On the derived Lusztig correspondence

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This is joint work with Emmanuel Letellier.

Let k be an algebraic closed field of positive characteristic and ℓ a prime number which is invertible in k

Let $G = GL_n$. Let T be the maximal torus of diagonal matrices and B = TU be the Borel subgroup of upper triangular matrices.

Let $N_G(T)$ be the normalizer of T in G and $W = N_G(T)/T$ be the Weyl group of G.

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Lusztig induction and restriction

Let

$$X = \{(x, t, gB) \in \mathfrak{g} \times G/B \mid g^{-1}xg \in t + \mathfrak{u} \subset \mathfrak{b}\}$$

and its two projections

$$\mathfrak{t} \stackrel{q}{\longleftrightarrow} X \stackrel{p}{\longrightarrow} \mathfrak{g} .$$

We have the Lusztig induction and restriction functors

$$\mathrm{Ind}: D^{\mathsf{b}}_{\mathsf{c}}(\mathfrak{t}) \to D^{\mathsf{b}}_{\mathsf{c}}(\mathfrak{g}), \ K \mapsto p_! q^* K$$

$$\operatorname{Res}: D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{g}) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{t}), \ K \mapsto q_{!}p^{*}K$$

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Cohomological correspondance

Notice that \mathfrak{g} , \mathfrak{t} and X are schemes over

$$\mathfrak{car} := \mathfrak{t}/\!/W = \operatorname{Spec}(k[\mathfrak{t}]^W) = \operatorname{Spec}(k[\mathfrak{g}]^G) = \mathfrak{g}/\!/G$$

and that p and q are car-morphisms. Concretely, car is the affine space of unitary polynomials of degree n and $\mathfrak{t} \subset \mathfrak{g} \to \mathfrak{car}$ takes t or x to its characteristic polynomial.

Then we have the following commutative diagram



where f = (q, p).

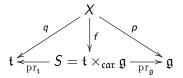
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Operator with kernel version

Lemma

- $X \xrightarrow{f} S \xrightarrow{\operatorname{pr}_{\mathfrak{g}}} \mathfrak{g}$ is the Stein factorization of p.
- $\operatorname{pr}_{\mathfrak{g}}: S \to \mathfrak{g}$ is a Galois ramified covering with group W.
- $f: X \to S$ is a small resolution of singularities of S.

In particular

$$\mathrm{IC}_{S}=f_{!}\mathbb{Q}_{\ell,S}\in D^{\mathsf{b}}_{\mathsf{c}}(S).$$

The functors Ind and Res may be also defined by

 $\mathrm{Ind}(K) = \mathrm{pr}_{\mathfrak{g},!}\left(\mathrm{pr}_{\mathfrak{t}}^*K \otimes \mathrm{IC}_{\mathcal{S}}\right)$

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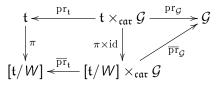
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W-equivariant version

Let $\mathcal{G} = [\mathfrak{g}/G]$ (adjoint action). We have the commutative diagram



where π and $\pi \times id$ are *W*-torsors.

IC_S descends to the intersection complex $IC_{[t/W] \times_{cat} \mathcal{G}}$.

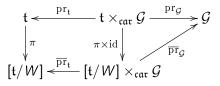
Therefore the functors Res and Ind factorize as

Ind = $I \circ \pi_{I}$ and $\text{Res} = \pi^* \circ \text{R}$

with I : $D^{b}_{c}([\mathfrak{t}/W]) \to D^{b}_{c}(\mathcal{G})$ and $R : D^{b}_{c}(\mathcal{G}) \to D^{b}_{c}([\mathfrak{t}/W])$.

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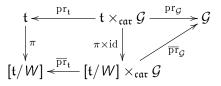
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Over the regular semisimple open subset

The previous construction is neat, but not perfect.

Over cat^{rss} , the regular semisimple open subset (unitary degree *n* polynomials with *n* distincts roots), our correspondance is

$$\mathfrak{cat}^{\mathsf{rss}} = [\mathfrak{t}^{G\operatorname{-}\mathsf{reg}}/W] \xleftarrow{\operatorname{pr}_\mathfrak{t}} \mathcal{S}^{\mathsf{rss}} \overset{\operatorname{pr}_\mathfrak{g}}{=} \mathcal{G}^{\mathsf{rss}} = [\mathfrak{t}^{G\operatorname{-}\mathsf{reg}}/\operatorname{N}_G(\mathcal{T})]$$

and I^{rss} and R^{rss} are the pullback and the direct image by the morphism $\mathcal{G}^{rss} \to \mathfrak{cat}^{rss}$.

So it is natural to try to replace $[\mathfrak{t}/W]$ by $[\mathfrak{t}/N_{\mathcal{G}}(\mathcal{T})]$.

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Stack version

We have:

• the quotient car-stacks

$$\mathcal{G} = [\mathfrak{g}/G], \quad \mathcal{T} = [\mathfrak{t}/T], \quad \mathcal{B} = [X/G] = [\mathfrak{b}/B]$$

for the adjoint actions (trivial action for T);

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where q is induced by the projections $\mathfrak{b} \to \mathfrak{t}$ and $B \to T$ and p is induced by the inclusions $\mathfrak{b} \subset \mathfrak{g}$ and $B \subset G$.

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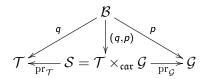
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Cohomological correspondence for the Lusztig functors

We have the Lusztig induction and restriction functors

$$\mathrm{Ind}: D^{\mathsf{b}}_{\mathsf{c}}(\mathcal{T}) \to D^{\mathsf{b}}_{\mathsf{c}}(\mathcal{G}), \ \mathcal{K} \mapsto p_! q^* \mathcal{K} = \mathrm{pr}_{\mathcal{G}, !}(\mathrm{pr}_{\mathcal{T}}^* \mathcal{K} \otimes \mathcal{N})$$

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is the cohomological kernel.

The new construction is perfect over the regular semisimple open subset of \mathfrak{cat} , but the kernel $\mathcal N$ is not anymore an intersection complex.

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Main question

We denote by $\pi : \mathcal{T} = [\mathfrak{t}/T] \to \overline{\mathcal{T}} := [\mathfrak{t}/N_G(\mathcal{T})]$ the map induced by $\mathcal{T} \subset N_G(\mathcal{T})$. It is a *W*-torsor.

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(2) Does the functors I and R are inverse equivalences of categories between $D_c^b(\overline{\mathcal{T}})$ and $D_c^b(\mathcal{G})$?

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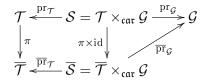
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Main question : descent of \mathcal{N} ?

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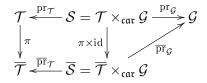
Can we equip \mathcal{N} with a W-equivariant structure and thus descend it to a complex $\overline{\mathcal{N}}$ on $\overline{\mathcal{T}} \times_{cat} \mathcal{G}$.

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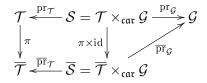
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For any locally closed smooth subset $C \subset \mathfrak{car}$, denote by an $(-)_C$ the restriction to C of any object over \mathfrak{car} .

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Our result

Let us consider the stratification

$$\mathfrak{car} = \bigcup_{\nu}^{\cdot} C_{\nu},$$

 $u = (\nu_1 \ge \nu_2 \ge \cdots)$ partition of *n*, C_{ν} the set of polynomials $c(z) = (z - t_1)^{\nu_1} (z - t_2)^{\nu_2} \cdots$ with $t_i \ne t_j$ if $i \ne j$.

Theorem

The complex $\mathcal{N}|C_{\nu}$ can be equipped with a W-invariant structure for any ν . We thus have functors $I_{C_{\nu}}$ and $R_{C_{\nu}}$ such that

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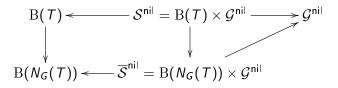
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For the trivial partition, C_(n) = A¹ and over C_(n) of our main diagram becomes the product by A¹ of



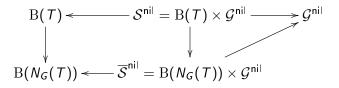
where $\mathcal{G}^{\mathsf{ni}|}$ is the nilpotent cone. Therefore $D^{\mathsf{b}}_{\mathsf{c}}(\mathrm{B}(N_G(\mathcal{T})))\cong D^{\mathsf{b}}_{\mathsf{c}}(\mathcal{G}^{\mathsf{ni}|}).$

Laura Rider has computed $D_c^b(\mathcal{G}^{nil})$ by other means.

• By a descent to Levi argument, we are essentially reduced to prove that $\mathcal{N}^{\text{nil}} = \mathcal{N}|\mathcal{S}^{\text{nil}}$ descends to $\overline{\mathcal{S}}^{\text{nil}}$.

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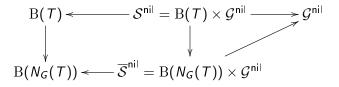
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Laura Rider has computed $D_c^b(\mathcal{G}^{nil})$ by other means.

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Descent of $\mathcal{N}^{\mathsf{nil}}(\mathsf{I})$

In the rest of the talk, I will explain how we descend \mathcal{N}^{nil} to $B(N_G(\mathcal{T})) \times \mathcal{G}^{nil}$. Here are the main steps:

• $B(T) \times \mathcal{G}^{nil}$ can be embedded into $B(T) \times \mathcal{G}$ and \mathcal{N}^{nil} is the restriction to $B(T) \times \mathcal{G}^{nil}$ of

$$\mathcal{P} = \hat{p}_! \delta_! \mathbb{Q}_{\ell, \mathcal{B}}$$

where

$$\delta: \mathcal{B} \xrightarrow{\text{diag}} \mathcal{B} \times \mathcal{B} \longrightarrow B(\mathcal{T}) \times \mathcal{B} = \hat{\mathcal{B}}$$

is a *T*-torsor and

$$\hat{p} = \mathrm{id} \times p : \hat{\mathcal{B}} \to \mathrm{B}(\mathcal{T}) \times \mathcal{G} = \hat{\mathcal{G}}$$

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- \mathcal{P} is thus the *outcome* of a *Postnikov diagram*, whose vertices of the *base* of are intersection complexes.
- By a weight argument, we prove that the base of that Postnikov diagram is *rigid*, i.e. the Postnikov diagram is completely determined by its base.
- Therefore, in order to descend the Postnikov diagram, it is enough to descend its base.
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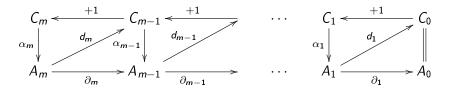
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Postnikov diagrams

In a triangulated category \mathcal{D} , a *Postnikov diagram* Λ is a diagram



where the upper triangles are all distinguished and the bottom triangles are all commutative. Notice that $\partial_{i-1} \circ \partial_i = 0, \forall i$.

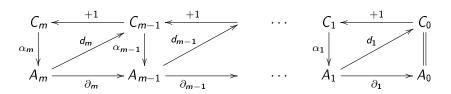
The base $\Lambda_{
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$$A_m \xrightarrow{\partial_m} A_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

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The Postnikov diagram $\Lambda(K)$

If \mathcal{D} is equipped with a non degenerate t-structure, then we can define a Postnikov diagram $\Lambda(K)$ for any $K \in \mathcal{D}^{[n,n+m]}$ as follows.

For each integer $0 \le i \le m$, we have the usual distinguished triangle

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Those distinguished triangles define a Postnikov diagram $\Lambda(K)$: we put

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The *T*-torsor $\delta : \mathcal{B} \to \hat{\mathcal{B}} = B(T) \times \mathcal{B}$ has a Chern class morphism $X^*(T) \to H^2(\hat{\mathcal{B}}, \mathbb{Q}_\ell)(1)$ which may be viewed as a morphism

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$\mathcal{P} = \hat{p}_* \delta_! \mathbb{Q}_{\ell, \mathcal{B}}$ is the outcome of the Postnikov diagram $\Lambda = \hat{p}_* \Lambda(\delta_! \mathbb{Q}_{\ell, \mathcal{B}}).$

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Rigid Postinikow complex

A complex

$$A = (A_m \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0)$$

in \mathcal{D} is a *rigid Postinikow complex* if it satisfies

$$\operatorname{Hom}(A_j, A_i) = 0, \ \forall j < i,$$

and

$$\operatorname{Hom}(A_j,A_i[-k])=0, \ \forall j>i, \ k\geq 1.$$

Theorem

Given a rigid Postinikow complex A in \mathcal{D} , then there exists a unique (up to a unique isomorphism) Postnikov diagram Λ in \mathcal{D} such that $\Lambda_{\rm b} = A$.

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All the stacks and stack morphisms that we have introduce are defined over a finite field $k_o \subset k$ and cohomological weights make sense.

Proposition (Purity)

 $\operatorname{Ext}_{\mathcal{G}}^{2i+1}(\operatorname{IC}_{\mathcal{G}}(\mathcal{L}), \operatorname{IC}_{\mathcal{G}}(\mathcal{L})) = 0$ and $\operatorname{Ext}_{\mathcal{G}}^{2i}(\operatorname{IC}_{\mathcal{G}}(\mathcal{L}), \operatorname{IC}_{\mathcal{G}}(\mathcal{L}))$ is pure of weight 2*i* for all *i*.

Since $\hat{\mathcal{G}} = B(T) \times \mathcal{G}$ and since the statement of the proposition is true if we replace \mathcal{G} by B(T), by Künneth formula we get:

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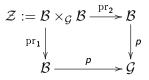
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Proof of the purity property (I)

Consider the cartesian diagram



Since $\mathrm{IC}_{\mathcal{G}}(\mathcal{L}) = p_! \mathbb{Q}_{\ell,\mathcal{B}_1}$ by the base change theorem we have

$$\operatorname{Ext}^{j}(\operatorname{IC}_{\mathcal{G}}(\mathcal{L}),\operatorname{IC}_{\mathcal{G}}(\mathcal{L}))=H^{j}(\mathcal{Z},\operatorname{pr}_{2}^{!}\mathbb{Q}_{\ell,\mathcal{B}}),$$

and therefore, by Poincaré duality, we have

$$\operatorname{Ext}^{j}(\operatorname{IC}_{\mathcal{G}}(\mathcal{L}), \operatorname{IC}_{\mathcal{G}}(\mathcal{L})) = H_{c}^{-j}(\mathcal{Z}, \mathbb{Q}_{\ell})^{\vee},$$

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The canonical morphisms $\mathcal{B} o \operatorname{B}(B)$ and $\mathcal{G} o \operatorname{B}(G)$ induce a morphism

$$\varphi: \mathcal{Z} \to \mathcal{B}(B) \times_{\mathcal{B}(G)} \mathcal{B}(B) = [B \setminus G/B].$$

Denote

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We have $[B \setminus BwB/B] = B(B_w)$ with $B_w = TU_w = B \cap wBw^{-1}$, and $\mathcal{Z}_w = [\mathfrak{b}_w/B_w]$ where \mathfrak{b}_w is the Lie algebra of B_w .

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Denote

$$\mathcal{Z} = igcup_{w \in W} \mathcal{Z}_w$$

the inverse image by φ of the stratification

$$[B \setminus G/B] = \bigcup_{w \in W} [B \setminus BwB/B].$$

Lemma

We have $[B \setminus BwB/B] = B(B_w)$ with $B_w = TU_w = B \cap wBw^{-1}$, and $\mathcal{Z}_w = [\mathfrak{b}_w/B_w]$ where \mathfrak{b}_w is the Lie algebra of B_w .

Proof of the purity property (III)

Choosing a suitable total order $\{w_0, w_1, ...\}$ on W, we get a decreasing filtration

$$\mathcal{Z} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots \supset \mathcal{Z}_{|W|-1} \supset \mathcal{Z}_{|W|} = \emptyset$$

by closed substacks such that $Z_i \setminus Z_{i+1} = Z_{w_i}$ for all *i*.

By a standard spectral sequence argument, the purity property follows from

$$H^i_c(\mathcal{Z}_w, \mathbb{Q}_\ell) = H^{i-2\dim(\mathfrak{t})}_c(\mathrm{B}(\mathcal{T}), \mathbb{Q}_\ell)(-\dim(\mathfrak{t}))$$

and the purity of the cohomology with compact supports of $\mathrm{B}(\mathcal{T}).$

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Postnikov rigidity of $\Lambda_{\rm b}$

By the standard exact sequence relating cohomology of a stack over k_o to the cohomology of its extension to k, we get:

Corollary For all j < i, we have

$$\operatorname{Hom}(\operatorname{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2j](-j), \operatorname{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2i](-i)) = 0$$

and for all j > i and $k \ge 1$, we have

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(indice o for taking into account the k_o -structure).

Proposition

The Postnikov diagram $\Lambda = \Lambda(\mathcal{P})$, $\mathcal{P} = \hat{p}_{!}\delta_{!}\mathbb{Q}_{\ell,\mathcal{B}}$, is the unique Postnikov diagram defined over k_{o} that completes Λ_{b} .

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Therefore, in order to show that \mathcal{P} , which is the outcome of Λ , descends from $B(\mathcal{T}) \times_{car} \mathcal{G}$ to $B(N_G(\mathcal{T})) \times_{car} \mathcal{G}$, it is enough to prove that Λ_b descends.

The intersection complex $\mathrm{IC}_{\hat{\mathcal{C}}}(\hat{\mathcal{L}})$ descends to $\mathrm{B}(N_{\mathcal{G}}(\mathcal{T})) \times_{\mathrm{cat}} \mathcal{G}$.

We are thus reduced to prove that

 $\hat{p}_*(c_{\delta}) : \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}) \to X_*(\mathcal{T}) \otimes \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}})[2](1)$

is W-equivariant.

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Key propositions

Proposition

If we now denote by δ the T-torsor $\dot{\mathcal{B}} = [\mathfrak{b}/U] \rightarrow [\mathfrak{b}/B] = \mathcal{B}$, then its Chern class

$$c_{\delta}: X^*(\mathcal{T})
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is W-equivariant for the Springer action on the target.

Proposition

The functoriality morphism

 $p_*: H^2(\mathcal{B}, \mathbb{Q}_{\ell}) = \operatorname{Hom}_{\mathcal{B}}(\mathbb{Q}_{\ell, \mathcal{B}}, \mathbb{Q}_{\ell, \mathcal{B}}[2]) \to \operatorname{Hom}_{\mathcal{G}}(p_*\mathbb{Q}_{\ell, \mathcal{B}}, p_*\mathbb{Q}_{\ell, \mathcal{B}}[2])$

is W-equivariant for the actions on the source and the target induced by the W-action on $p_*\mathbb{Q}_{\ell,\mathcal{B}}=\mathrm{IC}_\mathcal{G}(\mathcal{L}).$

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Proof of *W*-equivariance of c_{δ}

The *T*-torsor $\dot{\mathcal{B}} \to \mathcal{B}$ is the pull-back of the canonical *T*-torsor $\operatorname{Spec}(k) \to \operatorname{B}(T)$ by the morphism $\mathcal{B} = [\mathfrak{b}/B] \to \operatorname{B}(T)$ induced by $\mathfrak{b} \to \operatorname{Spec}(k)$ and $B \to T$.

Moreover the restriction map $H^2(\mathcal{B}(\mathcal{T}), \mathbb{Q}_\ell) \to H^2(\mathcal{B}, \mathbb{Q}_\ell)$ is a W-equivariant isomorphism.

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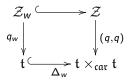
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For $w \in W$, we have the strata $\mathcal{Z}_w = [\mathfrak{b}_w/B_w]$ and a commutative square

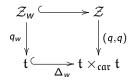


where $\Delta_w(t) = (t, w(t))$.

It follows that the cohomology sheaves $(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}}$ is the abutment of a spectral sequence starting in E_2 with the cohomology sheaves of the

 $\Delta_{w,*}q_{w,!}\mathbb{Q}_{\ell,\mathcal{Z}_w}=\Delta_{w,*}\mathbb{Q}_{\ell,\mathfrak{t}}\otimes R\Gamma_{\mathsf{c}}(\mathrm{B}(\mathcal{T}),\mathbb{Q}_{\ell}).$

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As $H^{\mathrm{odd}}_{\mathsf{c}}(\mathrm{B}(\mathcal{T}),\mathbb{Q}_{\ell})=0$, the spectral sequence degenerate in E_2 .

It follows that $R^{\text{odd}}(q,q)_! \mathbb{Q}_{\ell,\mathcal{Z}} = 0$ and that each $R^{2i}(q,q)_! \mathbb{Q}_{\ell,\mathcal{Z}}$ is successive extensions of the $\Delta_{w,*} \mathbb{Q}_{\ell,t} \otimes H^{2i}_c(B(\mathcal{T}), \mathbb{Q}_{\ell})$.

Consequently, up to a shift, $R^{2i}(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}}$ is a perverse sheaf on $t \times_{car} t$ which is the middle extension of its restriction to the open subset $t^{reg} \times_{car} t^{reg}$.

We thus have a $(W \times W)$ -equivariant morphism

 $R^{2i}(q,q)_{!}\mathbb{Q}_{\ell,\mathcal{Z}} = \Delta_{*}\mathbb{Q}_{\ell} \otimes H_{\mathsf{c}}^{-2i}(\mathrm{B}(\mathcal{T}),\mathbb{Q}_{\ell})$

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for $j \neq 2n = 2\dim(\mathfrak{t})$.

As a consequence, we get a (W imes W)-equivariant isomorphism

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