Errata and Addenda to "Odds and ends on finite group actions and traces"

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Abstract

We wish to correct a few typographical errors in [3], complete an argument in [3, Proposition 2.7], and take the opportunity to point out a few additional consequences of [3] on fixed-point sets of finite group actions and complement some results of Esnault-Nicaise [2] in the case of affine spaces.

Corrections to [3]:

- Proposition 2.7: Some details are missing in the proof. A G-stable open cover exists only when the action of G is admissible. To reduce to this case, we proceed by induction on the dimension, which allows us to shrink U to any dense open subset. It then suffices to apply Chow's lemma.
- Setup 3.1, second paragraph: Read $D^b_c(\mathcal{X}, E)$ for $D^b_c(X, E)$.
- Proof of Theorem 3.2, line 4: Read H for G.
- Proposition 5.6: Read $\mathcal{F}$ for F.
- Proposition 7.1: The first sentence of the proof is superfluous.
- Above Corollary 7.11: A tilde is missing in the definition of "mod ℓ cohomology N-sphere". The condition should be $R\tilde{\Gamma}(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-N]$.

In the context of mod ℓ étale cohomology we proved in [3, Section 7] results of Smith theory type for fixed points of finite ℓ-group actions. In this note we examine variants for actions of other types of finite groups and complement some results of Esnault-Nicaise [2] in the case of standard affine spaces.

Proposition 1. *Let k be an algebraically closed field of characteristic p, X be an algebraic space separated and of finite type over k, equipped with an action of a finite group G whose order is not divisible by p. Assume that X is mod ℓ acyclic and G is an extension by an ℓ-group H of an extension by a cyclic group C of an ℓ'-group, for some primes $\ell \neq p$ and ℓ'. Then $\chi(X^G) \equiv 1 \pmod{\ell'}$. In particular, X^G is nonempty.*

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Proof. By [3, Theorem 7.3], X^H is mod ℓ acyclic. In particular, $H^0(X^H, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^i(X^H, \mathbb{Q}_\ell) = 0$ for $i > 0$. Thus, by [3, Corollary 5.11], $\chi((X^H)^C) = 1$. Therefore, by [3, Proposition 7.1], $\chi(X^G) \equiv 1 \pmod{\ell'}$. □

Lemma 2. *Let F be a perfect field, C be a connected smooth curve over F such that $\chi(C_{\bar{F}}) = 1$, where $\bar{F}$ is an algebraic closure of F. Then C is an affine line over F.*

Proof. Let X be a smooth compactification of $C_{\bar{F}}$. Then

$$1 = \chi(C_{\bar{F}}) = 2 - 2g - n,$$

where g is the genus of X and $n = \#(X - C_{\bar{F}})$. Thus $g = 0$ and $n = 1$. It follows that C is an affine line over F. □

Lemma 3. *Let F be a field of characteristic p, X be a smooth algebraic space over F endowed with an action of a finite group G of order not divisible by p. Then X^G is smooth.*

Proof. The case where X is a scheme is well-known (see [1, 3.4] for a generalization). We may assume F is separably closed. As in the case of schemes, the lemma follows from the linearizability of the action of G on $\hat{\mathcal{O}}_{X,x}$ for every closed point x of X^G [1, 3.3]. □

Proposition 4. *Let F be a perfect field of characteristic p, X be a smooth algebraic space separated and of finite type over F of dimension ≤ 2 [1], equipped with an action of a finite solvable group G of order n not divisible by p. Assume that $X_{\bar{F}}$ is mod ℓ acyclic for every prime number ℓ dividing n, where $\bar{F}$ is an algebraic closure of F. Then $X^G = X$ or $X^G \simeq \mathbb{A}^1_F$ or $X^G \simeq \mathbb{A}^0_F$.*

Proof. By induction on n, we may assume that $G \neq \{1\}$ is an ℓ-group. In this case, by [3, Theorem 7.3], $X^G_{\bar{F}}$ is mod ℓ acyclic. Moreover, by Lemma 3, X^G is smooth. There are three cases.

(a) $\dim X^G = 2$. This implies $X^G = X$ because X is connected by the mod ℓ acyclicity assumption, hence integral.

(b) $\dim X^G = 1$. This implies $X^G \simeq \mathbb{A}^1_F$ by Lemma 2.

(c) $\dim X^G = 0$. This implies $X^G \simeq \mathbb{A}^0_F$. □

For $X = \mathbb{A}^2_F$, the above result can be strengthened as follows.

Proposition 5. *Let F be a field of characteristic p, G be a finite group of order not divisible by p. For every action of G on $X = \mathbb{A}^2_F$, $X^G \simeq \mathbb{A}^m_F$ for some $0 \leq m \leq 2$.*

For solvable G, this is [2, Corollary 5.14]. Similarly to [2, Question 7.1], we consider the subgroup $\mathrm{Aff}_n(F) < \mathrm{Aut}(\mathbb{A}^n_F)$ of affine automorphisms and the subgroup

$$J_n(F) = \{(P_1, \dots, P_n) \in \mathrm{Aut}(\mathbb{A}^n_F) \mid P_i \in F[X_1, \dots, X_i], 1 \leq i \leq n\}$$

of triangular automorphisms. Note that $P_i = a_i X_i + Q_i$, where $a_i \in F^\times$ and $Q_i \in F[X_1, \dots, X_{i-1}]$. For $n \geq 1$, we have a split short exact sequence of groups

$$0 \to F[X_1, \dots, X_{n-1}] \xrightarrow{\alpha} J_n(F) \xrightarrow{\beta} F^\times \times J_{n-1}(F) \to 1,$$

[1] It is well-known that such an algebraic space is a quasi-projective scheme over F.

where $\alpha(P) = (X_1, \dots, X_{n-1}, X_n + P)$ and $\beta(P_1, \dots, P_n) = (a_n, (P_1, \dots, P_{n-1}))$, the homomorphism $F^\times \times J_{n-1}(F) \to J_n(F)$ sending $(a, (P_1, \dots, P_{n-1}))$ to $(P_1, \dots, P_{n-1}, aX_n)$ giving a splitting. In particular, $J_n(F)$ is solvable.

Lemma 6. *Let F be a field of characteristic p, G be a finite subgroup of $\mathrm{Aff}_n(F)$ or $J_n(F)$, of order not divisible by p. Then $(\mathbb{A}^n_F)^G \simeq \mathbb{A}^m_F$ for some $0 \le m \le n$.*

Proof. Since $J_n(F)$ is solvable, if G factors through $J_n(F)$, we proceed by induction to reduce to the case where G is cyclic. By [4, Theorem 2.2], G is then conjugate in $J_n(F)$ to a subgroup of $\mathrm{Aff}_n(F) \cap J_n(F)$. Therefore, in all cases, we may assume that G is a subgroup of $\mathrm{Aff}_n(F)$. In this case, for any point x of $\mathbb{A}^n(F)$,

$$\frac{1}{\#G} \sum_{g \in G} xg$$

belongs to $(\mathbb{A}^n)^G(F)$, so that G is conjugate to a subgroup of $\mathrm{GL}_n(F)$. The assertion is then obvious. □

Proof of Proposition 5. By the theorem of Jung and van der Kulk [7], $\mathrm{Aut}(\mathbb{A}^2_F)$ is the amalgamated product of $\mathrm{Aff}_2(F)$ and $J_2(F)$ over $\mathrm{Aff}_2(F) \cap J_2(F)$. Thus, by [6, cor. du théorème 8, p. 54], the image of G is conjugate to a subgroup of $\mathrm{Aff}_2(F)$ or $J_2(F)$. Therefore, the assertion follows from Lemma 6. □

Remark 7. It follows from the above proof and the classification of finite subgroups of $\mathrm{PGL}_2(F)$ of order not divisible by p [5, Proposition 16, page 281] that, in the situation of Proposition 5, the possible composition factors of the image of G in $\mathrm{Aut}(X)$ are cyclic groups of prime order and the alternating group A_5.

Proposition 8. *Let k be an algebraically closed field of characteristic p, X be a smooth algebraic space separated and of finite type over k of dimension 3, equipped with an action of a finite nilpotent group G of order $n > 1$ not divisible by p. Assume that X is mod ℓ acyclic for every prime number ℓ dividing n. Then $\chi(X^G) = 1$. In particular, X^G is nonempty.*

Proof. We decompose G as $\prod_\ell G_\ell$, where G_ℓ is a ℓ-group. Let $d_\ell = \dim X^{G_\ell}$. There are two cases.

(a) $d_\ell \le 1$ for some ℓ. By [3, Theorem 7.3], X^{G_ℓ} is mod ℓ acyclic. As in the proof of Proposition 4, by Lemmas 2 and 3, X^{G_ℓ} is either an affine line or a point. Applying Proposition 4 to X^{G_ℓ}, we see that X^G is either an affine line or a point.

(b) $d_\ell \ge 2$ for all ℓ. For those ℓ such that $d_\ell = 2$, choose a cyclic subgroup H_ℓ of G_ℓ acting nontrivially on X. By [3, Theorem 7.3] and Lemma 3, X^{H_ℓ} is connected and smooth, hence integral. Moreover, $\dim X^{H_\ell} \le 2$, as $\dim X = 3$, and X is connected by the mod ℓ acyclicity assumption, hence integral. Thus $X^{G_\ell} = X^{H_\ell}$. It follows that $X^G = X^H$, where $H = \prod_{d_\ell = 2} H_\ell$ is a cyclic group. Therefore, by [3, Corollary 5.11], $\chi(X^G) = \chi(X^H) = 1$. □

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