Tsinghua 2010 **Abelian Varieties** Problem Set 5

In this note, k denotes always a field. If X is an abelian variety over k, then $\hat{X}$ denotes the dual abelian variety of X.

1. Let $f, g : X \to Y$ be homomorphisms of abelian varieties, and $\hat{f}, \hat{g} : \hat{Y} \to \hat{X}$ be the induced morphisms between their dual abelian varieties. Show that $(f+g)^\wedge = \hat{f} + \hat{g}$, i.e. passing to the dual abelian varieties defines a homomorphism of groups $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\hat{Y}, \hat{X})$, where Hom means the homomorphisms in the category of commutative k-group schemes.

2. Let X be an abelian variety over k, and L be a line bundle over X. We denote by $\phi_L : X \to \hat{X}$ the homomorphism $x \mapsto T_x^*(L) \otimes L^{-1}$. Consider the homomorphism $(\phi_L, 1_X) : X \to \hat{X} \times X$. Prove that $(\phi_X, 1_X)^*(\mathcal{P}) \simeq L \otimes (-1_X)^*(L)$, where $\mathcal{P}$ is the Poincaré sheaf on $\hat{X} \times X$.

3. Let X be an abelian variety, Z be a closed subscheme of $\hat{X}$ with $Z \neq \hat{X}$. We denote by $\mathcal{P}_Z$ the restriction of the Poincaré sheaf to $Z \times X$. If $0 \in Z$, then we assume that Z is smooth around 0 (actually, we just need that $Z \subset \hat{X}$ is a regular imbedding around 0). Prove that $\chi(\mathcal{P}_Z) = 0$. (Use base change theorem and the explicit description of the complex which computes the direct image $Rp_{1*}(\mathcal{P})$, where $p_1 : \hat{X} \times X \to \hat{X}$ is the natural projection.)

4. Let X, Y be abelian varieties of different dimensions, and L be a line bundle on $X \times Y$ such that $L|_{\{0\} \times Y} \simeq \mathcal{O}_Y$ and $L|_{X \times \{0\}} \simeq \mathcal{O}_X$. Prove that $\chi(L) = 0$, where $\chi(L)$ is the Euler-Poincaré characteristic of L. (Use dual abelian varieties to interpret L.)

5. Let $f : X \to Y$ be an isogeny of abelian varieties, L be an ample line bundle on Y, and $\phi_L : X \to \hat{X}$ be the polarization associated with L, i.e. for $x \in X$, $\phi_L(x)$ is class of $T_x^*(L) \otimes L^{-1}$ in $\hat{X} \simeq \underline{\mathrm{Pic}}^0_{X/k}$. Show that $\phi_{f^*(L)} = \hat{f} \circ \phi_L \circ f$; in particular, $\deg(\phi_{f^*(L)}) = (\deg f)^2 \deg(\phi_L)$.

6. (Mumford §16) Let X be an abelian variety of dimension $g \geq 1$, L be a line bundle on X, and M be the line bundle $m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1})$ on $X \times X$. We denote by $K(L)$ the maximal closed subscheme of X such that $M|_{K(L) \times X}$ is trivial.

(i) Show that $R^q p_{1*}(M)$ is supported on $K(L)$.

(ii) Assume $K(L)$ is finite. Prove that $R^q p_{1*}(M) = 0$ for $q \neq g$, and $R^g p_{1*}(M) \simeq i_* \mathcal{O}_{K(L)}$, where $i : K(L) \to X$ is the natural closed immersion. (One can use the higher direct image $R^q p_{1*}(\mathcal{P})$ of the Poincaré sheaf $\mathcal{P}$.)

7. Let X, Y be abelian varieties over k, $\mathrm{Hom}(X,Y)$ be the group of homomorphisms between X and Y. Show that $\mathrm{Hom}(X,Y)$ is a torsion free $\mathbf{Z}$-module, and an element $f \in \mathrm{Hom}(X,Y)$ can be written as $f = n \cdot g$ where $n \in \mathbf{Z}_{>0}$ if and only if $X[n] \in \mathrm{Ker}(f)$.

8. Assume k algebraically closed of characteristic $p > 0$. Let X be an abelian variety over k of dimension g. Assume that X is principally polarized, i.e. there is an ample line bundle L on X such that the associated polarization $\phi_L : X \to \hat{X}$ is an isomorphism. Show that for any positive integer n, we have an isomorphism of finite group schemes $A[p^n] \simeq \mu_{p^n}^r \times G_n \times (\mathbf{Z}/p^n)^r$ where $r \leq g$ is a positive integer, and G_n is a connected finite group scheme of order $p^{2n(g-r)}$ with $G_n^D = G_n$. (Hint: Prove first that the Cartier dual of $A[p^n]$ is itself, and use the fact that any commutative finite group scheme G over an algebraically closed field can be decomposed as $G = G^0 \times G^{\mathrm{et}}$ with G^0 connected and G^{et}.)