On the derived Lusztig correspondence

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CNRS and Université Paris-Sud

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Notations

This is joint work with Emmanuel Letellier.

Let k be an algebraic closed field of positive characteristic and ℓ a prime number which is invertible in k

Let $G = \mathrm{GL}_n$. Let T be the maximal torus of diagonal matrices and $B = TU$ be the Borel subgroup of upper triangular matrices.

Let $\mathrm{N}_G(T)$ be the normalizer of T in G and $W = \mathrm{N}_G(T)/T$ be the Weyl group of G.

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Lusztig induction and restriction

Let

$$X = \{(x, t, gB) \in \mathfrak{g} \times G/B \mid g^{-1}xg \in t + \mathfrak{u} \subset \mathfrak{b}\}$$

and its two projections

$$\mathfrak{t} \xleftarrow{q} X \xrightarrow{p} \mathfrak{g}\ .$$

We have the Lusztig induction and restriction functors

$$\mathrm{Ind} : D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{t}) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{g}),\ K \mapsto p_! q^* K$$

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between bounded derived categories of constructible ℓ-adic sheaves.

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Cohomological correspondance

Notice that $\mathfrak{g}$, $\mathfrak{t}$ and X are schemes over

$$\mathfrak{car} := \mathfrak{t}//W = \mathrm{Spec}(k[\mathfrak{t}]^W) = \mathrm{Spec}(k[\mathfrak{g}]^G) = \mathfrak{g}//G$$

and that p and q are $\mathfrak{car}$-morphisms. Concretely, $\mathfrak{car}$ is the affine space of unitary polynomials of degree n and $\mathfrak{t} \subset \mathfrak{g} \to \mathfrak{car}$ takes t or x to its characteristic polynomial.

Then we have the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "q"'] \arrow[d, "f"] \arrow[dr, "p"] & \\
\mathfrak{t} & S = \mathfrak{t} \times_{\mathfrak{car}} \mathfrak{g} \arrow[l, "\mathrm{pr}_{\mathfrak{t}}"] \arrow[r, "\mathrm{pr}_{\mathfrak{g}}"'] & \mathfrak{g}
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Operator with kernel version

Lemma

- $X \xrightarrow{f} S \xrightarrow{\mathrm{pr}_{\mathfrak{g}}} \mathfrak{g}$ *is the Stein factorization of* p*.*
- $\mathrm{pr}_{\mathfrak{g}} : S \to \mathfrak{g}$ *is a Galois ramified covering with group* W*.*
- $f : X \to S$ *is a small resolution of singularities of* S*.*

In particular

$$\mathrm{IC}_S = f_! \mathbb{Q}_{\ell, S} \in D^b_c(S).$$

The functors Ind and Res may be also defined by

$$\mathrm{Ind}(K) = \mathrm{pr}_{\mathfrak{g},!}\left(\mathrm{pr}_{\mathfrak{t}}^* K \otimes \mathrm{IC}_S\right)$$

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W-equivariant version

Let $\mathcal{G} = [\mathfrak{g}/G]$ (adjoint action). We have the commutative diagram

$$\begin{tikzcd}
\mathfrak{t} \arrow[d, "\pi"] & \mathfrak{t} \times_{\mathfrak{car}} \mathcal{G} \arrow[l, "\mathrm{pr}_{\mathfrak{t}}"'] \arrow[r, "\mathrm{pr}_{\mathcal{G}}"] \arrow[d, "\pi\times\mathrm{id}"'] & \mathcal{G} \\
{[\mathfrak{t}/W]} & {[\mathfrak{t}/W] \times_{\mathfrak{car}} \mathcal{G}} \arrow[l, "\overline{\mathrm{pr}}_{\mathfrak{t}}"'] \arrow[ur, "\overline{\mathrm{pr}}_{\mathcal{G}}"'] &
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where π and $\pi \times \mathrm{id}$ are W-torsors.

IC_S descends to the intersection complex $\mathrm{IC}_{[\mathfrak{t}/W]\times_{\mathfrak{car}}\mathcal{G}}$.

Therefore the functors Res and Ind factorize as

$$\mathrm{Ind} = \mathrm{I} \circ \pi_! \quad \text{and} \quad \mathrm{Res} = \pi^* \circ \mathrm{R}$$

with $\mathrm{I} : D^b_c([\mathfrak{t}/W]) \to D^b_c(\mathcal{G})$ and $\mathrm{R} : D^b_c(\mathcal{G}) \to D^b_c([\mathfrak{t}/W])$.

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Over the regular semisimple open subset

The previous construction is neat, but not perfect.

Over $\mathfrak{car}^{\mathrm{rss}}$, the regular semisimple open subset (unitary degree n polynomials with n distincts roots), our correspondance is

$$\mathfrak{car}^{\mathrm{rss}} = [\mathfrak{t}^{G\text{-reg}}/W] \xleftarrow{\mathrm{pr}_{\mathfrak{t}}} \mathcal{S}^{\mathrm{rss}} \overset{\mathrm{pr}_{\mathfrak{g}}}{=} \mathcal{G}^{\mathrm{rss}} = [\mathfrak{t}^{G\text{-reg}}/\mathrm{N}_G(T)]$$

and I^{rss} and R^{rss} are the pullback and the direct image by the morphism $\mathcal{G}^{\mathrm{rss}} \to \mathfrak{car}^{\mathrm{rss}}$.

So it is natural to try to replace $[\mathfrak{t}/W]$ by $[\mathfrak{t}/\mathrm{N}_G(T)]$.

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Stack version

We have:

- the quotient $\mathfrak{car}$-stacks

$$\mathcal{G} = [\mathfrak{g}/G], \quad \mathcal{T} = [\mathfrak{t}/T], \quad \mathcal{B} = [X/G] = [\mathfrak{b}/B]$$

for the adjoint actions (trivial action for T);

- the commutative diagram

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where q is induced by the projections $\mathfrak{b} \to \mathfrak{t}$ and $B \to T$ and p is induced by the inclusions $\mathfrak{b} \subset \mathfrak{g}$ and $B \subset G$.

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Cohomological correspondence for the Lusztig functors

We have the Lusztig induction and restriction functors

$$\mathrm{Ind} : D^b_c(\mathcal{T}) \to D^b_c(\mathcal{G}),\ K \mapsto p_! q^* K = \mathrm{pr}_{\mathcal{G},!}(\mathrm{pr}^*_{\mathcal{T}} K \otimes \mathcal{N})$$

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where

$$\mathcal{N} = (q,p)_! \mathbb{Q}_{\ell,\mathcal{B}} \in D^b_c(\mathcal{T} \times_{\mathfrak{car}} \mathcal{G})$$

is the cohomological kernel.

The new construction is perfect over the regular semisimple open subset of $\mathfrak{car}$, but the kernel $\mathcal{N}$ is not anymore an intersection complex.

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Main question

We denote by $\pi : \mathcal{T} = [\mathfrak{t}/T] \to \overline{\mathcal{T}} := [\mathfrak{t}/\mathrm{N}_G(T)]$ the map induced by $T \subset \mathrm{N}_G(T)$. It is a W-torsor.

Question

(1) Does the functors Res *and* Ind *factorize as*

$$\mathrm{Ind} = \mathrm{I} \circ \pi_! \quad \textit{and} \quad \mathrm{Res} = \pi^* \circ \mathrm{R}$$

for some triangulated functors

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(2) Does the functors I *and* R *are inverse equivalences of categories between* $D^b_c(\overline{\mathcal{T}})$ *and* $D^b_c(\mathcal{G})$ *?*

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Main question : descent of $\mathcal{N}$?

We have the following commutative diagram

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where $\pi : \mathcal{T} \to \overline{\mathcal{T}}$ and $\pi \times \mathrm{id}$ are W-torsors.

Part (1) of our question can be rephrased as:

Can we equip $\mathcal{N}$ with a W-equivariant structure and thus descend it to a complex $\overline{\mathcal{N}}$ on $\overline{\mathcal{T}} \times_{\mathfrak{car}} \mathcal{G}$.

Our functors I and R would then be defined by

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C version

For any locally closed smooth subset $C \subset \mathfrak{car}$, denote by an $(-)_C$ the restriction to C of any object over $\mathfrak{car}$.

Notice that $f_C : X_C \to S_C$ is a semi-small resolution of singularities of S_C.

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$$\mathcal{N}|\mathcal{S}_C = (q,p)_{C,!}\mathbb{Q}_{\ell,\mathcal{B}_C},$$

which are compatible with Ind and Res.

Our result

Let us consider the stratification

$$\mathfrak{car} = \dot{\bigcup_{\nu}} C_\nu,$$

$\nu = (\nu_1 \geq \nu_2 \geq \cdots)$ partition of n, C_ν the set of polynomials $c(z) = (z - t_1)^{\nu_1}(z - t_2)^{\nu_2} \cdots$ with $t_i \neq t_j$ if $i \neq j$.

Theorem

The complex $\mathcal{N}|C_\nu$ can be equipped with a W-invariant structure for any ν. We thus have functors I_{C_ν} and R_{C_ν} such that

$$\mathrm{Ind}_{C_\nu} = \mathrm{I}_{C_\nu} \circ \pi_{C_\nu,!} \quad \textit{and} \quad \mathrm{Res}_{C_\nu} = \pi^*_{C_\nu} \circ \mathrm{R}_{C_\nu}.$$

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Remarks

- For the trivial partition, $C_{(n)} = \mathbb{A}^1$ and over $C_{(n)}$ of our main diagram becomes the product by $\mathbb{A}^1$ of

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\mathrm{B}(T) \arrow[d] & \mathcal{S}^{\mathsf{nil}} = \mathrm{B}(T) \times \mathcal{G}^{\mathsf{nil}} \arrow[l] \arrow[r] \arrow[d] & \mathcal{G}^{\mathsf{nil}} \\
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Descent of $\mathcal{N}^{\mathsf{nil}}$ (I)

In the rest of the talk, I will explain how we descend $\mathcal{N}^{\mathsf{nil}}$ to $\mathrm{B}(N_G(T)) \times \mathcal{G}^{\mathsf{nil}}$. Here are the main steps:

- $\mathrm{B}(T) \times \mathcal{G}^{\mathsf{nil}}$ can be embedded into $\mathrm{B}(T) \times \mathcal{G}$ and $\mathcal{N}^{\mathsf{nil}}$ is the restriction to $\mathrm{B}(T) \times \mathcal{G}^{\mathsf{nil}}$ of

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- $\mathcal{P}$ is thus the *outcome* of a *Postnikov diagram*, whose vertices of the *base* of are intersection complexes.
- By a weight argument, we prove that the base of that Postnikov diagram is *rigid*, i.e. the Postnikov diagram is completely determined by its base.
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Postnikov diagrams

In a triangulated category $\mathcal{D}$, a *Postnikov diagram* Λ is a diagram

$$\begin{tikzcd}
C_m \arrow[d, "\alpha_m"'] & C_{m-1} \arrow[l, "+1"'] \arrow[d, "\alpha_{m-1}"'] & {} \arrow[l, "+1"'] & \cdots & C_1 \arrow[d, "\alpha_1"'] & C_0 \arrow[l, "+1"'] \arrow[d, equal] \\
A_m \arrow[r, "\partial_m"'] \arrow[ur, "d_m"] & A_{m-1} \arrow[r, "\partial_{m-1}"'] \arrow[ur, "d_{m-1}"] & {} & \cdots & A_1 \arrow[r, "\partial_1"'] \arrow[ur, "d_1"] & A_0
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where the upper triangles are all distinguished and the bottom triangles are all commutative. Notice that $\partial_{i-1} \circ \partial_i = 0, \ \forall i$.

The *base* Λ_{b} of Λ is the *complex*

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If $\mathcal{D}$ is equipped with a non degenerate t-structure, then we can define a Postnikov diagram $\Lambda(K)$ for any $K \in \mathcal{D}^{[n,n+m]}$ as follows.

For each integer $0 \leq i \leq m$, we have the usual distinguished triangle

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Those distinguished triangles define a Postnikov diagram $\Lambda(K)$: we put

$$C_i := \tau_{\leq n+i} K[n-i], \qquad A_i := \mathcal{H}^{n+i} K[-2i],$$

we define the distinguished triangles of the Postnikov diagram by rotating the above triangles and we set $\partial_i = \alpha_{i-1} \circ d_i$.

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The T-torsor $\delta : \mathcal{B} \to \hat{\mathcal{B}} = \mathrm{B}(T) \times \mathcal{B}$ has a Chern class morphism $X^*(T) \to H^2(\hat{\mathcal{B}}, \mathbb{Q}_\ell)(1)$ which may be viewed as a morphism

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Rigid Postinikow complex

A complex

$$A = (\ A_m \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0\ )$$

in $\mathcal{D}$ is a *rigid Postinikow complex* if it satisfies

$$\mathrm{Hom}(A_j, A_i) = 0,\ \forall j < i,$$

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Theorem

Given a rigid Postinikow complex A in $\mathcal{D}$, then there exists a unique (up to a unique isomorphism) Postnikov diagram Λ in $\mathcal{D}$ such that $\Lambda_{\mathrm{b}} = A$.

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Purity property

All the stacks and stack morphisms that we have introduce are defined over a finite field $k_o \subset k$ and cohomological weights make sense.

Proposition (Purity)

$\mathrm{Ext}^{2i+1}_{\mathcal{G}}(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L})) = 0$ *and* $\mathrm{Ext}^{2i}_{\mathcal{G}}(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L}))$ *is pure of weight* $2i$ *for all* i*.*

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$\mathrm{Ext}^{2i+1}_{\hat{\mathcal{G}}}(\mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}), \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}})) = 0$ *and* $\mathrm{Ext}^{2i}_{\hat{\mathcal{G}}}(\mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}), \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}))$ *is pure of weight* $2i$ *for all* i.

Purity property

All the stacks and stack morphisms that we have introduce are defined over a finite field $k_o \subset k$ and cohomological weights make sense.

Proposition (Purity)

$\mathrm{Ext}^{2i+1}_{\mathcal{G}}(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L})) = 0$ and $\mathrm{Ext}^{2i}_{\mathcal{G}}(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L}))$ is pure of weight $2i$ for all i.

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Proof of the purity property (I)

Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{Z} := \mathcal{B} \times_{\mathcal{G}} \mathcal{B} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{B} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "p"] & \mathcal{G} \end{tikzcd}$$

Since $\mathrm{IC}_{\mathcal{G}}(\mathcal{L}) = p_! \mathbb{Q}_{\ell,\mathcal{B}}$, by the base change theorem we have

$$\mathrm{Ext}^j(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L})) = H^j(\mathcal{Z}, \mathrm{pr}_2^! \mathbb{Q}_{\ell,\mathcal{B}}),$$

and therefore, by Poincaré duality, we have

$$\mathrm{Ext}^j(\mathrm{IC}_{\mathcal{G}}(\mathcal{L}), \mathrm{IC}_{\mathcal{G}}(\mathcal{L})) = H_c^{-j}(\mathcal{Z}, \mathbb{Q}_\ell)^\vee,$$

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The canonical morphisms $\mathcal{B} \to \mathrm{B}(B)$ and $\mathcal{G} \to \mathrm{B}(G)$ induce a morphism

$$\varphi : \mathcal{Z} \to \mathrm{B}(B) \times_{\mathrm{B}(G)} \mathrm{B}(B) = [B \backslash G / B].$$

Denote

$$\mathcal{Z} = \dot{\bigcup_{w \in W}} \mathcal{Z}_w$$

the inverse image by φ of the stratification

$$[B \backslash G / B] = \dot{\bigcup_{w \in W}} [B \backslash BwB / B].$$

Lemma

We have $[B \backslash BwB / B] = \mathrm{B}(B_w)$ with $B_w = TU_w = B \cap wBw^{-1}$, and $\mathcal{Z}_w = [\mathfrak{b}_w / B_w]$ where $\mathfrak{b}_w$ is the Lie algebra of B_w.

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Proof of the purity property (III)

Choosing a suitable total order $\{w_0, w_1, \dots\}$ on W, we get a decreasing filtration

$$\mathcal{Z} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots \supset \mathcal{Z}_{|W|-1} \supset \mathcal{Z}_{|W|} = \emptyset$$

by closed substacks such that $\mathcal{Z}_i \backslash \mathcal{Z}_{i+1} = \mathcal{Z}_{w_i}$ for all i.

By a standard spectral sequence argument, the purity property follows from

$$H^i_c(\mathcal{Z}_w, \mathbb{Q}_\ell) = H_c^{i-2\dim(\mathfrak{t})}(\mathrm{B}(T), \mathbb{Q}_\ell)(-\dim(\mathfrak{t}))$$

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Postnikov rigidity of Λ_{b}

By the standard exact sequence relating cohomology of a stack over k_o to the cohomology of its extension to k, we get:

Corollary

For all $j < i$, we have

$$\mathrm{Hom}(\mathrm{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2j](-j), \mathrm{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2i](-i)) = 0$$

and for all $j > i$ and $k \geq 1$, we have

$$\mathrm{Hom}(\mathrm{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2j](-j), \mathrm{IC}_{\hat{\mathcal{G}}_o}(\mathcal{L}_o)[-2i-k](-i)) = 0$$

(indice o for taking into account the k_o-structure).

Proposition

The Postnikov diagram $\Lambda = \Lambda(\mathcal{P})$, $\mathcal{P} = \hat{p}_! \delta_! \mathbb{Q}_{\ell,\mathcal{B}}$, is the unique Postnikov diagram defined over k_o that completes Λ_{b}.

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Descent of $\mathcal{P}$

Therefore, in order to show that $\mathcal{P}$, which is the outcome of Λ, descends from $\mathrm{B}(T) \times_{\mathfrak{car}} \mathcal{G}$ to $\mathrm{B}(N_G(T)) \times_{\mathfrak{car}} \mathcal{G}$, it is enough to prove that Λ_{b} descends.

The intersection complex $\mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}})$ descends to $\mathrm{B}(N_G(T)) \times_{\mathfrak{car}} \mathcal{G}$.

We are thus reduced to prove that

$$\hat{p}_*(c_\delta) : \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}) \to X_*(T) \otimes \mathrm{IC}_{\hat{\mathcal{G}}}(\hat{\mathcal{L}})[2](1)$$

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Key propositions

Proposition

If we now denote by δ the T-torsor $\dot{\mathcal{B}} = [\mathfrak{b}/U] \to [\mathfrak{b}/B] = \mathcal{B}$, then its Chern class

$$c_\delta : X^*(T) \to H^2(\mathcal{B}, \mathbb{Q}_\ell) = H^2(\mathcal{G}, \mathrm{IC}_{\mathcal{G}}(\mathcal{L}))$$

is W-equivariant for the Springer action on the target.

Proposition

The functoriality morphism

$$p_* : H^2(\mathcal{B}, \mathbb{Q}_\ell) = \mathrm{Hom}_{\mathcal{B}}(\mathbb{Q}_{\ell,\mathcal{B}}, \mathbb{Q}_{\ell,\mathcal{B}}[2]) \to \mathrm{Hom}_{\mathcal{G}}(p_*\mathbb{Q}_{\ell,\mathcal{B}}, p_*\mathbb{Q}_{\ell,\mathcal{B}}[2])$$

is W-equivariant for the actions on the source and the target induced by the W-action on $p_*\mathbb{Q}_{\ell,\mathcal{B}} = \mathrm{IC}_{\mathcal{G}}(\mathcal{L})$.

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Proof of W-equivariance of c_δ

The T-torsor $\dot{\mathcal{B}} \to \mathcal{B}$ is the pull-back of the canonical T-torsor $\mathrm{Spec}(k) \to \mathrm{B}(T)$ by the morphism $\mathcal{B} = [\mathfrak{b}/B] \to \mathrm{B}(T)$ induced by $\mathfrak{b} \to \mathrm{Spec}(k)$ and $B \to T$.

Moreover the restriction map $H^2(\mathrm{B}(T), \mathbb{Q}_\ell) \to H^2(\mathcal{B}, \mathbb{Q}_\ell)$ is a W-equivariant isomorphism.

Our statement follows from the fact that the Chern class morphism

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Proof of W-equivariance of p_* (I)

We have seen that

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The morphism

$$p_* : H^2(\mathcal{B}, \mathbb{Q}_\ell) \to \mathrm{Hom}_{\mathcal{G}}(p_*\mathbb{Q}_{\ell,\mathcal{B}}, p_*\mathbb{Q}_{\ell,\mathcal{B}}[2])$$

is dual to the restriction to the diagonal $\mathcal{B} \subset \mathcal{B} \times_{\mathcal{G}} \mathcal{B} = \mathcal{Z}$

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For $w \in W$, we have the strata $\mathcal{Z}_w = [\mathfrak{b}_w / B_w]$ and a commutative square

$$\begin{tikzcd} \mathcal{Z}_w \arrow[r, hook] \arrow[d, "q_w"'] & \mathcal{Z} \arrow[d, "(q,q)"] \\ \mathfrak{t} \arrow[r, hook, "\Delta_w"'] & \mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t} \end{tikzcd}$$

where $\Delta_w(t) = (t, w(t))$.

It follows that the cohomology sheaves $(q,q)_! \mathbb{Q}_{\ell,\mathcal{Z}}$ is the abutment of a spectral sequence starting in E_2 with the cohomology sheaves of the

$$\Delta_{w,*} q_{w,!} \mathbb{Q}_{\ell,\mathcal{Z}_w} = \Delta_{w,*} \mathbb{Q}_{\ell,\mathfrak{t}} \otimes R\Gamma_c(\mathrm{B}(T), \mathbb{Q}_\ell).$$

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Proof of W-equivariance of p_* (III)

As $H_c^{\mathrm{odd}}(\mathrm{B}(T), \mathbb{Q}_\ell) = 0$, the spectral sequence degenerate in E_2.

It follows that $R^{\mathrm{odd}}(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}} = 0$ and that each $R^{2i}(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}}$ is successive extensions of the $\Delta_{w,*}\mathbb{Q}_{\ell,\mathfrak{t}} \otimes H_c^{2i}(\mathrm{B}(T), \mathbb{Q}_\ell)$.

Consequently, up to a shift, $R^{2i}(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}}$ is a perverse sheaf on $\mathfrak{t} \times_{\mathrm{car}} \mathfrak{t}$ which is the middle extension of its restriction to the open subset $\mathfrak{t}^{\mathrm{reg}} \times_{\mathrm{car}} \mathfrak{t}^{\mathrm{reg}}$.

We thus have a $(W \times W)$-equivariant morphism

$$R^{2i}(q,q)_!\mathbb{Q}_{\ell,\mathcal{Z}} = \Delta_*\mathbb{Q}_\ell \otimes H_c^{-2i}(\mathrm{B}(T), \mathbb{Q}_\ell)$$

where $\Delta = (\Delta_w)_{w \in W} : W \times \mathfrak{t} \to \mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}$ is the normalization morphism.

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Proof of W-equivariance of p_* (IV)

But

$$H^j_c(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_{\ell, W\times\mathfrak{t}}) = 0$$

for $j \neq 2n = 2\mathrm{dim}(\mathfrak{t})$.

As a consequence, we get a $(W \times W)$-equivariant isomorphism

$$\begin{aligned} H^{-2i}_c(\mathcal{Z}, \mathbb{Q}_\ell) &= H^{-2i}_c(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, (q,q)_! \mathbb{Q}_{\ell,\mathcal{Z}}) \\ &= H^{2n}_c(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_\ell) \otimes H^{-2n-2i}_c(\mathrm{B}(T), \mathbb{Q}_\ell). \end{aligned}$$

The restriction map $H^{-2i}_c(\mathcal{Z}, \mathbb{Q}_\ell) \to H^{-2i}_c(\mathcal{B}, \mathbb{Q}_\ell)$ is induced by the restriction map

$$\Delta^*_1 : H^{2n}_c(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_\ell) \to H^{2n}_c(\mathfrak{t}, \mathbb{Q}_\ell)$$

Therefore it is W-equivariant for the diagonal action.

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for $j \neq 2n = 2\mathrm{dim}(\mathfrak{t})$.

As a consequence, we get a $(W \times W)$-equivariant isomorphism

$$\begin{aligned} H_c^{-2i}(\mathcal{Z}, \mathbb{Q}_\ell) &= H_c^{-2i}(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, (q,q)_! \mathbb{Q}_{\ell, \mathcal{Z}}) \\ &= H_c^{2n}(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_\ell) \otimes H_c^{-2n-2i}(\mathrm{B}(T), \mathbb{Q}_\ell). \end{aligned}$$

The restriction map $H_c^{-2i}(\mathcal{Z}, \mathbb{Q}_\ell) \to H_c^{-2i}(\mathcal{B}, \mathbb{Q}_\ell)$ is induced by the restriction map

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Proof of W-equivariance of p_* (IV)

But

$$H_c^j(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_{\ell, W \times \mathfrak{t}}) = 0$$

for $j \neq 2n = 2\dim(\mathfrak{t})$.

As a consequence, we get a $(W \times W)$-equivariant isomorphism

$$\begin{aligned} H_c^{-2i}(\mathcal{Z}, \mathbb{Q}_\ell) &= H_c^{-2i}(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, (q,q)_! \mathbb{Q}_{\ell, \mathcal{Z}}) \\ &= H_c^{2n}(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_\ell) \otimes H_c^{-2n-2i}(\mathrm{B}(T), \mathbb{Q}_\ell). \end{aligned}$$

The restriction map $H_c^{-2i}(\mathcal{Z}, \mathbb{Q}_\ell) \to H_c^{-2i}(\mathcal{B}, \mathbb{Q}_\ell)$ is induced by the restriction map

$$\Delta_1^* : H_c^{2n}(\mathfrak{t} \times_{\mathfrak{car}} \mathfrak{t}, \Delta_* \mathbb{Q}_\ell) \to H_c^{2n}(\mathfrak{t}, \mathbb{Q}_\ell)$$

Therefore it is W-equivariant for the diagonal action.