Lectures on Commutative Algebra

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## Contents

**Convention** v

**7 UFDs** 1

**8 Primary decomposition** 3

**9 DVRs and Dedekind domains** 5

**10 Completions** 11

**11 Dimension theory** 17

**Summary of properties of rings** 23
Convention

Rings are assumed to be commutative.
Chapter 7

UFDs

**Theorem 7.1** (Fundamental theorem of arithmetic, cf. Euclid’s Elements 9.14). Every nonzero integer \( n \in \mathbb{Z} \) can be factorized uniquely (up to permutation of factors) as a product of primes \( n = \pm p_1 \cdots p_m \).

**Definition 7.2.** Let \( R \) be a domain.

1. \( x \in R \) is irreducible if \( x \neq 0, x \notin R^\times \), and \( x = yz \) implies \( y \in R^\times \) or \( z^\times \).
2. \( x, y \in R \) are associates if there exists \( u \in R^\times \) such that \( x = uy \).
3. \( R \) is a unique factorization domain (UFD, or factorial ring) if every \( x \in R \) satisfying \( x \neq 0 \) and \( x \notin R \) admits a factorization \( x = a_1 \cdots a_m \) with \( a_i \) irreducible, and if \( x = b_1 \cdots b_n \) with \( b_j \) irreducible, then \( m = n \) and there exists a bijection \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that \( a_i \) and \( b_{\sigma(i)} \) are associates for every \( i \). We say that \( x \in R \) is prime if \( xR \) is a prime ideal. Prime elements are irreducible. The converse holds in a UFD.

**Theorem 7.3.** Let \( R \) be a domain. Then \( R \) is a UFD if and only if the following conditions are satisfied:

1. The ascending chain condition for principle ideals of \( R \).
2. Irreducible elements in \( R \) are prime.

**Lemma 7.4.** Assume that the ascending chain condition holds for principle ideals of a ring \( R \). Then every \( x \in R \) satisfying \( x \neq 0 \) and \( x \notin R \) admits a factorization \( x = a_1 \cdots a_m \) with \( a_i \) irreducible.

**Corollary 7.5.** A PID is a UFD.

**Example 7.6.** (1) \( \mathbb{Z}, \mathbb{Z}[\sqrt{-1}] \) are PIDs, hence UFDs.

2. \( \mathbb{Z}[\sqrt{-5}] \) is not a UFD. Indeed, \( 2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \).

3. If \( R \) is a UFD, \( R[X] \) is a UFD. Note however that neither \( \mathbb{Z}[X] \) nor \( k[X, Y] \) (where \( k \) is a field) is a PID.

4. If \( R \) is a nonzero UFD, \( R[X_1, X_2, \ldots] \) is a UFD, but not a Noetherian ring.

5. In \( R = \bigcup_{n=1}^{\infty} k[X^{1/n}] \), factorization does not exist in general. In particular, \( R \) does not satisfy the ascending chain condition for principal ideals.
Chapter 8

Primary decomposition

We largely followed [AM, Chapters 4 and 7]. Most other sources define primes associated to an ideal \( I \) of a ring \( R \) differently, as prime ideals of the form \((I : x)\). The two definitions coincide when \( R \) is Noetherian.
Chapter 9
DVRs and Dedekind domains

We studied Artinian rings, which are Noetherian rings of dimension 0. In this chapter, we study the next simplest case, Noetherian domains of dimension 1. We start by applying primary decomposition to such domains.

**Proposition 9.1.** Let $R$ be a Noetherian domain of dimension 1. Every nonzero ideal $I \subseteq R$ can be uniquely written as a product of primary ideals whose radicals are distinct.

We would like to further decompose primary ideals into prime powers. We first look at the local case.

**Discrete valuation rings**

Recall that the value group of a valuation ring $R$ is $K^\times / R^\times$, where $K = \text{Frac}(R)$.

**Definition 9.2.** A discrete valuation ring (DVR) is a valuation ring whose value group is isomorphic to $\mathbb{Z}$.

**Proposition 9.3.** Let $R$ be a valuation ring. The following are equivalent:

1. $R$ is a DVR.
2. $R$ is a PID.
3. $R$ is Noetherian.

For (2) $\Rightarrow$ (1), we use the following.

**Lemma 9.4.** Let $R$ be a local ring of dimension $> 0$. Assume that the maximal ideal $m$ of $R$ is principal and $\bigcap_{n=1}^\infty m^n = 0$. Then $R$ is a DVR. The assumption $\bigcap_{n=1}^\infty m^n = 0$ holds if $R$ is a Noetherian domain.

We will see later that the assumption $\bigcap_{n=1}^\infty m^n = 0$ holds for $R$ Noetherian.

A generator of the maximal ideal of a DVR is called a **uniformizer**.

**Example 9.5.** (1) Let $K = \mathbb{Q}$ and let $p$ be a rational prime. The $p$-adic valuation $v_p: K^\times \to \mathbb{Z}$ is defined by $v_p(p^a u^v) = a$, where $a, u, v \in \mathbb{Z}$, $(u, p) = (v, p) = 1$.

Every valuation of $K^\times$ is equivalent to $v_p$ (Ostrowski's theorem).
(2) Let \( k \) be a field and let \( K = k(X) \). Similarly to (1), for every irreducible polynomial \( f \in k[X] \), the \( f \)-adic valuation \( v_f: K^\times \to \mathbb{Z} \) is defined by
\[
v_f(f^n u) = 1,
\]
where \( a \in \mathbb{Z}, u, v \in k[X], (u, f) = (v, f) = 1 \). Every valuation of \( K^\times \) is equivalent to \( v_f \) or \( v_{1/X} \).

(3) Let \( k \) be a field and let \( K = \bigcup_n k(X^{1/n}) \). We have a non-discrete rank 1 valuation \( v: K^\times \to \mathbb{Q} \) whose restriction to \( k(X^{1/n}) \) is \( \frac{1}{n} v_{X^{1/n}} \) with \( v_{X^{1/n}} \) defined in (2). The valuation ideal \( m = (X^{1/n})_{n \geq 1} \) is not finitely generated.

(4) Let \( F \) be a field and let \( v_F: F^\times \to \Gamma \). Let \( K = F(X) \) (or \( F((X)) \)). Then \( K^\times \to \mathbb{Z} \times \Gamma \) carrying \( f = \sum_{n \geq N} a_n X^n \) with \( a_N \neq 0 \) to \( (N, v(a_N)) \) is a valuation, of rank > 1 for \( v_F \) nontrivial. Here \( \mathbb{Z} \times \Gamma \) is equipped with the lexicographical order.

(5) Consider the particular case of (4) where \( F = k(Y) \) and \( v_F = v_Y \). Let \( m \) be the valuation ideal. Then \( m = YR \) is principal, but \( \bigcap_n m^n = (X/Y^n)_{n \geq 1} \) is not finitely generated.

**Definition 9.6.** We say that a ring \( R \) is **normal** if for every prime \( p, \) \( R_p \) is an integrally closed domain.

A normal domain is synonymous to an integrally closed domain.

**Proposition 9.7.** Let \( R \) be a Noetherian local domain that is not a field. Let \( m \) be the maximal ideal and let \( k = m/m^2 \). The following conditions are equivalent:

1. \( R \) is a DVR.
2. \( R \) is normal of dimension one.
3. \( m \) is principal.
4. \( \dim_k(m/m^2) = 1 \).
5. Every nonzero ideal of \( R \) is a power of \( m \).

For (2)⇒(3), we use the following.

**Lemma 9.8.** Let \( R \) be a normal local domain and assume that the maximal ideal \( m \) is finitely generated and there exists \( a, b \in R \) such that \( m = (aR : b) \). Then \( m \) is principal.

The proposition holds in fact for \( R \) a Noetherian local ring of dimension > 0. The proof uses the Krull intersection theorem (Corollary 10.28).

**Dedekind domains**

**Theorem 9.9.** Let \( R \) be a Noetherian domain of dimension one. The following are equivalent:

1. \( R \) is normal.
2. Every primary ideal of \( R \) is a power of a prime ideals.
3. Every local ring \( R_p \) \( (p \neq 0) \) is a DVR.

**Definition 9.10.** A **Dedekind domain** is a Noetherian domain of dimension one satisfying the above equivalent conditions.

**Corollary 9.11.** Every nonzero ideal of a Dedekind domain has a unique factorization as a product of prime ideals.
Remark 9.12. The following converse of Corollary 9.11 holds: An integral domain of which every nonzero ideal is a product of prime ideals is a Dedekind domain [M2, Theorem 11.6].

Example 9.13. A PID is a Dedekind domain.

More examples are given by taking integral closure.

Proposition 9.14. Let $A$ be a normal domain. Let $L$ be a finite separable extension of $K = \text{Frac}(A)$ and let $B$ be the integral closure of $A$ in $L$. Then $B$ is contained in a finitely generated $A$-submodule of $L$.

Corollary 9.15. Let $A$ be a Dedekind domain. Let $L$ be a finite separable extension of $K = \text{Frac}(A)$. Then the integral closure $B$ of $A$ in $L$ is a Dedekind domain.

Remark 9.16. More generally, if $A$ is a Noetherian domain of dimension 1 (not necessarily normal) and if $L$ is a finite (not necessarily separable) extension of $K = \text{Frac}(A)$, then the normalization $B$ of $A$ in $L$ is a Dedekind domain (even when $B$ is not finite over $A$). This follows from the Krull-Akizuki Theorem [M2, Theorem 11.7].

Example 9.17. Let $K$ be a number field (namely, a finite extension of $\mathbb{Q}$). Then the ring of integers $\mathcal{O}_K$ (namely, the integral closure of $\mathbb{Z}$ in $K$) is a Dedekind domain by Corollary 9.15.

Example 9.18. In particular, for $K = \mathbb{Q}(\sqrt{-5})$, $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain. The prime ideal $p = (2, 1 + \sqrt{-5})$ is not principal. Indeed, since $p^2 = 2R$, if $p = \alpha R$ for $\alpha = a + b\sqrt{-5}$, $a,b \in \mathbb{Z}$, then $(N_{K/\mathbb{Q}} \alpha)^2 = N_{K/\mathbb{Q}} 2 = 4$, so that $a^2 + 5b^2 = N_{K/\mathbb{Q}} \alpha = 2$, which is impossible.

We have $3R = qq'$, $(1 + \sqrt{-5})R = pq$, $(1 - \sqrt{-5})R = pq'$, where $q = (3, 1 + \sqrt{-5})$, $q' = (3, 1 - \sqrt{-5})$.

The maximal ideal of the DVR $R_p$ is $(1 + \sqrt{-5})R_p$.

Example 9.19. Let $R = \mathbb{Z}[\sqrt{5}]$. This is a Noetherian domain of dimension one, but not normal. The integral closure of $R$ in $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[\frac{1 + \sqrt{5}}{2}]$. Consider the prime ideal $p = (2, 1 + \sqrt{5})$ of $R$. We have $p^2 = 2p \subseteq 2R \subseteq p$. The ideal $2R$ is $p$-primary, but not a power of $p$.

Definition 9.20. We say that a domain $A$ is Japanese (or N-2) if for every finite extension $L$ of $K = \text{Frac}(A)$, the integral closure of $A$ in $L$ if finite over $A$. We say that a Noetherian ring $R$ is Nagata (or universally Japanese) if every finitely generated integral $R$-algebra is Japanese.

Grothendieck uses “Japanese” and “universally Japanese” [G, Sections 0.23, IV.7.6, IV.7.7], while Matsumura uses “N-2” and “Nagata” [M1, Chapter 12].


(2) Every field is Nagata. Every Dedekind domain of fraction field of characteristic 0 is Nagata.

(3) Nagata constructed a DVR that is not Nagata (not Japanese?).
Invertible modules

The nonzero ideals of Dedekind domain is a free commutative monoid with maximal ideals forming a basis. We now consider the associated free abelian group. There are a number of generalizations to commutative rings.

**Proposition 9.22.** Let $R$ be a ring $M$ be a finitely presented module. The following are equivalent:

1. For every prime ideal $p$ of $R$, the $R_p$-module $M_p$ is isomorphic to $R_p$.
2. For every maximal ideal $m$ of $R$, the $R_m$-module $M_m$ is isomorphic to $R_m$.
3. The evaluation map $u: M^* \otimes_R M \to R$ is an isomorphism, where $M^* = \text{Hom}_R(M, R)$.
4. There exists an $R$-module $N$ such that $M \otimes_R N$ is isomorphic to $R$.

**Definition 9.23.** A finitely generated $R$-module $M$ is said to be invertible if it satisfies the above conditions. The Picard group $\text{Pic}(R)$ of a ring $R$ is the abelian group of isomorphism classes of invertible $R$-modules $M$ with group law given by tensor product. The class of $M$ is denoted $\text{cl}(M)$.

The identity element of $\text{Pic}(R)$ is $\text{cl}(R)$ and $\text{cl}(M)^{-1} = \text{cl}(M^*)$.

**Remark 9.24.** The proposition holds more generally for $M$ finitely generated. The equivalent conditions then imply that $M$ is projective and finitely presented. Invertible $R$-modules are also called projective $R$-modules of rank 1. See [B2, II.5].

**Remark 9.25.** For a local ring $R$, $\text{Pic}(R) = 0$.

Let $R$ be a domain and let $K = \text{Frac}(R)$.

**Lemma 9.26.** Every invertible $R$-module $M$ is an $R$-submodule of $K$.

**Definition 9.27.** An $R$-submodule of $K$ is called a fractional ideal of $R$ if there exists $x \in R$, $x \neq 0$ such that $xI \subseteq R$.

**Example 9.28.** (1) Every ideal of $R$ is a fractional ideal of $R$.
(2) For every $x \in K$, $xR$ is a fractional ideal of $R$. Such fractional ideals are said to be principal. A principal fractional ideal is free of rank $\leq 1$. Conversely, any free fractional ideal is principal.

**Remark 9.29.** Every finitely generated $R$-submodule of $K$ is a fractional ideal. Conversely, if $R$ is Noetherian, then every fractional ideal if finitely generated.

$R$-submodules of $K$ form a commutative monoid, with identity element $R$ and $IJ = \{\sum_{i=1}^n a_i b_i \mid a_i \in I, \; b_i \in J\}$. We write $I^{-1} = \{x \in K \mid xI \subseteq R\}$.

**Proposition 9.30.** Let $I$ be an $R$-submodule of $K$. The following are equivalent:

1. $I$ is finitely generated and for every prime ideal $p$, $I_p \simeq R_p$.
2. $I$ is finitely generated and for every maximal ideal $m$, $I_m \simeq R_m$.
3. $I$ is finitely generated and for every prime ideal $p$, $I_p(I_p)^{-1} \simeq R_p$.
4. $I$ is finitely generated and for every maximal ideal $m$, $I_m(I_m)^{-1} \simeq R_m$.
5. $II^{-1} = R$.
6. There exists an $R$-submodule $J$ of $K$ such that $IJ = R$. 

Definition 9.31. An *invertible* (fractional) ideal is a fractional ideal satisfying the above conditions. We let $\text{CaDiv}(R)$ (for Cartier divisors) denote the abelian group of invertible ideals.

**Proposition 9.32.** We have an exact sequence $1 \to R^\times \to K^\times \xrightarrow{\cdot R} \text{CaDiv}(R) \xrightarrow{\text{cl}} \text{Pic}(R) \to 1$.

**Remark 9.33.** For $K$ a number field, $\mathcal{O}_K^\times$ is a finitely generated abelian group and $\text{Pic}(\mathcal{O}_K^\times)$ is a finite abelian group, called the *class group* of $K$.

**Proposition 9.34.** Let $R$ be a UFD. Then $\text{Pic}(R) = 1$.

**Lemma 9.35.** Let $R$ be a Noetherian domain and let $p$ be an invertible prime ideal. Then $R_p$ is a DVR.

**Theorem 9.36.** Let $R$ be a domain that is not a field. Then the following are equivalent:

1. $R$ is a Dedekind domain.
2. Every nonzero ideal of $R$ is invertible.
3. Every nonzero fractional ideal of $R$ is invertible.

**Corollary 9.37.** A Dedekind UFD is a PID.

**Proposition 9.38.** Let $R$ be a Dedekind domain. Then $\text{CaDiv}(R)$ is a free abelian group with maximal ideals forming a basis.

**Definition 9.39.** The *height* of a prime ideal $p$ of a ring $R$ is the supremum of the length $n$ of chains $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = p$ of prime ideals.

**Remark 9.40.** We have $\text{ht}(p) = \dim(R_p)$, $\dim(R) \leq \text{ht}(p) + \dim(R/p)$, and $\dim(R) = \sup_p \text{ht}(p) = \sup_p \dim(R/p)$.

**Remark 9.41.** (1) For any ring $R$, one can define an abelian group $\text{CaDiv}(R)$ and there is an exact sequence $1 \to R^\times \to K^\times \to \text{CaDiv}(R) \to \text{Pic}(R)$.

(2) For any ring $R$, the group $\text{Div}(R)$ of *Weil divisors* is defined to be the free abelian group generated by the primes ideals of $p$ of height 1. For $R$ a Noetherian domain, there is a homomorphism $\text{CaDiv}(R) \to \text{Div}(R)$, which is injective for $R$ normal and an isomorphism for $R$ locally factorial (namely, $R_p$ is a UFD for all $p$).

We end this chapter with a criterion of normality.

**Proposition 9.42.** Let $R$ be a Noetherian domain. The following conditions are equivalent:

1. $R$ is normal.
2. For every prime ideal $p$ associated to a nonzero principal ideal of $R$, $R_p$ is a DVR.
3. $R = \bigcap_{\text{ht}(p)=1} R_p$ and for each $p$ of height 1, $R_p$ is a DVR.

For (2)⇒(3), we use the following.

**Lemma 9.43.** Let $R$ be a Noetherian domain. Then $R = \bigcap_p R_p$, where $p$ runs through prime ideals associated to nonzero principal ideals of $R$. 
Chapter 10

Completions

Topologies and completions

Definition 10.1. A topological group is a group $G$ equipped with a topology such that the maps

$$G \times G \to G \quad (x, y) \mapsto xy \text{ (multiplication)}$$
$$G \to G \quad x \mapsto x^{-1} \text{ (inversion)}$$

are continuous. A topological ring is a ring $R$ equipped with a topology such that the maps

$$R \times R \to R \quad (x, y) \mapsto x + y$$
$$R \times R \to R \quad (x, y) \mapsto xy$$

are continuous. A topological $R$-module is an $R$-module $M$ equipped with a topology such that the maps

$$M \times M \to M \quad (x, y) \mapsto x + y$$
$$R \times M \to M \quad (r, x) \mapsto rx$$

are continuous.

Let $G$ be a topological abelian group. In the sequel we write the group law additively. For any $a \in G$, the translation map $T_a : G \to G, g \mapsto g + a$ is a homeomorphism.

Lemma 10.2. Let $H$ be the intersection of neighborhoods of 0 in $G$. Then $H$ is a subgroup of $G$, closure of $\{0\}$. Moreover, the following conditions are equivalent:

1. $G$ is Hausdorff.
2. Every point of $G$ is closed.
3. $H = 0$.

Proof. That $H$ is a subgroup of $G$ follows from the continuity of group operations. For $x \in G$, $x \in H$ if and only if $0 \in x - U$ for all neighborhoods $U$ of 0, which is equivalent to $x \in \{0\}$. Then (3) is equivalent to 0 being a closed point of $G$. Thus $(1) \Rightarrow (2) \Rightarrow (3)$. Conversely, if 0 is a closed point of $G$, then the diagonal $\Delta \subseteq G \times G$ is a closed subset, namely $G$ is Hausdorff. Indeed, $\Delta = d^{-1}(0)$, where $d : G \times G \to G$ is the continuous map defined by $d(x, y) = x - y$. \qed
To define the completion of a topological abelian group in full generality, we need the following generalization of sequences. We will soon restrict to cases where the topology is first-countable, for which sequences suffices.

**Definition 10.3.** A directed set is a set $I$ equipped with a preorder $\leq$ (a reflexive and transitive binary relation: $i \leq i$; $i \leq j$ and $j \leq k$ implies $i \leq k$) such that for each pair $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A net in a set $X$ is a collection $(x_i)_{i \in I}$ of elements of $X$, where $I$ is a directed set. Given a subset $U \subseteq X$, we say that $(x_i)_{i \in I}$ eventually belongs to $U$ if there exists $i_0 \in I$ such that $x_i \in U$ for all $i \geq i_0$. A net $(x_i)_{i \in I}$ in a topological spaces $X$ converges to $x \in X$ if for it eventually belongs to every neighborhood $U$ of $x$ in $X$.

Limits of nets in $X$ are unique (whenever they exists) if and only if every point of $X$ is closed.

**Definition 10.4.** Let $G$ be a topological abelian group. A net $(x_i)_{i \in I}$ in $G$ is called a Cauchy net if for every neighborhood $U$ of $0 \in G$, there exists $i_0 \in I$ such that for all $i, j \geq i_0$, $x_i - x_j \in U$ (in other words, the net $(x_i - x_j)_{(i,j) \in I \times I}$ converges to 0). We say that $G$ is complete if every Cauchy net converges to a unique point of $G$.

As usual, convergent nets are Cauchy nets. Our definition of complete includes Hausdorff (unlike in Bourbaki).

**Proposition 10.5.** Let $G$ be a topological abelian group. There exists a topological abelian group $\hat{G}$ and a continuous homomorphism $\phi: G \to \hat{G}$ such that for every continuous homomorphism $f: G \to H$ of topological abelian groups with $H$ complete, there exists a unique continuous homomorphism $g: \hat{G} \to H$ such that $f = g\phi$.

$\hat{G}$ is called the completion of $G$.

**Remark 10.6.** (1) Similar results hold for topological rings and modules, but not for topological groups [B1, Exercice X.3.16].

(2) By the universal property, $\hat{G}$ is unique up to unique isomorphism (of topological groups). The image of $\phi$ is dense. Moreover, the assignment $G \mapsto \hat{G}$ is functorial.

(3) By the proof, $\ker(\phi)$ is the intersection of the neighborhoods of $0 \in G$. Thus $G$ is Hausdorff if and only if $\phi$ is injective.

(4) By the proof, if $H$ is a subgroup of $G$ equipped with the subspace topology, then $\hat{H}$ can be identified with a topological subgroup of $\hat{G}$. The subgroup $\hat{H} < \hat{G}$ is closed (since $\hat{H}$ is complete and $\hat{G}$ is Hausdorff) and is the closure of the image of $H \to \hat{G}$.

The following is an immediate consequence of the universal property.

**Corollary 10.7.** $\phi_G: \hat{G} \to \hat{G}$ is an isomorphism (of topological groups).

Assume that $0 \in G$ admits a fundamental system of neighborhoods consisting of subgroups $(G_\lambda)_{\lambda \in A}$ of $G$ (indexed by inverse inclusion). Note that $G_\lambda$ is open. The projection $G \to G/G_\lambda$ induces $\hat{G} \to G/G_\lambda$, with $G/G_\lambda$ equipped with the discrete topology.
Proposition 10.8. The map $\hat{G} \to \lim_\Lambda G/G_\Lambda$ is an isomorphism of topological groups.

Here $\lim_\Lambda G/G_\Lambda \subseteq \prod_\Lambda G/G_\Lambda$ is equipped with the subspace topology.

Proposition 10.9. Let $0 \to G' \to G \xrightarrow{\pi} G'' \to 0$ be an exact sequence of abelian groups. Equip $G'$ with the subspace topology and $G''$ with the quotient topology. Then we have an exact sequence of groups $0 \to \widehat{G'} \to \widehat{G} \xrightarrow{\widehat{\pi}} \widehat{G''}$. Moreover, $\widehat{\pi}$ is surjective if $\Lambda = \mathbb{Z}_{\geq 0}$.

Assume $\Lambda = \mathbb{Z}_{\geq 0}$ in the sequel.

Corollary 10.10. We have $G/G_n \simeq \hat{G}/\hat{G}_n$.

Let $R$ be a ring and let $M$ be an $R$-module. The $I$-adic topology on $R$ is given by $R \supseteq I \supseteq I^2 \supseteq \ldots$, and the $I$-adic topology on $M$ is given by $M \supseteq IM \supseteq I^2 M \supseteq \ldots$.

Example 10.11. (1) For $R = \mathbb{Z}$ and $I = p\mathbb{Z}$, $\hat{R} = \mathbb{Z}_p$ is the ring of $p$-adic integers.

(2) For $R = R_0[X_1, \ldots, X_n]$ and $I = (X_1, \ldots, X_n)$, $\hat{R} = R_0[[X_1, \ldots, X_n]]$ is the ring of formal power series.

(3) For $R = \mathbb{Z}_p[X_1, \ldots, X_n]$ and $I = (p)$, $\hat{R} = \mathbb{Z}_p((X_1, \ldots, X_n))$ is the ring of convergent power series $\sum_{(i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}$ satisfying $a_{i_1, \ldots, i_n} \to 0$ for $i_1 + \cdots + i_n \to \infty$.

(4) Let $k$ be field of characteristic $\neq 2$ and $R = k[x, y]/(y^2 - (1 + x))$, $I = (x)$. Then $\hat{R} \simeq k[[x]] \oplus k[[y]]$, carrying $y$ to $(\sqrt{1 + x}, -\sqrt{1 + x})$, where $\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \in k[[x]]$. An important generalization of this is Hensel’s Lemma (exercise).

(5) Let $k$ be as above and $R = k[x, y]/(y^2 - x^2(1 + x))$, $I = (x)$. Then $\hat{R} \simeq k[[x]][y]/(y - x\sqrt{1 + x})(y + x\sqrt{1 + x})$.

Proposition 10.12. Let $R$ be a ring and let $I \subseteq R$ be an ideal such that $R$ is $I$-adically complete. Then $I \subseteq \text{rad}(R)$.

Filtrations

Let $M$ be an $R$-module and let $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ be a decreasing filtration by $R$-submodules.

Definition 10.13. Let $I \subseteq R$ be a ideal. We say that the filtration $(M_i)$ is an $I$-filtration if $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. We say that the filtration $(M_i)$ is a stable $I$-filtration if moreover there exists $N$ such that $IM_n = M_{n+1}$ for all $n \geq N$.

Lemma 10.14. Let $(M_n)$ and $(M'_n)$ be stable $I$-filtrations of $M$. Then they have bounded difference: There exists an integer $N \geq 0$ such that $M_{n+N} \subseteq M'_n$ and $M'_{n+N} \subseteq M_n$ for all $n \geq 0$. Hence stable $I$-filtrations determine the same topology on $M$ as the $I$-adic topology.

Proof. We may assume $M'_n = I^n M$. Then $I^n M = I^n M_0 \subseteq M_n$ for all $n \geq 0$. If $IM_n = M_{n+1}$ for all $n \geq N$, then $M_{n+N} = I^n M_N \subseteq I^n M$ for all $n \geq N$. \qed
Proposition 10.15 (Artin-Rees lemma). Let \( R \) be a Noetherian ring, \( I \subseteq R \) an ideal, \( M \) a finitely generated \( R \)-module, \((M_n)\) a stable \( I \)-filtration of \( M \), \( M' \subseteq M \) an \( R \)-submodule. Then \((M' \cap M_n)\) is a stable \( I \)-filtration of \( M' \). In particular, there exists an integer \( N \geq 0 \) such that \( (I^{N+n}M) \cap M' = I^n((I^N M) \cap M') \) for all \( n \geq 0 \).

Corollary 10.16. The \( I \)-adic topology on \( M' \) coincides with the subspace topology induced from the \( I \)-adic topology of \( M \).

We will prove the Artin-Rees lemma after some constructions.

Definition 10.17. A graded ring is a ring \( R \) together with an isomorphism of abelian groups \( R \cong \bigoplus_{n=0}^{\infty} R_n \) such that \( R_m R_n \subseteq R_{m+n} \) for all \( m, n \geq 0 \). A graded \( R \)-module is an \( R \)-module together with an isomorphism of abelian groups \( M \cong \bigoplus_{n=0}^{\infty} M_n \) such that \( R_m M_n \subseteq M_{m+n} \).

It follows that \( R_0 \) is a ring and each \( R_n \) and each \( M_n \) are \( R_0 \)-modules.

An element \( x \in M \) is said to be homogeneous if \( x \in M_n \) for some \( n \). For \( x = \sum_n x_n \) with \( x_n \in M_n \), the \( x_n \)'s are called the homogeneous components of \( x \).

Definition 10.18. Let \( R \) be a ring and let \( I \subseteq R \) be an ideal. The blowup algebra is the graded ring (in fact a graded \( R \)-algebra) \( B_I R = \bigoplus_{n=0}^{\infty} I^n \). For an \( R \)-module and an \( I \)-filtration \( \mathcal{F} = (M_n) \), we define the graded \( B_I R \)-module \( B_{\mathcal{F}} M = \bigoplus_{n=0}^{\infty} M_n \).

Proposition 10.19. (1) Assume \( M_n \) is a finitely generated \( R \)-module for all \( n \geq 0 \). Then \( B_{\mathcal{F}} M \) is a finitely generated \( B_I R \)-module if and only if \( \mathcal{F} \) is \( I \)-stable.

(2) Assume that \( R \) is a Noetherian ring. Then \( B_I R \) is a Noetherian ring.

The Artin-Rees lemma has many consequences.

Proposition 10.20. Let \( R \) be a Noetherian ring and let \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence of finitely generated \( R \)-modules. For any ideal \( I \subseteq R \), taking \( I \)-adic completion gives an exact sequence \( 0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0 \).

Proposition 10.21. Let \( R \) be a ring, \( I \subseteq R \) an ideal, \( M \) a finitely generated \( R \)-module. The homomorphism \( \phi : M \to \hat{M} \) induces a surjection \( \psi : \hat{R} \otimes_R M \to \hat{M} \), where \( \hat{\cdot} \) denotes the \( I \)-adic completion. In particular, \( \hat{M} = \hat{R} \phi(M) \). For \( M \) Noetherian, \( \psi \) is an isomorphism.

Corollary 10.22. Let \( R \) be a ring and \( I \subseteq R \) a finitely generated ideal. Then \( \hat{I}^n = \hat{I}^n \). Moreover, for any finitely generated \( R \)-module \( M \), \( \hat{M} \) has the \( I \)-adic topology as an \( R \)-module and the \( \hat{I} \)-adic topology as an \( \hat{R} \)-module.

Corollary 10.23. Let \( R \) be a ring and let \( \mathfrak{m} \subseteq R \) a finitely generated maximal ideal. Then the \( \mathfrak{m} \)-adic completion \( \hat{R} \) is a local ring of maximal ideal \( \mathfrak{m} \). Moreover, \( R \to \hat{R} \) factorizes through \( R_{\mathfrak{m}} \to \hat{R} \) that identifies \( \hat{R} \) as the \( \mathfrak{m} R_{\mathfrak{m}} \)-adic completion of \( R_{\mathfrak{m}} \).

Corollary 10.24. Let \( R \) be a Noetherian ring and \( I \subseteq R \) an ideal. Then the \( I \)-adic completion \( \hat{R} \) is a flat \( R \)-algebra.
Theorem 10.25 (Krull). Let $R$ be a Noetherian ring, $I \subseteq R$ an ideal, $M$ a finitely generated $R$-module. Then $\text{Ker}(M \to \hat{M}) = \bigcap_{n=0}^{\infty} I^n M$ consists of those $x \in M$ killed by some $r \in 1 + I$.

Corollary 10.26. Let $R$ be a Noetherian domain and let $I \varsubsetneq R$ be a proper ideal. Then $\bigcap_{n=0}^{\infty} I^n = 0$.

Corollary 10.27. Let $R$ be a Noetherian ring, $I \subseteq \text{rad}(R)$ and ideal of $R$, $M$ a finitely generated $R$-module. Then $\bigcap_{n=0}^{\infty} I^n M = 0$. In other words, the $I$-adic topology on $M$ is Hausdorff. Moreover, every $R$-submodule $M'$ of $M$ is closed.

Corollary 10.28. Let $R$ be a Noetherian local ring and let $m$ be the maximal ideal of $R$. Then $\bigcap_{n=0}^{\infty} m^n = 0$. This allows to extend Proposition 9.7 to Noetherian local rings.

Proposition 10.29. Let $R$ be a Noetherian ring and $I \subseteq R$ an ideal. The following conditions are equivalent:

1. $I \subseteq \text{rad}(R)$.
2. Every ideal of $R$ is closed for the $I$-adic topology.
3. The $I$-adic completion $\hat{R}$ of $R$ is a faithfully flat $R$-algebra.

We conclude this chapter with the following.

Theorem 10.30. Let $R$ be a Noetherian ring and $I \subseteq R$ an ideal. The $I$-adic completion $\hat{R}$ of $R$ is a Noetherian ring.

Corollary 10.31. Let $R$ be a Noetherian ring. Then the ring $R[[X_1, \ldots, X_n]]$ of formal power series is Noetherian.

Associated graded rings

Definition 10.32. Let $R$ be a ring, $I \subseteq R$ an ideal. The associated graded ring $\text{gr}_I R = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$. For a graded $R$-module $M$ with a $I$-filtration $\mathcal{F} = (M_n)$, the associated graded $\text{gr}_I R$-module $\text{gr}_I M = \bigoplus_{n=0}^{\infty} M_n / M_{n+1}$.

Remark 10.33. The associated graded ring is related to the blowup algebra by $\text{gr}_I R \simeq B_I R \otimes_R R/I$. Similarly, $\text{gr}_\mathcal{F} M \simeq B_\mathcal{F} M \otimes_R R/I$.

Proposition 10.34. (1) If every $M_n$ is a finitely generated $R$-module and $\mathcal{F}$ is $I$-stable, then $\text{gr}_\mathcal{F} M$ is a finitely generated $\text{gr}_I R$-module.

2. If $R$ is a Noetherian ring, then $\text{gr}_I R$ is a Noetherian ring and $\text{gr}_I R \simeq \text{gr}_I \hat{R}$.

The proof of Theorem 10.30 relies on a partial converse of the implication $R$ Noetherian $\Rightarrow \text{gr}_I R$ Noetherian.

Lemma 10.35. Let $\phi: A \to B$ be a homomorphism of filtered abelian groups.

1. If $\text{gr}(\phi)$ is injective, then $\hat{\phi}$ is injective.

2. If $\text{gr}(\phi)$ is surjective, then $\hat{\phi}$ is surjective.
Proposition 10.36. Let $R$ be a ring, $I \subseteq R$ an ideal such that $R$ is $I$-adically complete, $M$ an $R$-module, $\mathcal{F} = (M_n)$ an $I$-filtration of $M$ such that $\bigcap_n M_n = 0$. If $\text{gr}_\mathcal{F} M$ is a finitely generated $\text{gr}_I R$-module, then $M$ is a finitely generated $R$-module.

Corollary 10.37. If $\text{gr}_\mathcal{F} M$ is a Noetherian $\text{gr}_I R$-module, then $M$ is a Noetherian $R$-module.
Chapter 11

Dimension theory

Hilbert functions

**Proposition 11.1.** Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded ring. Then $R$ is Noetherian if and only if $R_0$ is Noetherian and $R$ is a finitely generated $R_0$-algebra.

**Corollary 11.2.** Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a finitely generated graded $R$-module. Then $M_n$ is a finitely generated $R_0$-module for each $n \geq 0$.

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a finitely generated graded $R$-module.

**Definition 11.3.** Let $\lambda$ be an additive function (namely $\lambda(N) = \lambda(N') + \lambda(N'')$ for every exact sequence $0 \to N' \to N \to N'' \to 0$) on the class of finitely generated $R_0$-modules with values in $\mathbb{Z}$. The Poincaré series of $M$ with respect to $\lambda$ is $P(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \in \mathbb{Z}[[t]]$.

It follows from additivity that $\lambda(0) = 0$.

**Theorem 11.4** (Hilbert, Serre). We have $P(M, t) \in \mathbb{Q}(t)$. More precisely, if $R = R_0[x_1, \ldots, x_r]$, then $P(M, t) = f(t)/\prod_{i=1}^{r} (1 - t^{k_i})$ with $f(t) \in \mathbb{Z}[t]$.

We let $D(M)$ denote the order of pole of $P(M, t)$ at $t = 1$ (we put $D(M) = 0$ if $P(M, t)$ has no pole at $t = 1$). It is a measurement of the size of $M$. In the sequel we assume $k_i = 1$ for all $i$.

**Definition 11.5.** A numerical polynomial is a polynomial $\phi(z) \in \mathbb{Q}[z]$ such that $\phi(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}$, $n \gg 0$.

**Example 11.6.** For $d \in \mathbb{Z}_{\geq 0}$, $\left(\frac{z}{d}\right) = \frac{1}{d^d} z(z-1) \cdots (z-d+1)$ is a numerical polynomial.

**Remark 11.7.** One can show that every numerical polynomial $\phi$ is a $\mathbb{Z}$-linear combination of the above. It follows that $\phi(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

**Corollary 11.8.** Assume $R = R_0[R_1]$. Let $D = D(M)$. Then there exists a unique numerical polynomial $\phi_M$ of degree $D-1$ such that $\phi_M(z) = \lambda(M_n)$ for $n \geq N+1-D$, where $N = \deg(1-t)^D P(M, t)$. We adopt the convention that $\deg(0) = -1$. 
Definition 11.9. The function \( n \mapsto \lambda(M_n) \) is called the Hilbert function and \( \phi_M \) is called the Hilbert polynomial.

In the sequel we assume \( R_0 \) is Artinian and we take \( \lambda(N) = \lg(N) \) to be the length of \( N \).

Remark 11.10. In this case, for \( M \neq 0 \), \( P(M,t) \) is not zero and \( t = 1 \) is not a zero of \( P(M,t) \). In fact, if \( D(M) = 0 \), \( P(M,1) = \sum_{n=0}^{\infty} \lg(M_n) > 0 \).

Example 11.11. Let \( R_0 \) be an Artinian ring and let \( R = R_0[X_0, \ldots, X_r] \), graded by degree. Then \( \lg(R_n) = \lg(R_0)((n+r)^r) \) (where \( \binom{n}{a} = \binom{a}{b} \) for \( a \geq b \) and \( \binom{a}{b} = 0 \) for \( a < b \)). We have \( \phi_R(z) = \binom{z+r}{r} \) with leading term \( \frac{1}{r!}z^r \).

Example 11.12. Let \( k \) be a field and \( F \in k[X_0, \ldots, X_r] \) a homogeneous polynomial of degree \( s \). Let \( R = k[X_0, \ldots, X_r]/(F) \), graded by degree. Then \( \lg(R_n) = \binom{n+r}{r} - \binom{n+s+r}{r}s \), so that \( \phi_R(z) = \binom{z+r}{r} - \binom{z+s+r}{r} = \sum_{i=1}^{s} \binom{z^{-i+r}}{r-i} \). The leading term is \( \frac{1}{(r-1)!}z^{r-1} \).

Proposition 11.13. Let \( R = \bigoplus_{n=0}^{\infty} R_n \) be a Noetherian graded ring with \( R_0 \) Artinian and \( R = R_0[R_1] \) and \( M \neq 0 \) a finitely generated graded module. Let \( k > 0 \) and \( x \in R_k \) an \( M \)-regular element (\( xM = 0 \) implies \( m = 0 \)). Then \( D(M/xM) = D(M) - 1 \).

Remark 11.14. In algebraic geometry, to a Noetherian graded ring \( R = \bigoplus_{n=0}^{\infty} R_n \) with \( R_0 \) Artinian and \( R = R_0[R_1] \) and a finitely generated graded \( R \)-module \( M \), one associates a scheme \( \text{Proj}(R) \) and a coherent sheaf \( \tilde{M} \) on \( \text{Proj}(R) \). Then \( M_n \simeq H^0(\text{Proj}(R), \tilde{M(n)}) \), where \( M(n)_k = M_{n+k} \), and

\[
\phi_M(n) = \sum_{i} (-1)^i \lg(H^i(\text{Proj}(R), \tilde{M(n)}))
\]

is the Euler characteristic of \( \tilde{M(n)} \).

Dimension theory of Noetherian local rings

Proposition 11.15. Let \( R \) be a Noetherian local ring of maximal ideal \( m \), \( q \) an \( m \)-primary ideal of \( R \), \( M \) a finitely generated \( R \)-module, \( F = (M_n) \) a stable \( q \)-filtration of \( M \).

1. \( M/M_n \) has finite length for all \( n \geq 0 \).
2. There exists a unique numerical polynomial \( \chi_F^M \) such that \( \lg(M/M_n) = \chi_F^M(n) \) for \( n \gg 0 \). Moreover, \( \deg(\chi_F^M) = D(\text{gr}_F M) \leq r \), where \( r \) denotes the least number of generators of \( q \).
3. The degree and leading coefficient of \( \chi_F^M \) depend only \( M \) and \( q \), not on \( F \).

For \( F = (q^n M) \), we write \( \chi^M_q \) for \( \chi^M_F \). For \( M = R \), we write \( \chi_q \) for \( \chi^R_q \).

Corollary 11.16. There exists a unique polynomial of degree \( D(\text{gr}_q R) \leq r \) such that \( \lg(R/q^n) = \chi_q(n) \) for \( n \gg 0 \).

Proposition 11.17. \( \deg(\chi_q) = \deg(\chi_m) \).
Notation 11.18. We write $d(R)$ for $\deg(\chi_m) = D(\text{gr}_m R)$. We denote by $\delta(R)$ the least number of generators of $m$-primary ideals of $R$.

Theorem 11.19. Let $R$ be a Noetherian local ring. We have $d(R) = \dim(R) = \delta(R)$.

We will show $\dim(R) \leq d(R)$ and $\delta(R) \leq \dim(R)$. We start with an analogue of Proposition 11.13 for Noetherian local rings.

Proposition 11.20. Let $M$ be a finitely generated $R$-module, $x \in R$ such that $\text{Ker}(M \xrightarrow{x} M) = 0$, $M' = M/xM$. Then $\deg(\chi_M') \leq \deg(\chi_M) - 1$.

Corollary 11.21. Let $x \in R$ that is not a unit or zero-divisor. Then $d(R/xR) \leq d(R) - 1$.

We will show later that equality holds in this case (Corollary 11.33).

Proposition 11.22. $\dim(R) \leq d(R)$.

Proposition 11.23. $\delta(R) \leq \dim(R)$.

This finishes the proof of Theorem 11.19. The dimension theorem has many consequences.

Example 11.24. Let $R_0$ be a nonzero Artinian ring and $R = R_0[X_1, \ldots, X_d]$, $m = (X_1, \ldots, X_d)$. We have $\text{gr}_{mR_0}(R_m) \simeq \text{gr}_m(R_m) \simeq R$. We have seen $\phi_R(z) = \lg(R_0)(\frac{z + d - 1}{d - 1})$. Thus $\dim(R_m) = d(R_m) = D(R) = d$.

Corollary 11.25. Let $R$ be a Noetherian local ring of maximal ideal $m$, $\hat{R}$ its $m$-adic completion. Then $\dim(R) = \dim(\hat{R})$.

Corollary 11.26. Every prime ideal in a Noetherian ring $R$ has finite height. In other words, $R$ satisfies the descending chain condition for prime ideals. In particular, if $R$ is a Noetherian local ring, then $\dim(R) < \infty$.

Remark 11.27. Nagata constructed a Noetherian ring $R$ with $\dim(R) = \infty$.

Definition 11.28. The embedding dimension $\text{emb.dim}(R)$ of a Noetherian local ring $R$ of maximal ideal $m$ is $\dim_k(m/m^2)$, where $k = R/m$.

By Nakayama’s lemma, $\text{emb.dim}(R)$ is the least number of generators of $m$.

Corollary 11.29. $\dim(R) \leq \text{emb.dim}(R)$.

Corollary 11.30. Let $R$ be a Noetherian ring, $x_1, \ldots, x_r \in R$. Every isolated prime ideal $p$ belonging to $(x_1, \ldots, x_r)$ has height $\leq r$.

The case $r = 1$ is called Krull’s principal ideal theorem.

Corollary 11.31. Let $R$ be a Noetherian ring, $x \in R$ not a zero-divisor. Then every isolated prime ideal $p$ belonging to $(x)$ has height $1$. 

Systems of parameters

Let $R$ be a Noetherian local ring of dimension $d$, $\mathfrak{m}$ the maximal ideal of $R$.

**Definition 11.32.** $x_1, \ldots, x_d \in R$ is called a system of parameters if $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary.

**Corollary 11.33.** Let $x_1, \ldots, x_r \in \mathfrak{m}$. We have $\dim(R/(x_1, \ldots, x_r)) \geq \dim(R) - r$. Equality holds if $x_1, \ldots, x_r$ is part of a system of parameters of $R$.

**Proposition 11.34.** Let $x_1, \ldots, x_d \in R$ be a system of parameters and $q = (x_1, \ldots, x_d)$. Let $f \in R[X_1, \ldots, X_d]$ be a homogeneous polynomial of degree $s$. Assume $f(x_1, \ldots, x_d) \in q^{s+1}$. Then $f \in \mathfrak{m}R[X_1, \ldots, X_d]$.

Let $A = R/q$ and $\alpha: A[X_1, \ldots, X_d] \to \text{gr}_q(R)$ the homomorphism carrying $X_i$ to $x_i \mod q$. The proposition says $\ker(\alpha) \subseteq \mathfrak{m}A[X_1, \ldots, X_d]$.

**Corollary 11.35.** Assume that $R$ has a subfield $k$. Then any system of parameters $x_1, \ldots, x_d$ is algebraically independent over $k$.

**Theorem 11.36.** Let $k$ be a field, $R$ a finitely generated $k$-algebra that is a domain, $K = \text{Frac}(R)$. Then for every maximal ideal $\mathfrak{m}$ of $R$, $\dim(R) = \dim(R_{\mathfrak{m}}) = \text{tr.deg}(K/k)$, where tr.deg denotes the transcendence degree.

**Lemma 11.37.** Let $A \subseteq B$ be an extension of integral domains with $A$ integrally closed and $B$ integral over $A$. Let $\mathfrak{m}$ be a maximal ideal of $B$ and $\mathfrak{n} = \mathfrak{m} \cap A$. Then $\mathfrak{n}$ is maximal and $\dim(B_{\mathfrak{n}}) = \dim(A_{\mathfrak{m}})$.

Regular local rings

**Theorem 11.38.** Let $R$ be a Noetherian local ring of dimension $d$, $\mathfrak{m}$ its maximal ideal, $k = R/\mathfrak{m}$. The following conditions are equivalent:

1. We have an isomorphism $\text{gr}_\mathfrak{m}(R) \cong k[X_1, \ldots, X_d]$ of $k$-algebras.
2. $\dim_k(\mathfrak{m}^2) = d$.
3. $\mathfrak{m}$ is generated by $d$ elements.

**Definition 11.39.** A regular local ring is a Noetherian local ring $R$ satisfying the above conditions. A regular system of generators for $R$ is $x_1, \ldots, x_d$ such that $(x_1, \ldots, x_d) = \mathfrak{m}$ (where $d = \dim(R)$).

**Example 11.40.** (1) Regular local rings of dimension 0 are precisely fields. Regular local rings of dimension 1 are precisely DVRs.

2. Let $k$ be a field, $R = k[X_1, \ldots, X_d]$, $\mathfrak{m} = (X_1, \ldots, X_d)$. Then $R_{\mathfrak{m}}$ is a regular local ring. Indeed, $\text{gr}_\mathfrak{m}(R_{\mathfrak{m}}) \cong R$.

3. Let $R$ be a regular local ring of dimension $d$ and $x_1, \ldots, x_d$ a regular system of parameters. Then $A = R/(x_1, \ldots, x_r)$ is a regular ring of dimension $d - r$. Indeed, $\bar{x}_{r+1}, \ldots, \bar{x}_d$ is a regular system of parameters for $A$.

**Proposition 11.41.** Let $R$ be a ring, $I$ an ideal satisfying $\bigcap_{n=0}^{\infty} I^n = 0$. Assume that $\text{gr}_I(R)$ is a domain. Then $R$ is a domain.

**Corollary 11.42.** A regular local ring is a domain.

**Proposition 11.43.** Let $R$ be a Noetherian local ring of maximal ideal $\mathfrak{m}$. Then $R$ is regular if and only if the $\mathfrak{m}$-adic completion $\hat{R}$ is regular.
CM rings

**Definition 11.44.** Let $R$ be a ring, $M$ an $R$-module. A sequence $x_1, \ldots, x_n \in R$ is called $M$-regular if it satisfies the following conditions:

1. Multiplication by $x_i$ is an injection on $M/\sum_{j=1}^{i-1} x_j M$ for all $1 \leq i \leq n$.
2. $M/\sum_{j=1}^{n} x_j M \neq 0$.

The depth of $M$ is the supremum of the lengths of $M$-regular sequences.

We will only use $M$-regularity when $R$ is a Noetherian local ring and $M \neq 0$ is a finitely generated $R$-module. In this case, by Nakayama’s lemma, condition (2) is equivalent to $x_1, \ldots, x_n \in m$, where $m$ is the maximal ideal of $m$.

**Proposition 11.45.** Let $R$ be a Noetherian local ring, $x_1, \ldots, x_n$ an $R$-regular sequence. Then $\dim(R/(x_1, \ldots, x_n)) = \dim(R) - n$. In particular, $\text{depth}(R) \leq \dim(R)$.

**Definition 11.46.** A Cohen-Macaulay (CM) local ring is a Noetherian local ring satisfying $\text{depth}(R) = \dim(R)$.

**Example 11.47.** (1) Artinian local rings are CM local rings.

(2) Regular local rings are CM local rings. Indeed, any regular system of parameters is an $R$-regular sequence.

**Remark 11.48.** One can show that if $R$ is a regular (resp. CM) local ring, then $R_p$ is a regular (resp. CM) local ring for every prime ideal $p$.

**Definition 11.49.** A regular (resp. CM) ring is a Noetherian ring such that $R_p$ is a regular (resp. CM) local ring for every prime ideal $p$.

**Remark 11.50.** A regular ring is normal. More generally, Serre proved the following criterion of normality: A Noetherian ring $R$ is normal if and only if the following conditions are satisfied:

1. For every prime ideal $p$ of height $\leq 1$, $R_p$ is regular.
2. For every prime ideal $p$ of height $\geq 2$, $\text{depth}(R_p) \geq 2$. 
Summary of properties of rings

- Field → DVR or field → valuation ring → local ring
- PID or field → UFD → normal ring
- Dedekind domain or field → regular ring
- Artinian ring → CM ring → Noetherian ring
Bibliography


