text{for } \sigma \in \operatorname{Aut}\{0, \dots, p\}. \end{cases}$$

We let $C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}) \subseteq C^p(\mathcal{U}, \mathcal{F})$ denote the abelian subgroup consisting of the alternating elements. If we choose a total order on I, then $C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}) \simeq \prod_{i_0<\dots<i_p} \mathcal{F}(U_{i_0,\dots,i_p})$.

There are natural chain morphisms

$$C^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \underset{r}{\overset{i}{\rightleftarrows}} C^\bullet(\mathcal{U}, \mathcal{F})$$

where i is the inclusion and r is given by projection. We have $ri = \mathrm{id}$ and one can check that $ir - \mathrm{id} = dh + hd$ for some homotopy h. Thus i is a homotopy equivalence and we have

$$H^q C^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} \check{H}^q(\mathcal{U}, \mathcal{F}).$$

In particular, for $p \geqslant \#I$, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$, since $C^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$.

Date: 11.26

2.4 Serre's theorem on affine schemes

Theorem 2.4.1 (Serre)**.** *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module on an affine scheme X. Then $H^q(X, \mathcal{F}) = 0$ for all $q \geqslant 1$.*

Lemma 2.4.2. *Let X be an affine scheme and let $\mathcal{U}$ be a finite affine open cover. Let $\mathcal{F}$ be a quasi-coherent sheaf. Then $\check{H}^q(\mathcal{U}, \mathcal{F}) = 0$, for all $q \geqslant 1$.*

Proof. We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots .$$

Each $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = f_* f^{-1} \cdots f_* f^{-1} \mathcal{F}$ is quasi-coherent, where $f \colon \coprod_i U_i \to X$. The functor $\Gamma(X, -)$ carries exact sequences of quasi-coherent sheaves to exact sequences of modules. Therefore,

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

is exact. □

Lemma 2.4.3. *Let $\mathcal{J} \subseteq \mathrm{Shv}(X, \mathcal{O}_X)$ be the full subcategory consisting of $\mathcal{O}_X$-modules $\mathcal{F}$ such that for every affine over subset $U \subseteq X$ and every finite affine open cover $\mathcal{V}$ of U, we have $\check{H}^q(\mathcal{V}, \mathcal{F}) = 0$ for all $q \geqslant 1$. Then $\mathcal{J}$ is $\Gamma(X, -)$-injective.*

Proof. We check the axioms (a) and (b').

(a) It suffices to show that every injective $\mathcal{O}_X$-module $\mathcal{F}$ belongs to $\mathcal{J}$. For every open subset $U \subseteq X$, $\mathcal{F}|_U$ is flabby. It follows that we have $\check{H}^q(\mathcal{V}, \mathcal{F}) = 0$ for all $\mathcal{V}$ and all $q \geqslant 1$ by Corollary 2.3.7. Thus $\mathcal{F}$ belongs to $\mathcal{J}$.

(b') Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}, \mathcal{G} \in \mathcal{J}$. Let $U \subseteq X$ be an affine open. We have

$$H^1(U, \mathcal{F}) \simeq \operatorname*{colim}_{\mathcal{V}} \check{H}^1(\mathcal{V}, \mathcal{F}) = 0.$$

Here $\mathcal{V}$ runs through finite affine open covers of U and we used the fact that every open cover of U can be refined by a cover $\mathcal{V}$. Thus the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{Q}(U) \longrightarrow 0$$

is exact. In particular, the sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{Q}) \to 0$$

is exact. Moreover, we have a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{G}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{Q}) \longrightarrow 0,$$

which induces the exact sequence

$$\begin{tikzcd} \check{H}^q(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, equal] & \check{H}^q(\mathcal{V},\mathcal{Q}) \arrow[r] & \check{H}^{q+1}(\mathcal{V},\mathcal{F}) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd}$$

for $q \geq 1$. Thus $\mathcal{Q} \in \mathcal{J}$. □

Proof of Theoremt.Serre1. By Lemma 2.4.2, $\mathrm{QCoh}(X) \subseteq \mathcal{J}$. By Lemma 2.4.3, for all $\mathcal{F} \in \mathcal{J}$, we have $H^q(X,\mathcal{F}) = 0$ for all $q \geqslant 1$. □

We have the following converse of Theorem 2.4.1.

Theorem 2.4.4 (Serre)**.** *Let X be a quasi-compact scheme. Suppose that for all quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we have $H^1(X,\mathcal{I}) = 0$. Then X is affine.*

We will prove this later as a consequence of Theorem 2.5.9.

Combine Theorem 2.4.1 and Leray's theorem, we obtain:

Corollary 2.4.5. *Let X be a scheme and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X such that $U_{i_0,\dots,i_p}$ is affine for all $p \geqslant 0$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, for all q, the canonical map $\check{H}^q(\mathcal{U},\mathcal{F}) \to H^q(X,\mathcal{F})$ is an isomorphism.*

Remark 2.4.6. If the diagonal $\Delta_X\colon X \to X \times X$ is affine (for example if X is separated), then for all affine open $U, V \subseteq X$, $U \cap V$ is affine. Indeed

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\ U \times V \arrow[r] & X \times X \end{tikzcd}$$

is an Cartesian square. In this case, the corollary applies to every affine open cover of X.

Corollary 2.4.7. *Let X be a scheme and let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$, $\mathcal{Q} \in \mathrm{QCoh}(X)$, then $\mathcal{G} \in \mathrm{QCoh}(X)$.

Proof. We may assume that X is affine. Then the long exact sequence of cohomology has the form

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{Q}(X) \longrightarrow H^1(X,\mathcal{F}) = 0.$$

Thus we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(X)^\sim \arrow[r] \arrow[d, "\simeq"] & \mathcal{G}(X)^\sim \arrow[r] \arrow[d] & \mathcal{Q}(X)^\sim \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{Q} \arrow[r] & 0. \end{tikzcd}$$

The vertical arrow in the middle is an isomorphism by the five lemma. □

We let $D^+_{\mathrm{qcoh}}(X, \mathcal{O}_X) \subseteq D^+(X, \mathcal{O}_X)$ denote the full subcategory consisting of objects L such that $\mathcal{H}^i L \in \mathrm{QCoh}(X)$ for all i, where $\mathcal{H}^i$ denotes the i-th cohomology sheaf. The corollary implies that $D^+_{\mathrm{qcoh}}(X, \mathcal{O}_X) \subseteq D^+(X, \mathcal{O}_X)$ is triangulated subcategory. Indeed, if $L \to M \to N \to L[1]$ is a distinguished triangle with $L, M \in D^+_{\mathrm{qcoh}}$, then, by the long exact sequence

$$\mathcal{H}^i L \longrightarrow \mathcal{H}^i M \longrightarrow \mathcal{H}^i N \longrightarrow \mathcal{H}^{i+1} L \longrightarrow \mathcal{H}^{i+1} M$$

and the corollary, we have $N \in D^+_{\mathrm{qcoh}}$. The inclusion functor $\varphi\colon \mathrm{QCoh}(X) \subseteq \mathrm{Shv}(X, \mathcal{O}_X)$ is exact and induces a triangulated functor $\varphi\colon D^+\mathrm{QCoh}(X) \to D^+_{\mathrm{qcoh}}(X, \mathcal{O}_X)$.

Theorem 2.4.8. *(1) (Gabber)* $\mathrm{QCoh}(X)$ *admits enough injectives.*

(2) Assume either

- *X is Noetherian, or*
- *X is quasi-compact and Δ_X is affine.*

Then the functor

$$\varphi\colon D^+(\mathrm{QCoh}(X)) \to D^+_{\mathrm{qcoh}}(X, \mathcal{O}_X)$$

is an equivalence of category. Moreover, for $L \in D^+(\mathrm{QCoh}(X))$,

$$R\Gamma(X, \varphi -)(L) \simeq R\Gamma(X, \varphi L).$$

Remark 2.4.9. If X is Noetherian, then φ preserves injectives. In general, even for X affine, φ does not necessarily sends injectives to flabby sheaves.

We refer to [SP, 077P], [SGA6, II 3.5, Appendice I] and [TT, Propositions B.8, B.16] for more details.

Applications to Rf_*

Corollary 2.4.10. *Let $f\colon X \to S$ be an affine morphism and let $\mathcal{F} \in \mathrm{QCoh}(X)$. Then*

(1) $R^q f_* \mathcal{F} = 0$ for all $q \geqslant 1$.

(2) $H^q(X, \mathcal{F}) \simeq H^q(S, f_*\mathcal{F})$ for all q.

Proof. Recall that $R^q f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^q(f^{-1}(V), \mathcal{F})$. For V affine, $f^{-1}(V)$ is affine and $H^q(f^{-1}(V), \mathcal{F}) = 0$ for $q \geqslant 1$. It follows that $f_*\mathcal{F} \simeq Rf_*\mathcal{F}$ and $R\Gamma(X, \mathcal{F}) \simeq R\Gamma(S, Rf_*\mathcal{F}) \simeq R\Gamma(S, f_*\mathcal{F})$. □

Proposition 2.4.11. *Let $f\colon X \to S$ be a quasi-compact and quasi-separated morphism of schemes. For all $\mathcal{F} \in \mathrm{QCoh}(X)$ and all q, $R^q f_* \mathcal{F} \in \mathrm{QCoh}(S)$.*

Lemma 2.4.12 (Mayer-Vietoris). *Let $f\colon X \to S$ be a continuous map between topological spaces and let $X = U_1 \cup U_2$ be an open cover of X. Let $U = U_1 \cap U_2$. Let $f_i\colon U_i \to S$ and $g\colon U \to S$ denote the restrictions of f. For $L \in D^+(\mathrm{Shv}(X))$, we have a distinguished triangle*

$$Rf_*L \longrightarrow Rf_{1*}L \oplus Rf_{2*}L \longrightarrow Rg_*L \longrightarrow Rf_*L[1]$$

Proof. Up to replacing L by an injective resolution, we may assume $L \in K^+$ with L^i injective. It suffices to show the exactness of the sequence

$$0 \longrightarrow f_*L \longrightarrow f_{1*}L \oplus f_{2*}L \longrightarrow g_*L \longrightarrow 0.$$

Taking sections on an open subset $V \subseteq S$, we get

$$0 \to L^i(f^{-1}(V)) \to L^i(f_1^{-1}(V)) \oplus L^i(f_2^{-1}(V)) \xrightarrow{\alpha} L^i(f_1^{-1}(V) \cap f_2^{-1}(V)) \to 0.$$

The surjectivity of α follows from the fact that L^i is flabby and the remaining part of the exactness follows from the sheaf condition. $\square$

Proof of Proposition 2.4.11. We may assume that S is affine. Since X is quasi-compact, X can be covered by n affine opens for some n.

Case X separated. We proceed by induction on n. The case $n = 0$ is trivial. For $n > 0$, we have $X = U_1 \cup U_2$ with U_1 affine and U_2 covered by $n-1$ affine opens. In the notation of the lemma above, we have a distinguished triangle

$$Rf_*\mathcal{F} \longrightarrow Rf_{1*}\mathcal{F} \oplus Rf_{2*}\mathcal{F} \longrightarrow Rg_*\mathcal{F} \longrightarrow Rf_*\mathcal{F}[1].$$

By Corollary 2.4.10, $Rf_{1*}(\mathcal{F}) \simeq f_{1*}\mathcal{F}$ is quasi-coherent. Moreover, $Rf_{2*}\mathcal{F} \in D^+_{\mathrm{qcoh}}(S)$ by induction hypothesis. Since X is separated, $U_1 \cap U_2$ can be covered by $n-1$ affine opens, and consequently $Rg_*(\mathcal{F}) \in D^+_{\mathrm{qcoh}}(S)$ by induction hypothesis. It follows that $Rf_*(\mathcal{F}) \in D^+_{\mathrm{qcoh}}(S)$.

General Case. We proceed again by induction on n. The case $n = 0$ is trivial. For $n > 0$, write $X = U_1 \cap U_2$ with U_1 affine and U_2 covered by $n-1$ affine opens. Proceed as in separated case except that $Rg_*\mathcal{F} \in D^+_{\mathrm{qcoh}}(S)$ is deduced from the separated case applied to $U = U_1 \cap U_2 \subseteq U_1$. Note that U is quasi-compact and separated. $\square$

Flat base change

Given a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \qquad (*)$$

and an $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a base change morphism

$$g^*f_* \to f'_*h^*$$

given equivalently by

$$g^*f_* \to g^*f_*h_*h^* \xrightarrow{\sim} g^*g_*f'_*h^* \to f'_*h^*$$

or

$$g^*f_* \to f'_*f'^*g^*f_* \xrightarrow{\sim} f'_*h^*f^*f_* \to f'_*h^*.$$

We will give sufficient conditions for the base change morphism to be an isomorphism in the case of quasi-coherent sheaves on schemes.

Lemma 2.4.13. *Assume that* $(*)$ *is a Cartesian square of schemes and* f *is affine. Then for* $\mathcal{F} \in \mathrm{QCoh}(X)$*, the base change map* $g^*f_*\mathcal{F} \to f'_*h^*\mathcal{F}$ *is an isomorphism.*

Proof. We may assume $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B \otimes_A A')$. Assume $\mathcal{F} = \widetilde{M}$ for a B-module M. Then the left hand side is $(M \otimes_A A')^\sim$ and the right hand side is $M \otimes_B (B \otimes_A A')^\sim$ and the base change map is the canonical isomorphism. ☐

Proposition 2.4.14 (flat base change)**.** *Assume that* $(*)$ *is a Cartesian square of schemes,* f *is quasi-compact and quasi-separated, and* g *is flat. Then for* $\mathcal{F} \in \mathrm{QCoh}(X)$*, the base change map* $g^*Rf_*\mathcal{F} \to (Rf'_*)h^*\mathcal{F}$ *is an isomorphism.*

Since g^* is exact, it induces a functor $g^* \colon D^+(S, \mathcal{O}_S) \to D^+(S', \mathcal{O}_{S'})$. The same holds for h^*. The base change map is given by

$$g^*Rf_* \simeq R(g^*f_*) \to R(f'_*h^*) \to (Rf'_*)h^*$$

Proof. We first prove the case where g is an open immersion. We replace $\mathcal{F}$ by a resolution $L \in K^+$ with L^i injective. Then h^*L^i is injective and

$$g^*Rf_*L = g^*f_*L \xrightarrow{\sim} f'_*h^*L = (Rf'_*)h^*L$$

Having established the proposition for open immersions, we may assume that S is affine. Since f is quasi-compact, X is quasi-compact and can be covered by n affine open subsets. We then proceed by induction on n and apply Mayer-Vietoris as in Proposition 2.4.11 to reduce to the case where X is affine, which has been proved in Lemma 2.4.13. ☐

2.5 Cohomology of projective space

Theorem 2.5.1. *Let* A *be a ring,* $X = \mathbb{P}^d_A = \mathrm{Proj}(R)$*, where* $R = A[x_0, \dots, x_d]$*,* $d \geqslant 1$*. We regard* $\Gamma(X, -)$ *as a functor* $\mathrm{Shv}(X, \mathcal{O}_X) \to \mathrm{Mod}_A$*.*

- $H^q(X, \mathcal{O}_X(n)) = 0$*, for* $q \neq 0, d$ *and for all* n*.*
- $R \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ *as graded* A*-modules.*
- $H^d(X, \mathcal{O}_X(n))$ *is a free* A*-module with basis* $\{x_0^{k_0} \cdots x_d^{k_d} \mid k_i < 0,\ \sum k_i = n\}$*.*

In particular, for $n \geqslant 0$*,* $H^0(X, \mathcal{O}_X(n))$ *and* $H^d(X, \mathcal{O}_X(-n-d-1))$ *are both free of rank* $\binom{n+d}{d}$*.*

Recall that $\mathcal{O}_X(n) = \widetilde{R(n)}$. For the proof it is convenient to use derived tensor products of multiple complexes. We will not develop the theory in full generality but concentrate on what is necessary for the proof of theorem.

Definition 2.5.2. Let $\mathcal{A}$ be an additive category. The category $C^m(\mathcal{A})$ of m-uple complexes is defined recursively by $C^0(\mathcal{A}) = \mathcal{A}$ and $C^m(\mathcal{A}) = C(C^{m-1}(\mathcal{A}))$ for $m \geq 1$. We consider the total complex functor with respect to coproducts $\mathrm{tot}_\oplus \colon C^m(\mathcal{A}) \to C(\mathcal{A})$ defined by

$$\begin{aligned}(\mathrm{tot}_\oplus L)^n &= \bigoplus_{i_1+\cdots+i_m=n} L^{i_1,\dots,i_m}\\ d^{i_1,\dots,i_m} &= \sum_{j=1}^{n}(-1)^{i_1+\cdots+i_{j-1}} d_j^{i_1,\dots,i_m}\end{aligned}$$

The tensor product functor extends to the category of complexes:

$$\begin{gathered}C^-(\mathrm{Mod}_A) \times C^-(\mathrm{Mod}_A) \to C^2(\mathrm{Mod}_A) \xrightarrow{\mathrm{tot}_\oplus} C(\mathrm{Mod}_A)\\ (L, M) \mapsto L \otimes M\end{gathered}$$

where $(L \otimes M)^{ij} = L^i \otimes M^j$.

Lemma 2.5.3. *Let $L, M \in C^-(\mathrm{Mod}_A)$. Assume that L^i is flat for all i and L or M is acyclic. Then $\mathrm{tot}(L \otimes M)$ acyclic.*

Proof. Case where M is acyclic. Then $L^i \otimes M$ is acyclic for each i and hence $\mathrm{tot}(L \otimes M)$ is acyclic.

Case where L is acyclic. The complex L decomposes into short exact sequences

$$0 \longrightarrow Z^i L \longrightarrow L^i \longrightarrow Z^{i+1} L \longrightarrow 0.$$

By descending induction on i, one shows that $Z^i L$ is flat. Thus

$$0 \longrightarrow Z^i L \otimes M \longrightarrow L^i \otimes M \longrightarrow Z^{i+1} L \otimes M \longrightarrow 0$$

is exact, which implies $H_I^{i,\bullet}(L \otimes M) = 0$. Thus $\mathrm{tot}(L \otimes M)$ is acyclic. □

Proof of Theorem 2.5.1. Consider the cover $\mathcal{U} = \{U_i\}_{i=0}^d$ of X, where $U_i = D_+(x_i)$. Note that $U_{i_0,\dots i_p} = D(x_{i_0} \cdots x_{i_p})$ is affine. By Leray's theorem, we have $\check{H}^q(\mathcal{U}, \mathcal{O}(n)) \simeq H^q(X, \mathcal{O}(n))$.

We will compute the Čech cohomology. We have

$$C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{O}(n)) = \prod_{i_0<\cdots<i_p} (R_{x_{i_0}\cdots x_{i_p}})_n.$$

Let

$$C^p_{alt}(\mathcal{U}, \mathcal{O}(\bullet)) := \bigoplus_{n\in\mathbb{Z}} C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{O}(n)) = \bigoplus_{i_0<\cdots<i_p} (R_{x_{i_0}\cdots x_{i_p}})$$

and let $K^\bullet$ be $\bigoplus_{n\in\mathbb{Z}} C^\bullet_{alt}(\mathcal{U},\mathcal{O}(n))$:

$$\begin{tikzcd} K^\bullet = (R \arrow[r] & C^0_{alt}(\mathcal{U},\mathcal{O}(\bullet)) \arrow[r] \arrow[d, equal] & C^1_{alt}(\mathcal{U},\mathcal{O}(\bullet)) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & C^d_{alt}(\mathcal{U},\mathcal{O}(\bullet))) \arrow[d, equal] \\ & \bigoplus_i R_{x_i} & \bigoplus_{i<j} R_{x_ix_j} & & R_{x_0\cdots x_d} \end{tikzcd}$$

with R placed at degree 0. We have

$$R_{x_{i_0},\dots x_{i_p}} = \left(\bigotimes_{i\in\{i_0,\dots,i_p\}} A[x_i]_{x_i}\right) \otimes \left(\bigotimes_{i\notin\{\{i_0,\dots,i_p\}} A[x_i]\right)$$

and

$$K^\bullet = \mathrm{tot}\left(\bigotimes_{i=0}^{d}(A[x_i] \hookrightarrow A[x_i]_{x_i})\right)$$

with $A[x_i]$ in degree 0. Since

$$(A[x_i] \to A[x_i]_{x_i}) \to \bigoplus_{k_i<0} x_i^{k_i} A[-1]$$

is a quasi-isomorphism, we have a quasi-isomorphism

$$K^\bullet \to \bigotimes_{i=0}^{d} \bigoplus_{k_i<0} x_i^{k_i} A[-d-1]$$

by Lemma 2.5.3. The theorem follows. □

Definition 2.5.4 (Koszul complex)**.** Let A be a ring, F an A-module, and $v\colon A \to F$ a homomorphism of A-modules (determined by $v(1) \in F$). Define $K^\bullet(v) \in C^{\geqslant 0}(\mathrm{Mod}_A)$ by $K^p(v) = \bigwedge^p_A(F)$,

$$\begin{aligned} d^p\colon \bigwedge^{p} F &\to \bigwedge^{p+1} F \\ x &\mapsto v(1)\wedge x \end{aligned}$$

For an A-module M, we define $K^\bullet(v,M) := K(v)\otimes_A M$.

For $F = A^r$ and $v(1) = f \in A^r$, we write $K^\bullet(f)$ for $K^\bullet(v)$.

Example 2.5.5. Let $f_1,\dots,f_r \in A$. Let $X = \mathrm{Spec}(A)$. Then $\mathcal{U} = \{D(f_i)\}_i$ is an affine open cover of $U = \bigcup_{i=1}^r D(f_i) \subseteq X$. We have

$$\begin{aligned} K^\bullet(A \to \bigoplus_{i=1}^{r} A_{f_i}, M) &= \left(M \to \bigoplus_{i=1}^{r} M_{f_i} \to \bigoplus_{0\le i<j\le r} M_{f_if_j} \to \cdots \to M_{f_1\cdots f_r}\right) \\ &= \left(\Gamma(X,\widetilde{M}) \to C^0_{\mathrm{alt}}(\mathcal{U},\widetilde{M}) \to C^1_{\mathrm{alt}}(\mathcal{U},\widetilde{M}) \to \cdots \to C^r_{\mathrm{alt}}(\mathcal{U},\widetilde{M})\right), \end{aligned}$$

where M is placed at degree 0.

Date: 12.1

Finiteness and vanishing theorems

Let X be a locally Noetherian scheme.

Definition 2.5.6. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **coherent** if it is quasi-coherent and of finite type. We let $\mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X)$ denote the full subcategory consisting of all coherent $\mathcal{O}_X$-modules.

Theorem 2.5.7 (Serre)**.** *Let A be a Noetherian ring, $S = \mathrm{Spec}(A)$, $f\colon X \to S$ a projective morphism, and $\mathcal{F}$ a coherent sheaf on X.*

(1) (finiteness) For all q, $H^q(X, \mathcal{F})$ is a finitely generated A-module.

(2) (vanishing) Let $\mathcal{L}$ be an ample invertible sheaf. Then there exists $n_0 \geqslant 0$ such that $H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geqslant n_0$ and $q \geqslant 1$.

Note that $H^q(X, \mathcal{F}) = 0$ for $q \gg 0$ (independently of $\mathcal{F}$) by Grothendieck's theorem (Theorem 2.2.14) or the proposition below.

Proposition 2.5.8. *Let X be a scheme and $\mathcal{U} = \{U_i\}_{i=1}^d$ an open cover of X such that each $U_{i_0,\cdots,i_p}$ is affine. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^q(X, \mathcal{F}) = 0$ for all $q > d$.*

Proof. By Leray's theorem $H^q(X, \mathcal{F}) \simeq \check{H}^q_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}) = 0$ for $q > d$. □

Proof of Theorem 2.5.7. (1) Since f is projective, it factors through a closed immersion $i\colon X \hookrightarrow \mathbb{P}^d_A$. Then $H^q(X, \mathcal{F}) = H^q(\mathbb{P}^d_A, i_*\mathcal{F})$ and $i_*\mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}^d_A}$-module. Up to replacing X by $\mathbb{P}^d_A$, we may assume $X = \mathbb{P}^d_A$.

In this case, we proceed by descending induction on q. For $q > d$, $H^q(X, \mathcal{F}) = 0$. Assume that the assertion is proved for $q + 1$. By the ampleness of $\mathcal{O}_X(1)$, there exists an epimorphism $\mathcal{O}_X(-m)^r \to \mathcal{F}$, which extends to a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(-m)^r \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\mathcal{G}$ coherent. Taking cohomology, we get the exact sequence

$$H^q(X, \mathcal{O}(-m)^r) \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^{q+1}(X, \mathcal{G}).$$

Since $H^q(X, \mathcal{O}(-m)^r)$ is a finitely generated A-module by Theorem 2.5.1 and $H^{q+1}(X, \mathcal{G})$ is a finitely generated A-module by induction hypothesis, $H^q(X, \mathcal{F})$ is a finitely generated A-module. (Here we used the assumption that A is Noetherian.)

(2) Case $\mathcal{L}$ very ample. By assumption, we have a closed embedding $i\colon X \hookrightarrow \mathbb{P}^n_A$ with $\mathcal{L} \simeq i^*\mathcal{O}(1)$. Consequently, $i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \simeq i_*\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n} = i_*\mathcal{F} \otimes \mathcal{O}(n)$ and

$$H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^q(\mathbb{P}^n_A, i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})) \simeq H^q(\mathbb{P}^n_A, i_*\mathcal{F} \otimes \mathcal{O}(n)).$$

Since $i_*\mathcal{F}$ is a coherent sheaf, we may assume, up to replacing X by $\mathbb{P}^d_A$, that $X = \mathbb{P}^d_A$ and $\mathcal{L} = \mathcal{O}(1)$.

In this case, we proceed by descending induction on q. For $q > d$, $H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) = 0$. Assume the assertion proved for $q+1$. As in (1), we have a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(-m)^r \longrightarrow \mathcal{F} \longrightarrow 0,$$

which induces a short exact sequence

$$0 \longrightarrow \mathcal{G} \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}_X(n-m)^r \longrightarrow \mathcal{F} \otimes \mathcal{O}(n) \longrightarrow 0.$$

Taking cohomology, we the exact sequence

$$H^q(X, \mathcal{O}(n-m)^r) \longrightarrow H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) \longrightarrow H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(n)).$$

Since $H^q(X, \mathcal{O}(n-m)^r) = 0$ for $n > m$ by Theorem 2.5.1 and $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(n)) = 0$ for $n \gg 0$ by induction hypothesis, $H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) = 0$ for $n \gg 0$.

General case. There exists $m \geq 1$ such that $\mathcal{L}^{\otimes m}$ is very ample. We apply the very ample case to $(\mathcal{F} \otimes \mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes m})$, $0 \leqslant i \leqslant m-1$. For each i, there exists N_i such that $H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes mn+i}) = 0$ for $n \geqslant N_i$ and $q \geqslant 1$. Therefore, it suffices to take $n_0 = \max_{0 \leqslant i < m}\{mN_i + 1\}$. □

The vanishing theorem has the following converse.

Theorem 2.5.9. *Let X be a quasi-compact scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Assume that for every quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, there exists $n \geqslant 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$. Then $\mathcal{L}$ is ample.*

In the case where X is Noetherian, every quasi-coherent ideal is coherent.

Proof. Let $x \in X$ be a closed point. There exists an affine open neighborhood $U = \mathrm{Spec}(A) \ni x$ on which $\mathcal{L}$ is trivial. Let $Z = X \backslash U$ and $Z' = Z \cup \{x\}$, equipped with induced reduced closed subscheme structure. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_Z/\mathcal{I}_{Z'} \longrightarrow 0,$$

where $\mathcal{I}_Z/\mathcal{I}_{Z'} \simeq i_*\kappa(x)$, $i\colon \{x\} \hookrightarrow X$. By assumption, there exists $n \geqslant 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$. Twisting the short exact sequence by $\mathcal{O}(n)$ and taking cohomology, we get the exact sequence

$$\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \longrightarrow \kappa(x) \longrightarrow H^1(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes}) = 0.$$

Let $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n})$ be a pre-image of $1 \in \kappa(x)$. We may regard s as a section of $\mathcal{L}^{\otimes n}$ via the map $\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \hookrightarrow \Gamma(X, \mathcal{L}^{\otimes n})$. Then $X_s \subseteq X \backslash U = Z$. Since s is mapped to $1 \in \kappa(x)$, we have $x \in X_s$. Choose a trivialization $\mathcal{L}|_U \simeq \mathcal{O}_U$ and consider the induced map

$$\Gamma(U, \mathcal{L}^{\otimes n}) \xrightarrow{\sim} \Gamma(U, \mathcal{O}_U)$$
$$s \mapsto f.$$

Then $X_s = \mathrm{Spec}(A_f)$ is affine.

Let $S = \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{L}^{\otimes n})$. Then $Y = \bigcup_{S_+,\mathrm{homog}} X_s$ contains all closed points of X by the above. If $Y \neq X$, then $X \backslash Y$, which is a closed subset of X, contains at least one closed point. Thus $Y = X$. In other words, $\mathcal{L}$ is ample. □

Corollary 2.5.10. *Let A be a Noetherian ring, $f\colon X \to \operatorname{Spec}(A)$ a proper morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ ample if and only if for every quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, there exists $n \geqslant 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$.*

Proof of Theorem 2.4.4. By Theorem 2.5.9, $\mathcal{O}_X$ is ample. In other words, $X = \bigcup_{i=1}^n X_{f_i}$ with $f_i \in A = \Gamma(X, \mathcal{O}_X)$. The morphism

$$\mathcal{O}_X^n \xrightarrow{(f_1,\dots,f_n)} \mathcal{O}_X$$

is an epimorphism of sheaves of abelian groups, because it is so on each X_{f_i}. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Let $\mathcal{F}_i$ be the intersection of $\mathcal{F}$ with the direct sum of the first i summands on $\mathcal{O}_X^n$. Then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a quasi-coherent ideal sheaf. It follows that $H^1(X, \mathcal{F}) = 0$ and $\Gamma(X, \mathcal{O}_X^n) \twoheadrightarrow \Gamma(X, \mathcal{O}_X)$. Thus $f_1, \dots, f_n$ generate the unit ideal of A. Therefore, X is affine (exercise). □

The finiteness theorem has the following generalization.

Theorem 2.5.11. *Let X and S be locally Noetherian schemes and $f\colon X \to S$ a proper morphism. Let $\mathcal{F} \in \operatorname{Coh}(X)$. Then $R^q f_*(\mathcal{F}) \in \operatorname{Coh}(S)$ for all q.*

By contrast, for f affine, f_* does not preserve coherent sheaves in general.

Exercises

Problem 1. Let A be a ring. Let U and V be quasi-compact open subsets of $\mathrm{Spec}(A)$. Show that $U \cap V$ is quasi-compact.

Problem 2. An open subset of $\mathrm{Spec}(A)$ is called **principal** if it is of the form $D(f)$ for some $f \in A$.

(1) Find an open subset of $\mathrm{Spec}(\mathbb{Z}[X])$ that is not principal.

(2) Let A be a Dedekind domain whose ideal class group is torsion (e.g. A is the ring of integers of a number field). Show that every open subset of $\mathrm{Spec}(A)$ is principal.

Problem 3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. We let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ denote the presheaf on X carrying an open subset $U \subseteq X$ to $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf on X.

Problem 4. Let X be a topological space, U an open subset, and $j \colon U \to X$ the inclusion map.

(1) (Extension by the empty set) Let $\mathcal{F}$ be a sheaf of sets on U. Show that the presheaf on X

$$j_!^{\mathrm{Set}}\mathcal{F} \colon V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ \emptyset & V \not\subseteq U \end{cases}$$

is a sheaf. Compute the stalks of $j_!^{\mathrm{Set}}\mathcal{F}$.

(2) (Extension by zero) Let $\mathcal{F}$ be a sheaf of abelian groups on U. Let $j_!\mathcal{F}$ be the sheafification of the presheaf on X

$$j_!^{\mathrm{psh}}\mathcal{F} \colon V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & V \not\subseteq U. \end{cases}$$

Compute the stalks of $j_!\mathcal{F}$. Deduce that $j_! \colon \mathrm{Shv}(U, \mathrm{Ab}) \to \mathrm{Shv}(X, \mathrm{Ab})$ is an exact functor. Find an example for which $j_!^{\mathrm{psh}}\mathcal{F}$ is not a sheaf.

(**Remark.** $j_!^{\mathrm{Set}}$ is a left adjoint of $j^{-1} \colon \mathrm{Shv}(X, \mathrm{Set}) \to \mathrm{Shv}(U, \mathrm{Set})$ and $j_!$ is a left adjoint of $\mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Shv}(U, \mathrm{Ab})$.)

Problem 5.

(1) Show that a ring homomorphism $\phi\colon A \to B$ is a monomorphism if and only if ϕ is an injection. (**Hint.** Consider ring homomorphisms $\mathbb{Z}[X] \to A$ or the diagonal $\Delta_\phi\colon B \to B \times_A B$).

(2) Let $f\colon Y \to X$ be an epimorphism of schemes. Show that $f^\flat_X\colon \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is an injection and $f(Y)$ intersects with every nonempty closed subset Z of X. (**Hint** for the second assertion. Consider the scheme obtained by gluing two copies of X along $X\backslash Z$.)

(3) Use (b) to give an example of an injective ring homomorphism $\phi\colon A \to B$ such that $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is **not** an epimorphism of schemes.

Problem 6. We say that a continuous map $f\colon Y \to X$ is **dominant** if $f(Y)$ is dense in X. We say that a morphism $f\colon Y \to X$ of schemes is **scheme-theoretically dominant** if $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a monomorphism. (You may either admit the fact that a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(X, \mathrm{Ring})$ is a monomorphism if and only if $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an injection for every open subset U of X, or take this as a definition.)

(1) Show that a ring homomorphism $\phi\colon A \to B$ is an injection if and only if $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is scheme-theoretically dominant.

(2) Show that a scheme-theoretically dominant morphism $f\colon Y \to X$ is dominant. Show moreover that the converse holds for X reduced.

(3) Show that a scheme-theoretically dominant morphism that is surjective is an epimorphism of schemes. Deduce that any surjective morphism of schemes $f\colon Y \to X$ with X reduced is an epimorphism.

Problem 7.

(1) Show that a ring homomorphism $\phi\colon A \to B$ is an epimorphism if and only if $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a monomorphism of schemes.

(2) Let X be a scheme. Let $X' = \coprod_{x\in X} \mathrm{Spec}(\kappa(x))$, where $\kappa(x)$ denotes the residue field of $\mathcal{O}_{X,x}$ and let $f\colon X' \to X$ be the canonical morphism sending $x' = \mathrm{Spec}(\kappa(x))$ to x with $f^\sharp_{x'}\colon \mathcal{O}_{X,x} \to \kappa(x)$ given by the projection. Show that f is a monomorphism of schemes.

(3) Use (b) and Problem 6(c) to give an example of a morphism of affine schemes that is a monomorphism of schemes, an epimorphism of schemes, and a bijection, but not an isomorphism of schemes.

Problem 8. Let $\mathcal{P}$ be an infinite set and let $A \subseteq \prod_{p\in\mathcal{P}} \mathbb{F}_2$ be the subring consisting of $a = (a_p)$ such that $supp(a) := \{p \mid a_p \neq 0\}$ is a finite or cofinite subset of $\mathcal{P}$. (Recall that a **cofinite** subset is the complement of a finite subset.) Let $\mathfrak{m}_p$ be the kernel of the projection $A \to \mathbb{F}_2$ sending a to a_p and let $\mathfrak{m}_\infty = \bigoplus_{p\in\mathcal{P}} \mathbb{F}_2 \subseteq A$.

(1) Let $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$ be the one-point compactification of the discrete set $\mathcal{P}$. (In other words, the open subsets of $\mathcal{P}^*$ are the cofinite subsets of $\mathcal{P}^*$ and all the subsets of $\mathcal{P}$.) Show that $\mathfrak{m}_p$ and $\mathfrak{m}_\infty$ are maximal ideals of A and the map $\mathcal{P}^* \to \mathrm{Spec}(A)$ sending p to $\mathfrak{m}_p$ and ∞ to $\mathfrak{m}_\infty$ is a homeomorphism.

(2) Show that $A/\mathfrak{m}_\infty \simeq \mathbb{F}_2$.

Problem 9. Show that every nonempty quasi-compact T_0 space has a closed point.

Problem 10.

(1) Let X be a quasi-compact scheme. Let $A = \mathcal{O}_X(X)$ and $f \in A$. Show that the restriction map $A \to \mathcal{O}_X(X_f)$ factors through an injective homomorphism $\phi\colon A_f \to \mathcal{O}_X(X_f)$.

(2) Let X be a scheme admitting a finite cover $\{U_i\}$ by open affines such that each intersection $U_i \cap U_j$ is quasi-compact. Show that $\phi\colon A_f \to \mathcal{O}_X(X_f)$ is an isomorphism.

(3) Let X be a scheme such that there exist $f_1, \dots, f_n \in \mathcal{O}_X(X) = A$ with $\sum_{i=1}^n f_i A = A$ and X_{f_i} affine for all i. Show that X is affine.

Problem 11. Let $f\colon Y \to X$ be a morphism of schemes.

(1) Show that if f is locally of finite type, $U \simeq \mathrm{Spec}(A)$ is an affine open of X and $V \simeq \mathrm{Spec}(B)$ is an affine open of $f^{-1}(U)$, then B is a finitely-generated A-algebra.

(2) Show that if f is quasi-compact and U is a quasi-compact open subset of X, then $f^{-1}(U)$ is quasi-compact.

(3) Show that if f is affine and U is an affine open of X, then $f^{-1}(U)$ is an affine open of Y.

(4) Show that if f is finite and $U \simeq \mathrm{Spec}(A)$ is an affine open of X, then $f^{-1}(U) \simeq \mathrm{Spec}(B)$ with B a finite A-algebra.

Problem 12. (1) Let A be a Noetherian local ring of dimension ≥ 1. Show that the maximal ideal $\mathfrak{m}$ is the union of prime ideals of A of height 1. (**Hint.** Use Krull's principal ideal theorem. The weaker assertion with height 1 replaced by height ≤ 1 suffices for (b).)

(2) Let A be a Noetherian ring of dimension ≥ 2. Deduce from (a) that there are infinitely many prime ideals of A of height 1. (**Hint.** Use the prime avoidance lemma.)

(3) Deduce from (b) that every locally Noetherian scheme of dimension ≥ 2 has infinitely many points.

Problem 13. (1) Show that a morphism $f\colon Y \to X$ in a category admitting fiber products is a monomorphism if and only if the first projection $Y \times_X Y \to Y$ is an isomorphism. Show moreover that monomorphisms are stable under base change.

(2) Let k be a field. Use (a) to show that a ring homomorphism $\phi\colon k \to B$ is an epimorphism if and only if $B = 0$ or ϕ is an isomorphism. Deduce that a morphism of schemes $f\colon Y \to \mathrm{Spec}(k)$ is a monomorphism if and only if $Y = \emptyset$ or f is an isomorphism.

(3) Let $f\colon Y \to X$ be a monomorphism of schemes. Show that f is an injection and for every point $y \in Y$, the extension of residue fields $\kappa(y)/\kappa(f(y))$ is trivial.

Problem 14. Given a scheme X and a field K, we let $X(K)$ denote $\mathrm{Hom}(\mathrm{Spec}(K), X)$.

(1) Let $\phi\colon K \to L$ be a field embedding. Show that the induced map $X(\phi)\colon X(K) \to X(L)$ is an injection. (**Hint.** Use Problem 6 or the identification of $X(K)$ with $\{(x,\iota) \mid x \in X, \iota\colon \kappa(x) \to K\}$.)

(2) Show that a morphism of schemes $f\colon X \to Y$ is surjective if and only if for every field K, there exists a field extension L/K such that $f(L)\colon X(L) \to Y(L)$ is a surjection. (**Hint** for the "only if" part. One can start by showing that for every K and every $y \in Y(K)$, there exists a field embedding $\phi_y\colon K \to L_y$ such that $Y(\phi_y)(y) \in Y(L_y)$ belongs to the image of $f(L_y)$. A more direct proof is also possible.)

(3) Show that a morphism of schemes $f\colon X \to Y$ is radiciel if and only if the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is surjective. Deduce that every radiciel morphism is separated.

Problem 15.

(1) Let g and h be morphisms of schemes $X \to W$ and let E be their equalizer. Show that the morphism $E \to X$ is an immersion whose image is contained in the set-theoretic equalizer $E' = \{x \in X \mid g(x) = h(x)\}$.

(2) Deduce the following improvement of Problem 6(c): a scheme-theoretically dominant morphism $f\colon Y \to X$ such that $f(Y)$ intersects with every nonempty closed subset Z of X is an epimorphism.

(3) Let A be a local domain of dimension ≥ 2 with fraction field K and residue field k. Use (b) and Problem 7(b) to show that $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is a monomorphism of schemes and an epimorphism of schemes, but not a surjection.

Problem 16. (1) Let $i\colon Z \to X$ be a closed immersion of schemes. Let

$$W = \underline{\mathrm{Spec}}(\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X).$$

Show that the canonical morphism $X \coprod X \simeq \underline{\mathrm{Spec}}(\mathcal{O}_X \times \mathcal{O}_X) \to W$ is finite surjective. Describe the underlying topological space of W.

(**Remark.** This construction and its generalizations are called **pinching**.)

(2) Let $f\colon Y \to X$ be a quasi-compact morphism of schemes. Show that the ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$ is quasi-coherent and the closed subscheme Z of X defined by $\mathcal{I}$ is the smallest closed subscheme of X through which f factors. We call Z the **scheme-theoretic image** of f.

(3) Deduce that a quasi-compact morphism of schemes $f\colon Y \to X$ is an epimorphism if and only if f is scheme-theoretically dominant and $f(Y)$ intersects with every nonempty closed subset of X. (See also Problems 5(b) and 15(b).)

Problem 17. Let k be an algebraically closed field. In each of the following cases, compute the normalization $f\colon X^\nu \to X$ of X. Describe all fibers of f that are not geometrically irreducible or geometrically reduced. Is f a universal homeomorphism?

(1) $X = \mathrm{Spec}(k[x,y]/(y^7 - x^{2020}))$;

(2) $X = \mathrm{Spec}(k[x,y,z]/(xy^2 - z^2))$. (**Hint.** The answers depend on whether $\mathrm{char}(k) = 2$.)

Problem 18. (1) Show that an injective and closed morphism of schemes is affine.

(2) Deduce that an injective and universally closed morphism of schemes is integral.

Problem 19. (1) Show that a scheme X is separated if and only if there exists an affine open cover $\{U_i\}$ of X such that $U_i \cap U_j$ is affine and the canonical homomorphism

$$\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

is surjective for all i, j.

(2) Let R be a graded ring. Show that, for all $f, g \in R_+$ homogeneous, the canonical homomorphism $R_{(f)} \otimes R_{(g)} \to R_{(fg)}$ is surjective. Deduce that $\mathrm{Proj}(R)$ is separated.

Problem 20. Let R be a graded ring.

(1) Show that for any prime ideal $\mathfrak{p}$ of R, $\bigoplus_{d\geq 0}(\mathfrak{p} \cap R_d)$ is a homogeneous prime ideal of R. Deduce that any minimal prime ideal of R is homogeneous.

(2) Let T be the set of maximal points of $\mathrm{Spec}(R)$. Show that $T \cap \mathrm{Proj}(R)$ is the set of maximal points of $\mathrm{Proj}(R)$.

(3) Show that $\mathrm{Proj}(R)$ is normal if R is an integrally closed domain.

Problem 21. (1) Let A be a Noetherian ring and $\mathfrak{b}$ an ideal of A. We say that an ideal $\mathfrak{a}$ of A is **$\mathfrak{b}$-saturated** if $(\mathfrak{a} : \mathfrak{b}) = \mathfrak{a}$, where $(\mathfrak{a} : \mathfrak{b}) := \{x \in A \mid \mathfrak{b}x \subseteq \mathfrak{a}\}$. For any ideal $\mathfrak{a}$ of A, show that the sequence of ideals $(\mathfrak{a} : \mathfrak{b}^n)$, $n \geq 0$ is stationary and $(\mathfrak{a} :^\infty \mathfrak{b}) := \bigcup_{n\geq 0}(\mathfrak{a} : \mathfrak{b}^n)$ is the smallest $\mathfrak{b}$-saturated ideal containing $\mathfrak{a}$.

(**Remark.** We have $(\mathfrak{a} :^\infty \mathfrak{b})/\mathfrak{a} \simeq \Gamma_{V(\mathfrak{b})}(\mathrm{Spec}(A), \widetilde{A/\mathfrak{a}})$, where Γ_Z denotes the set of global sections supported in a closed subset Z.)

(2) For any primary ideal $\mathfrak{q}$ of A, show that

$$(\mathfrak{q} :^\infty \mathfrak{b}) = \begin{cases} \mathfrak{q} & \sqrt{\mathfrak{q}} \not\supseteq \mathfrak{b}, \\ A & \sqrt{\mathfrak{q}} \supseteq \mathfrak{b}. \end{cases}$$

Deduce that $(\sqrt{\mathfrak{a}} :^\infty \mathfrak{b}) = \bigcap_{\mathfrak{p}\in V(\mathfrak{a})\setminus V(\mathfrak{b})} \mathfrak{p}$.

(3) Let R be a graded ring. For any subset $Y \subseteq \mathrm{Proj}(R)$, let $I(Y) = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$. Show that $V_+(I(Y)) = \overline{Y}$ is the closure of Y in $\mathrm{Proj}(R)$.

(4) Assume that R is Noetherian. For any homogeneous ideal $\mathfrak{a}$ of R, show that $I(V_+(\mathfrak{a})) = (\sqrt{\mathfrak{a}} :^\infty R_+)$. Deduce that the maps V_+ and I induce a one-to-one order-reversing correspondence between R_+-saturated radical homogeneous ideals of R and closed subsets of $\mathrm{Proj}(R)$.

Problem 22. Let A be a ring and let $a, b \geq 1$ be integers. Show that the weighted projective line $\mathbb{P}_A(a, b)$ is canonically isomorphic to $\mathbb{P}^1_A$.

Problem 23. Let A be a ring and let $d \geq 2$ be an integer. Let $I \subseteq R = A[x_0, \dots, x_d]$ denote the homogeneous ideal of the d-uple embedding $\mathbb{P}^1_A \hookrightarrow \mathbb{P}^d_A$.

(1) Show that $I \cap R_2$ is a free A-module of rank $\binom{d}{2}$. Deduce that I cannot be generated by less than $\binom{d}{2}$ elements unless $A = 0$.

(2) Show that I is generated by $I \cap R_2$. (**Hint.** Show that $I \cap R_n$ is generated by $x_{i_0} \cdots x_{i_n} - x_{j_0} \cdots x_{j_n}$ with $i_0 + \cdots + i_n = j_0 + \cdots + j_n$. Proceed by induction on n to show that such elements are generated by $I \cap R_2$.)

(3) Assume $d = 3$. Let $J = (x_1^2 - x_0x_2, x_2^3 - x_0x_3^2) \subseteq I$. Check that $\sqrt{J} = \sqrt{I}$.

Problem 24. We say that a scheme X is **locally integral** if $\mathcal{O}_{X,x}$ is a domain for every $x \in X$. Show that the irreducible components of a locally integral scheme are disjoint. Deduce that a locally integral scheme with finitely many irreducible components is a finite coproduct of integral schemes.

Problem 25. Let k be a field.

(1) Let A be a finitely generated k-algebra that is a domain. Assume that $A_\mathfrak{p}$ is integrally closed for every prime ideal $\mathfrak{p}$ of height 1. Show that the integral closure of A is $\bigcap_\mathfrak{p} A_\mathfrak{p}$, where $\mathfrak{p}$ runs through height 1 prime ideals. (**Remark.** The assumption that A is a finitely generated k-algebra can be weakened to A being a universally catenary Japanese Noetherian domain. The universal catenarity cannot be dropped. See [EGA IV, Exemple 5.6.11].)

(2) Let R be a finitely generated graded k-algebra that is a domain generated by R_1 over R_0. Assume that R_+ has height ≥ 2 and $X = \mathrm{Proj}(R)$ is normal. Show that the canonical map $R \to \Gamma_*(\mathcal{O}_X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ identifies $\Gamma_*(\mathcal{O}_X)$ with the integral closure of R.

Problem 26. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Show that for any affine open U of X, $X_s \cap U$ is affine.

Problem 27. Let A be a ring. For an A-module M, we let $\mathbb{P}_A(M)$ denote $\mathrm{Proj}(\mathrm{Sym}_A(M))$. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of A-modules.

(1) Show that g induces a closed immersion $\mathbb{P}_A(g)\colon \mathbb{P}_A(M'') \to \mathbb{P}_A(M)$ and f induces an affine morphism $\mathbb{P}_A(g)\colon \mathbb{P}_A(M) \backslash \mathrm{im}(\mathbb{P}_A(g)) \to \mathbb{P}_A(M')$.

(2) Assume that the exact sequence splits. Show that $\mathbb{P}_A(g)$ can be identified with the projection $\mathbb{V}(\mathcal{O}_Y(-1) \otimes_A M'') \to Y$. Here $Y := \mathbb{P}_A(M')$, and, for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F}))$.

Problem 28. Let $f\colon X \to Y$ be a morphism of schemes with X quasi-compact. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on X and $\mathcal{M}$ an invertible sheaf on Y.

(1) Show that $X = \bigcup_{s\in S_{+,\mathrm{homog}}} X_s$ if and only if $\mathcal{L}^{\otimes n}$ is globally generated for some $n \geq 1$. Here $S = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. In this case we say that $\mathcal{L}$ is **semiample**.

(2) Show that if $\mathcal{L}$ is ample and $\mathcal{L}'$ is semiample, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is ample.

(3) Show that if $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then for $n \gg 0$, $\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M}^{\otimes n}$ is ample.

(4) Show that if $\mathcal{L}$ is f-very ample and $\mathcal{L}'$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is f-very ample.

(5) Show that if f is locally of finite type and $\mathcal{L}$ is ample, then there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is f-very ample for all $n \geq n_0$. (**Hint.** Use (d).)

Problem 29. (1) Let $f\colon X \to S$ be a separated morphism of schemes. Show that every section s of f is a closed immersion.

(2) Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $f\colon \mathbb{V}(\mathcal{E}) \to S$ and let $s\colon S \to \mathbb{V}(\mathcal{E})$ be the **zero section** of f, namely the section induced by $0\colon \mathcal{E} \to \mathcal{O}_S$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{V}(\mathcal{E})}$ be ideal sheaf corresponding to s. Show that $s^*\mathcal{I} \simeq \mathcal{E}$.

Problem 30. Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $P = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_S)$. Let Z_P and 0_P denote the closed subschemes defined respectively by the closed immersions $\mathbb{P}(\mathcal{E}) \to P$ and $\mathbb{P}(\mathcal{O}) \to P$ given by the projections $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{E}$ and $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{O}_S$. We call Z_P the **infinity locus** and 0_P the **zero section** of $P \to S$.

(1) Show that Z_P is an effective Cartier divisor of P and that $P \backslash Z_P$ can be identified with $\mathbb{V}(\mathcal{E})$. We call P the **projective closure** of $\mathbb{V}(\mathcal{E})$.

(2) Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})})$. Let Z_X and 0_X denote respectively the infinity locus and zero sections of $X \to \mathbb{P}(\mathcal{E})$. Construct an S-morphism $\pi\colon X \to P$ identifying X with the blowing up of P at 0_P such that $\pi^{-1}(0_P) = 0_X$ and $\pi^{-1}(Z_P) = Z_X$ as subschemes of X. Describe π in terms of the functors $(\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set}$ that X and P represent.

(3) Deduce that $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq \mathrm{Bl}_{0_P}(\mathbb{V}(\mathcal{E}))$ and $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)) \simeq P\backslash 0_P$. (For the last isomorphism, see also Problem 27(b).)

Problem 31. Let k be a field of characteristic $\neq 2$ and let $S = \mathrm{Spec}(k[x,y]/(y^2 - x^4))$. (The point $V(x,y)$ is called a **tacnode**.) Find a blowing up $S' \to S$ with S' normal.

Problem 32. Show that, in a triangulated category, the direct sum of two distinguished triangles is a distinguished triangle. (**Hint.** Let $T_i\colon X_i \xrightarrow{f_i} Y_i \to Z_i \to X_i[1]$, $i = 1, 2$ be distinguished triangles. Extend $f_1 \oplus f_2$ to a distinguished triangle T and construct a morphism from $T_1 \oplus T_2$ to T.)

Problem 33. Let $\mathcal{D}$ be a triangulated category.

(1) Show that for objects X and Y in $\mathcal{D}$, the triangle $X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$, where i and p are the canonical morphisms, is a distinguished triangle. (**Hint.** Use Problem 32.)

(2) Conversely, show that every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{D}$ with $h = 0$ is isomorphic to the distinguished triangle in (1).

Problem 34. Let $\mathcal{A}$ be an abelian category. For every $L \in D(\mathcal{A})$ and $n \in \mathbb{Z}$, construct a distinguished triangle $\tau^{\leq n}L \to L \to \tau^{\geq n+1}L \xrightarrow{h} (\tau^{\leq n}L)[1]$ in $D(\mathcal{A})$. Show that $H^i h = 0$ for all i. Give an example with h nonzero in $D(\mathcal{A})$.

Problem 35. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories admitting an F-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$. We say that $X \in \mathcal{A}$ is F**-acyclic** if $R^nFX = 0$ for all $n \geq 1$. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ spanned by F-acyclic objects.

(1) Show that $\mathcal{I}$ is F-injective.

In the rest of this problem, assume that there exists $N > 0$ such that $R^NFX = 0$ for all $X \in \mathcal{A}$.

(b) Show that $R^nFX = 0$ for all $X \in \mathcal{A}$ and $n \geq N$.

(c) Show that for every exact sequence $X_{N-1} \to \cdots \to X_1 \to Y \to 0$ in $\mathcal{A}$ with $R^jFX_i = 0$ for all $j \geq i$, Y is F-acyclic.

(d) Deduce that for every $L \in C(\mathcal{I})$ acyclic, FL is acyclic.

Problem 36 (Serre)**.** Let X be a quasi-compact scheme. Assume that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$. Proceed in the following steps to show that X is affine.

(1) Show that for every closed point $x \in X$, there exists $f \in \mathcal{O}_X(X)$ such that $x \in X_f$ and X_f is affine. (**Hint.** Choose an affine open neighborhood U of x and consider the short exact sequence $0 \to \mathcal{I}_{Z'} \to \mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_{Z'} \to 0$, where $Z = X \backslash U$ and $Z' = Z \cup \{x\}$ are equipped with the reduced closed subscheme structures.)

(2) Use Problem 9 to deduce that there exist $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ with $X = \bigcup_{i=1}^n X_{f_i}$ and X_{f_i} affine.

(3) Show that $f_1, \ldots, f_n$ generate the unit ideal in $\mathcal{O}_X(X)$. Conclude by Problem 10(c).

Problem 37. Let $F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be a triangulated functor carrying $D^{\geq 0}(\mathcal{A})$ into $D^{\geq 0}(\mathcal{B})$. Let $X \in D^{\geq 0}(\mathcal{A})$. Prove the existence of an isomorphism $H^0FH^0X \simeq H^0FX$ and an exact sequence

$$0 \to H^1FH^0X \to H^1FX \to H^0FH^1X \to H^2FH^0X \to H^2FX.$$

(**Hint.** Use the distinguished triangle $H^0X \to X \to \tau^{\geq 1}X \to (H^0X)[1]$.)

Problem 38. Let $\mathcal{G}$ be a sheaf of groups on a topological space X. A sheaf $\mathcal{F}$ of sets on X equipped with a (left) action of $\mathcal{G}$ is called a **$\mathcal{G}$-torsor** if

- For every open subset U of X and every pair of sections $s, t \in \mathcal{F}(U)$, there exists a unique $g \in \mathcal{G}(U)$ such that $gs = t$.
- $\mathcal{F}_x \neq \emptyset$ for all $x \in X$.

A morphism of $\mathcal{G}$-torsors is a morphism of sheaves preserving the $\mathcal{G}$-action.

(1) Show that every morphism of $\mathcal{G}$-torsors is an isomorphism. Let $\mathrm{Tors}(\mathcal{G})$ denote the set of isomorphism classes of $\mathcal{G}$-torsors.

(2) In the case with $\mathcal{G}$ abelian, establish a bijection between $\mathrm{Tors}(\mathcal{G})$ and $H^1(X, \mathcal{G})$. For every open cover $\mathcal{U}$ of X, describe the collection of $\mathcal{G}$-torsors corresponding to the image of the map $H^1(\mathcal{U}, \mathcal{G}) \to H^1(X, \mathcal{G})$.

(3) Let $\mathcal{O}_X$ be a sheaf of rings on X. Let $\mathrm{Loc}_n(\mathcal{O}_X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$ modules of rank n. Establish a bijection between $\mathrm{Loc}_n(\mathcal{O}_X)$ and $\mathrm{Tors}(\mathrm{GL}_n(\mathcal{O}_X))$, where $\mathrm{GL}_n(\mathcal{O}_X)$ denotes the sheaf of groups $U \mapsto \mathrm{GL}_n(\mathcal{O}_X(U))$. (**Hint.** For a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank n, consider $\mathcal{I}som_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F})$.)

(4) Establish a group isomorphism $\mathrm{Pic}(X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ denotes the abelian sheaf $U \mapsto \mathcal{O}_X(U)^\times$.

Problem 39. Let X be a quasi-compact quasi-separated topological space such that quasi-compact open subsets form a basis. The goal of this problem is to show that $H^q(X, -)$ commutes with filtered colimit: for every filtered system $(\mathcal{F}_i)$ of abelian sheaves on X, the canonical map

$$\operatorname*{colim}_i H^q(X, \mathcal{F}_i) \to H^q(X, \operatorname*{colim}_i \mathcal{F}_i)$$

is an isomorphism.

(1) Let Cov denote the collection of finite quasi-compact open covers of open subsets of X. Show that the full subcategory $\mathcal{J}$ of $\mathrm{Shv}(X)$ consisting of $\mathcal{G}$ satisfying $\check{H}^p(\mathcal{U}, \mathcal{G}) = 0$ for all $\mathcal{U} \in \mathrm{Cov}$ and $p \geq 1$ is $\Gamma(X, -)$-injective.

(2) Let $\mathcal{G}_i$ be a filtered system of flabby sheaves. Show that $\operatorname{colim}_i \mathcal{G}_i \in \mathcal{J}$.

(3) Conclude by induction on q. (**Hint.** Choose a functorial monomorphism $\mathcal{F}_i \to \mathcal{G}_i$ with $\mathcal{G}_i$ flabby.)

Problem 40.

(1) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf of $\mathcal{O}_X$ such that $\mathcal{I}^n = 0$. Assume that the closed subscheme $X_0 = (X, \mathcal{O}_X/\mathcal{I})$ of X defined by $\mathcal{I}$ is affine. Show that X is affine. (**Hint.** Show that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ using the filtration $(\mathcal{I}^m\mathcal{F})_{0\leq m\leq n}$.)

(2) Deduce that a Noetherian scheme X such that X_{red} is affine is affine.

Remark. (Yin Hang) The Noetherian assumption can be removed by applying Problem 10(c) and a limit argument.

(3) Show that a reduced scheme X admitting a finite cover by affine closed subschemes is affine.

Problem 41. Let $f\colon X \to Y$ be a finite surjective morphism of Noetherian schemes with X affine. Show that Y is affine. You may follows the following steps.

(1) In the case where X and Y are integral, show that there exists a coherent sheaf $\mathcal{M}$ on X and a morphism of $\mathcal{O}_Y$-modules $\alpha\colon \mathcal{O}_Y^r \to f_*\mathcal{M}$ with $r > 0$ which is an isomorphism at the generic point η_Y of Y.

(2) Deduce in the case of (a) that for every coherent sheaf $\mathcal{F}$ on Y, there exists a coherent sheaf $\mathcal{G}$ on X and a morphism of $\mathcal{O}_Y$-modules $f_*\mathcal{G} \to \mathcal{F}^r$ that is an isomorphism at η_Y. (**Hint.** Apply $\mathcal{H}om(-, \mathcal{F})$ to α.)

(3) Use Problem 40 to reduce to the integral case. Conclude by Serre's criterion (Problem 36) and Noetherian induction on Y.

Remark. This result is due to Chevalley in the case of schemes of finite type over a field. It holds in fact more generally without the Noetherian assumption, generalizing Problem 40.

Problem 42. Let X be a scheme proper over a field k. Assume that X is geometrically connected and geometrically reduced over k. Show that the canonical map $k \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

Problem 43. Let S be a scheme and let X and Y be schemes over S.

(1) Assume that X is integral and Y is of finite type over S. Let $s \in S$ be a point and let $x \in X$ and $y \in Y$ be points above s. Let $\phi\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a homomorphism of $\mathcal{O}_{S,s}$-algebras. Show that there exists an open neighborhood $U \subseteq X$ of x and a morphism $f\colon U \to Y$ over S such that $f(x) = y$ and $f_x^\sharp = \phi$.

(2) Assume that X is Noetherian normal of dimension 1 and Y is proper over S. Let $U \subseteq X$ be a dense open subset. Show that every S-morphism $U \to Y$ extends uniquely to an S-morphism $X \to Y$:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

(3) Deduce from Chow's lemma that a normal scheme of dimension 1 and proper over k is projective over k. (**Remark.** This holds in fact without the normality assumption.)

Problem 44. Let $X \to S$ and $Y \to S$ be morphisms of schemes and let $p\colon X \times_S Y \to X$ and $q\colon X \times_S Y \to Y$ be the projections. Show that the canonical morphism $p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S}$ is an isomorphism.

Problem 45. Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Consider the condition $(*)$: the sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

is exact and locally splits.

(1) Show that if f is formally smooth, then $(*)$ holds.

(2) Show that if $(*)$ holds and gf is formally smooth, then f is formally smooth.

Problem 46. (1) Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Show that if B is regular, then so is A. (**Hint.** By theorems of Serre and Auslander, a Noetherian local ring A is regular if and only if A has finite weak dimension, namely there exists an integer d such that $\mathrm{Tor}_n^A(M, N) = 0$ for all A-modules M, N and all $n > d$.)

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes, locally of finite presentation. Show that if f is flat and surjective and gf is smooth, then g is smooth.

Problem 47. (1) Let A be a ring and let $R = A[x_0, \ldots, x_n]/I$, where I is a finitely generated graded ideal. Show that $X = \mathrm{Proj}(R)$ is smooth over A if and only if $\mathrm{Spec}(R)\backslash V(R_+)$ is smooth over A. (**Hint.** Identify the latter with $\mathbb{V}(\mathcal{O}_X(1))\backslash 0_X$, where 0_X denotes the zero section.)

(2) Let $n \geq 1$ and $d \geq 3$ be integers and let k be a field of characteristic $p \mid d$. Show that Gabber's hypersurface $X = \mathrm{Proj}(k[x_0, \ldots, x_n]/(f))$ in $\mathbb{P}^n$, where $f = x_0^d + \sum_{i=0}^{n-1} x_i x_{i+1}^{d-1}$, is smooth over k.

Problem 48. Let k be an infinite field. Let X be a variety over k admitting a dominant rational map $\mathbb{P}_k^n \dashrightarrow X$ over k (such a variety said to be **unirational**). Show that $X(k)$ is dense in X.

Problem 49. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Show that we have isomorphisms

$$Rf_* R\mathcal{H}om_X(Lf^*N, M) \simeq R\mathcal{H}om_Y(N, Rf_*M),$$
$$R\operatorname{Hom}_X(Lf^*N, M) \simeq R\operatorname{Hom}_Y(N, Rf_*M),$$

functorial in $M \in D(X)$ and $N \in D(Y)$.

Problem 50. (1) Let $f_i\colon X_i \to S$, $i = 1, 2$ be quasi-compact quasi-separated morphisms of schemes. Let $X := X_1 \times_S X_2$ and $f := f_1 \times_S f_2\colon X \to S$. Assume that f_1 is flat. Prove the Künneth formula

$$Rf_{1*}M_1 \otimes^L Rf_{2*}M_2 \simeq Rf_*(M_1 \boxtimes^L_S M_2)$$

for $M_i \in D_{\mathrm{qcoh}}(X_i)$. Here $M_1 \boxtimes^L_S M_2 := Lp_1^*M_1 \otimes^L_{\mathcal{O}_X} Lp_2^*M_2$ and $p_i\colon X \to X_i$ is the projection. (You may admit the fact that the flat base change theorem extends to D_{qcoh}.)

(2) Let X_1 and X_2 be proper smooth schemes over a field k. Express the Hodge numbers $h^{p,q}$ of $X := X_1 \times_{\mathrm{Spec}(k)} X_2$ in terms of those of X_1 and X_2.

Problem 51. Let A be a ring and let $P = \mathbb{P}^n_A$, where $n \geq 0$ is an integer.

(1) Show that $H^q(P, \Omega^p_{P/A}(m)) = 0$ unless one of the following holds:

(i) $0 \leq p = q \leq n$ and $m = 0$, in which case $H^p(P, \Omega^p_{P/A}) \simeq A$;

(ii) $q = 0$ and $m > p$;

(iii) $q = n$ and $m < p - n$.

(**Hint**. Use the exact sequence

$$0 \to \Omega^p_{P/A} \to \bigwedge^p(\mathcal{O}_P(-1)^{\oplus n+1}) \to \Omega^{p-1}_{P/A} \to 0.$$

The fact $H^q(P, \Omega^p_{P/A}(m)) = 0$ for $q > 0$ and $m > 0$ is called Bott vanishing.)

Assume in the sequel that $A = k$ is a field.

(b) Compute $\dim_k H^q(P, \Omega^p_{P/k}(m))$.

(c) Let $X \subseteq P$ be a hypersurface of degree d smooth over k. Show that the canonical map $H^q(P, \Omega^p_{P/k}(m)) \to H^q(X, \Omega^p_{X/k}(m))$ is an isomorphism for $p + q < n - 1$ and $m < d$. Deduce that $H^q(X, \Omega^p_{X/k}(m)) = 0$ for $p + q > n - 1$ and $m > 0$. (**Remark.** For k of characteristic zero, the last statement is a special case of the Kodaira vanishing theorem.)

Problem 52. Let k be an algebraically closed field and let X be a smooth projective curve over k of genus g. The **gonality** of X, denoted $\operatorname{gon}(X)$, is defined to be the least integer $d \geq 1$ such that there exists a morphism $X \to \mathbb{P}^1_k$ over k of degree d.

(1) Show that $\operatorname{gon}(X) = \min\{\deg(\mathcal{L}) \mid h^0(\mathcal{L}) \geq 2\}$.

(2) Show that $\operatorname{gon}(X) \leq g + 1$.

Problem 53. Let k be an algebraically closed field.

(1) Let X be a smooth projective curve of genus g over k. Let D be an effective divisor on X of degree $\geq 2g$. Show that D is rationally equivalent to an effective divisor D' on X disjoint from D. (**Hint.** Apply the Riemann-Roch theorem to D and $D - x$ for every x in the support of D.)

(2) Deduce that a curve C over k is either proper or affine. (**Hint.** Use Problem 41 to reduce to the case where C is smooth. Then apply (a) to an effective divisor whose support is precisely $\overline{C} \backslash C$. Here $\overline{C}$ denotes a smooth compactification of C.)

Problem 54. Let X be a nonempty scheme proper over a field k. The **arithmetic genus** of X is defined to be $g_a(X) := (-1)^{\dim(X)}(\chi(\mathcal{O}_X) - 1)$.

(1) Let X be a hypersurface of degree d in $\mathbb{P}^n_k$. Show that $g_a(X) = \binom{d-1}{n}$.

(2) Assume that k is algebraically closed. Let X be a proper curve over k. Show that $g_a(X) = g(X^\nu) + \sum_{x \in X} \dim_k(\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x})$, where X^ν denotes the normalization of X and $\mathcal{O}^\nu_{X,x}$ denotes the normalization of $\mathcal{O}_{X,x}$, and x runs through the singular points of X. Deduce that $g_a(X) = 0$ implies $X \simeq \mathbb{P}^1_k$.

Problem 55. Let k be a field, $R = k[x_0, \ldots, x_n]$, and $X = \operatorname{Proj}(R/I)$, where $I \subseteq R$ is the ideal generated by a regular sequence of $c \leq n$ homogeneous elements of positive degrees.

(1) Show that X has dimension $n - c$. We call X a **complete intersection** in $\mathbb{P}^n_k$. (**Remark.** In fact a complete intersection in $\mathbb{P}^n_k$ can be characterized as a scheme-theoretic intersection of dimension $n - c$ of c hypersurfaces in $\mathbb{P}^n_k$.)

(2) Assume that $n - c \geq 1$. Show that $H^0(\mathbb{P}^n_k, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective and $H^i(X, \mathcal{O}(m)) = 0$ for all $m \in \mathbb{Z}$ and $0 < i < n - c$. Deduce that X is geometrically connected.

(3) Let X be a complete intersection of a hypersurface of degree d and a hypersurface of degree e in $\mathbb{P}^3_k$. Show that $g_a(X) = \frac{1}{2}de(d + e - 4) + 1$.

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