Tsinghua 2010 **Abelian Varieties** Problem Set 6

1. Let $f\colon X \to Y$ be a morphism of $\mathbb{C}$-schemes. Assume that f is surjective and locally of finite type. Show that $f(\mathbb{C})\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is surjective.

2. Let X be a smooth $\mathbb{C}$-scheme. Prove the following statements.

(a) The map $\pi_0(X^{\mathrm{an}}) \to \pi_0(X)$ induced by the canonical map $X^{\mathrm{an}} \to X$ is a bijection. In particular, X is connected if and only if X^{an} is connected. (*Hint* for the "only if" part: reduce first to the affine case, then to the proper case, and then look at global sections.)

(b) If X is separated, then X^{an} is Hausdorff.

(c) If X is proper, then X^{an} is compact. (*Hint*: use Chow's lemma and Problem 1).

3. Show that any complex Lie group is Hausdorff.

4. (Rigidity) (a) Let X, Y, Z be complex manifolds, X connected and compact, Y connected, Z Hausdorff. Let $f\colon X \times Y \to Z$ be a holomorphic map, $y_0 \in Y$ and $z_0 \in Z$ be such that $f(X \times \{y_0\}) = \{z_0\}$. Show that there exists a holomorphic map $g\colon Y \to Z$ such that $f = g \circ p_2$, where $p_2\colon X \times Y \to Y$ is the projection.

(b) Let X be a complex torus, G be a complex Lie group, $f\colon X \to G$ be a holomorphic map. Show that the map $h\colon X \to G$ given by $h(x) = f(x)(f(0))^{-1}$ is a holomorphic homomorphism.

5. Let $\mathcal{C}$ be a category that admits finite products, $\mathcal{A}$ be an additive category, $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be a functor, $n \geq 0$ be an integer. For objects $X_0, \dots, X_n$ of $\mathcal{C}$, let

$$\alpha_{F,X_0,\dots,X_n}\colon \bigoplus_{i=0}^{n} F(X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n) \to F(X_0 \times \cdots \times X_n)$$

be the morphism induced by the projections

$$p_i\colon X_0 \times \cdots \times X_n \to X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n,$$

where $\widehat{X_i}$ means the omission of X_i. We say that F is *of order* n if $\alpha_{F,X_0,\dots,X_n}$ is an epimorphism for all $X_0, \dots, X_n$.

(a) Show that if F is of order n, then it is of order m for all $m \geq n$.

A *rational point* of an object X of $\mathcal{C}$ is defined to be a section of the projection $X \to e$, where e is the final object of $\mathcal{C}$. Assume in (b) and (c) below that $\mathcal{A}$ is an abelian category.

(b) Let $s_0, \dots, s_n$ be rational points of $X_0, \dots, X_n$, respectively, and let

$$\beta_{F,X_0,\dots,X_n,s_0,\dots,s_n}\colon F(X_0 \times \cdots \times X_n) \to \bigoplus_{i=0}^{n} F(X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n)$$

be the morphism induced by $\sigma_i\colon X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n \to X_0 \times \cdots \times X_n$, which is the base change of s_i. Show that $F(X_0 \times \cdots \times X_n) = \mathrm{Im}(\alpha) \oplus \mathrm{Ker}(\beta)$.

(c) Assume that every object of $\mathcal{C}$ has a rational point. Let $F_1 \to F_2 \to F_3$ be an exact sequence of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Show that, if F_1 and F_3 are both of order n, then so is F_2.

(d) (Theorem of the cube) Let $\mathcal{C}$ be the category of connected compact complex manifolds, $\mathcal{A}$ be the category of abelian groups. Show that the functors $X \mapsto H^n(X, \mathbb{Z})$ and $X \mapsto H^n(X, \mathcal{O}_X)$ are of order n. Use the exponential sequence to deduce that $X \mapsto \mathrm{Pic}(X)$ is of order 2.

6. Let V be a finite-dimensional complex vector space, Ω (resp. $\bar{\Omega}$) be the space of $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) forms on V, A be the space of alternating $\mathbb{R}$-bilinear maps $V \times V \to \mathbb{C}$.

(a) Show that the image of the $\mathbb{C}$-linear map $\Omega \otimes_{\mathbb{C}} \bar{\Omega} \to A$ given by $(f, g) \mapsto E_{f,g}$, where $E_{f,g}(v, w) = f(v)g(w) - f(w)g(v)$ for $v, w \in V$, is the subspace of A consisting of maps E satisfying

$$E(v, w) = E(iv, iw), \quad \text{for all } v, w \in V. \tag{1}$$

(b) Show that $H \mapsto \Im H$ (imaginary part of H) gives an $\mathbb{R}$-linear bijection from the space of Hermitian forms on V to the space of alternating $\mathbb{R}$-bilinear forms $E \colon V \times V \to \mathbb{R}$ satisfying (1). The inverse is given by $E \mapsto H_E$, where $H_E(v, w) = E(iv, w) + iE(v, w)$ for $v, w \in V$.