Lectures on Commutative Algebra

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Contents

Convention v

7 UFDs 1

8 Primary decomposition 3

9 DVRs and Dedekind domains 5

10 Completions 11

11 Dimension theory 17

Summary of properties of rings 23

Convention

Rings are assumed to be commutative.

Chapter 7

UFDs

Theorem 7.1 (Fundamental theorem of arithmetic, cf. Euclid's Elements 9.14)**.** *Every nonzero integer $n \in \mathbb{Z}$ can be factorized uniquely (up to permutation of factors) as a product of primes $n = \pm p_1 \cdots p_m$.*

Definition 7.2. Let R be a domain.

(1) $x \in R$ is *irreducible* if $x \neq 0$, $x \notin R^\times$, and $x = yz$ implies $y \in R^\times$ or $z^\times$.

(2) $x, y \in R$ are *associates* if there exists $u \in R^\times$ such that $x = uy$.

(3) R is a *unique factorization domain* (UFD, or factorial ring) if every $x \in R$ satisfying $x \neq 0$ and $x \notin R$ admits a factorization $x = a_1 \cdots a_m$ with a_i irreducible, and if $x = b_1 \cdots b_n$ with b_j irreducible, then $m = n$ and there exists a bijection $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates for every i).

We say that $x \in R$ is *prime* if xR is a prime ideal. Prime elements are irreducible. The converse holds in a UFD.

Theorem 7.3. *Let R be a domain. Then R is a UFD if and only if the following conditions are satisfied:*

(1) The ascending chain condition for principle ideals of R.

(2) Irreducible elements in R are prime.

Lemma 7.4. *Assume that the ascending chain condition holds for principle ideals of a ring R. Then every $x \in R$ satisfying $x \neq 0$ and $x \notin R$ admits a factorization $x = a_1 \cdots a_m$ with a_i irreducible.*

Corollary 7.5. *A PID is a UFD.*

Example 7.6. (1) $\mathbb{Z}$, $\mathbb{Z}[\sqrt{-1}]$ are PIDs, hence UFDs.

(2) $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. Indeed, $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

(3) If R is a UFD, $R[X]$ is a UFD. Note however that neither $\mathbb{Z}[X]$ nor $k[X, Y]$ (where k is a field) is a PID.

(4) If R is a nonzero UFD, $R[X_1, X_2, \ldots]$ is a UFD, but not a Noetherian ring.

(5) In $R = \bigcup_{n=1}^{\infty} k[X^{1/n}]$, factorization does not exist in general. In particular, R does not satisfy the ascending chain condition for principal ideals.

Chapter 8

Primary decomposition

We largely followed [AM, Chapters 4 and 7]. Most other sources define primes associated to an ideal I of a ring R differently, as prime ideals of the form $(I : x)$. The two definitions coincide when R is Noetherian.

Chapter 9

DVRs and Dedekind domains

We studied Artinian rings, which are Noetherian rings of dimension 0. In this chapter, we study the next simplest case, Noetherian *domains* of dimension 1. We start by applying primary decomposition to such domains.

Proposition 9.1. *Let R be a Noetherian domain of dimension* 1*. Every nonzero ideal $I \subseteq R$ can be uniquely written as a product of primary ideals whose radicals are distinct.*

We would like to further decompose primary ideals into prime powers. We first look at the local case.

Discrete valuation rings

Recall that the value group of a valuation ring R is $K^\times/R^\times$, where $K = \mathrm{Frac}(R)$.

Definition 9.2. A *discrete valuation ring* (DVR) is a valuation ring whose value group is isomorphic to $\mathbb{Z}$.

Proposition 9.3. *Let R be a valuation ring. The following are equivalent:*

(1) R is a DVR.
(2) R is a PID.
(3) R is Noetherian.

For (2) $\Rightarrow$ (1), we use the following.

Lemma 9.4. *Let R be a local ring of dimension > 0. Assume that the maximal ideal $\mathfrak{m}$ of R is principal and $\bigcap_{n=1}^\infty \mathfrak{m}^n = 0$. Then R is a DVR. The assumption $\bigcap_{n=1}^\infty \mathfrak{m}^n = 0$ holds if R is a Noetherian domain.*

We will see later that the assumption $\bigcap_{n=1}^\infty \mathfrak{m}^n = 0$ holds for R Noetherian.

A generator of the maximal ideal of a DVR is called a *uniformizer*.

Example 9.5. (1) Let $K = \mathbb{Q}$ and let p be a rational prime. The p-adic valuation $v_p\colon K^\times \to \mathbb{Z}$ is defined by $v_p(p^a \frac{u}{v}) = a$, where $a, u, v \in \mathbb{Z}$, $(u,p) = (v,p) = 1$. Every valuation of $K^\times$ is equivalent to v_p (Ostrowski's theorem).

(2) Let k be a field and let $K = k(X)$. Similarly to (1), for every irreducible polynomial $f \in k[X]$, the f-adic valuation $v_f \colon K^\times \times \to \mathbb{Z}$ is defined by $v_f(f^a \frac{u}{v}) = 1$, where $a \in \mathbb{Z}$, $u, v \in k[X]$, $(u, f) = (v, f) = 1$. Every valuation of $K^\times$ is equivalent to v_f or $v_{1/X}$.

(3) Let k be a field and let $K = \bigcup_n k(X^{1/n})$. We have a non-discrete rank 1 valuation $v \colon K^\times \to \mathbb{Q}$ whose restriction to $k(X^{1/n})$ is $\frac{1}{n} v_{X^{1/n}}$ with $v_{X^{1/N}}$ defined in (2). The valuation ideal $\mathfrak{m} = (X^{1/n})_{n \geq 1}$ is not finitely generated.

(4) Let F be a field and let $v_F \colon F^\times \to \Gamma$. Let $K = F(X)$ (or $F((X))$). Then $K^\times \to \mathbb{Z} \times \Gamma$ carrying $f = \sum_{n \geq N} a_n X^n$ with $a_N \neq 0$ to $(N, v(a_N))$ is a valuation, of rank > 1 for v_F nontrivial. Here $\mathbb{Z} \times \Gamma$ is equipped with the lexicographical order.

(5) Consider the particular case of (4) where $F = k(Y)$ and $v_F = v_Y$. Let $\mathfrak{m}$ be the valuation ideal. Then $\mathfrak{m} = YR$ is principal, but $\bigcap_n \mathfrak{m}^n = (X/Y^n)_{n \geq 1}$ is not finitely generated.

Definition 9.6. We say that a ring R is *normal* if for every prime $\mathfrak{p}$, $R_\mathfrak{p}$ is an integrally closed domain.

A normal domain is synonymous to an integrally closed domain.

Proposition 9.7. *Let R be a Noetherian local domain that is not a field. Let $\mathfrak{m}$ be the maximal ideal and let $k = \mathfrak{m}/\mathfrak{m}^2$. The following conditions are equivalent:*
(1) R is a DVR.
(2) R is normal of dimension one.
(3) $\mathfrak{m}$ is principal.
(4) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
(5) Every nonzero ideal of R is a power of $\mathfrak{m}$.

For (2)⇒(3), we use the following.

Lemma 9.8. *Let R be a normal local domain and assume that the maximal ideal $\mathfrak{m}$ is finitely generated and there exists $a, b \in R$ such that $\mathfrak{m} = (aR : b)$. Then $\mathfrak{m}$ is principal.*

The proposition holds in fact for R a Noetherian local ring of dimension > 0. The proof uses the Krull intersection theorem (Corollary 10.28).

Dedekind domains

Theorem 9.9. *Let R be a Noetherian domain of dimension one. The following are equivalent:*
(1) R is normal.
(2) Every primary ideal of R is a power of a prime ideals.
(3) Every local ring $R_\mathfrak{p}$ ($\mathfrak{p} \neq 0$) is a DVR.

Definition 9.10. A *Dedekind domain* is a Noetherian domain of dimension one satisfying the above equivalent conditions.

Corollary 9.11. *Every nonzero ideal of a Dedekind domain has a unique factorization as a product of prime ideals.*

Remark 9.12. The following converse of Corollary 9.11 holds: An integral domain of which every nonzero ideal is a product of prime ideals is a Dedekind domain [M2, Theorem 11.6].

Example 9.13. A PID is a Dedekind domain.

More examples are given by taking integral closure.

Proposition 9.14. *Let A be a normal domain. Let L be a finite* separable *extension of $K = \mathrm{Frac}(A)$ and let B be the integral closure of A in L. Then B is contained in a finitely generated A-submodule of L.*

Corollary 9.15. *Let A be Dedekind domain. Let L be a finite separable extension of $K = \mathrm{Frac}(A)$. Then the integral closure B of A in L is a Dedekind domain.*

Remark 9.16. More generally, if A is a Noetherian domain of dimension 1 (not necessarily normal) and if L is a finite (not necessarily separable) extension of $K = \mathrm{Frac}(A)$, then the normalization B of A in L is a Dedekind domain (even when B is not finite over A). This follows from the Krull-Akizuki Theorem [M2, Theorem 11.7].

Example 9.17. Let K be a number field (namely, a finite extension of $\mathbb{Q}$). Then the ring of integers $\mathcal{O}_K$ (namely, the integral closure of $\mathbb{Z}$ in K) is a Dedekind domain by Corollary 9.15.

Example 9.18. In particular, for $K = \mathbb{Q}(\sqrt{-5})$, $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain. The prime ideal $\mathfrak{p} = (2, 1 + \sqrt{-5})$ is not principal. Indeed, since $\mathfrak{p}^2 = 2R$, if $\mathfrak{p} = \alpha R$ for $\alpha = a + b\sqrt{-5}$, $a, b \in \mathbb{Z}$, then $(N_{K/\mathbb{Q}}\alpha)^2 = N_{K/\mathbb{Q}}2 = 4$, so that $a^2 + 5b^2 = N_{K/\mathbb{Q}}\alpha = 2$, which is impossible.

We have $3R = \mathfrak{q}\mathfrak{q}'$, $(1+\sqrt{-5})R = \mathfrak{p}\mathfrak{q}$, $(1-\sqrt{-5})R = \mathfrak{p}\mathfrak{q}'$, where $\mathfrak{q} = (3, 1+\sqrt{-5})$, $\mathfrak{q}' = (3, 1 - \sqrt{-5})$.

The maximal ideal of the DVR $R_\mathfrak{p}$ is $(1 + \sqrt{-5})R_\mathfrak{p}$.

Example 9.19. Let $R = \mathbb{Z}[\sqrt{5}]$. This is a Noetherian domain of dimension one, but not normal. The integral closure of R in $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Consider the prime ideal $\mathfrak{p} = (2, 1 + \sqrt{5})$ of R. We have $\mathfrak{p}^2 = 2\mathfrak{p} \subseteq 2R \subseteq \mathfrak{p}$. The ideal $2R$ is $\mathfrak{p}$-primary, but not a power of $\mathfrak{p}$.

Definition 9.20. We say that a domain A is *Japanese* (or N-2) if for every finite extension L of $K = \mathrm{Frac}(A)$, the integral closure of A in L if finite over A. We say that a Noetherian ring R is *Nagata* (or universally Japanese) if every finitely generated integral R-algebra is Japanese.

Grothendieck uses "Japanese" and "universally Japanese" [G, Sections 0.23, IV.7.6, IV.7.7], while Matsumura uses "N-2" and "Nagata" [M1, Chapter 12].

Example 9.21.
(1) Every normal domain of perfect fraction field is Japanese by Proposition 9.14.
(2) Every field is Nagata. Every Dedekind domain of fraction field of characteristic 0 is Nagata.
(3) Nagata constructed a DVR that is not Nagata (not Japanese?).

Invertible modules

The nonzero ideals of Dedekind domain is a free commutative monoid with maximal ideals forming a basis. We now consider the associated free abelian group. There are a number of generalizations to commutative rings.

Proposition 9.22. *Let R be a ring M be a finitely presented module. The following are equivalent:*

(1) For every prime ideal $\mathfrak{p}$ of R, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is isomorphic to $R_\mathfrak{p}$.
(2) For every maximal ideal $\mathfrak{m}$ of R, the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is isomorphic to $R_\mathfrak{m}$.
(3) The evaluation map $u\colon M^* \otimes_R M \to R$ is an isomorphism, where $M^* = \mathrm{Hom}_R(M, R)$.
(4) There exists an R-module N such that $M \otimes_R N$ is isomorphic to R.

Definition 9.23. A finitely generated R-module M is said to be *invertible* if it satisfies the above conditions. The *Picard group* $\mathrm{Pic}(R)$ of a ring R is the abelian group of isomorphism classes of invertible R-modules M with group law given by tensor product. The class of M is denoted $\mathrm{cl}(M)$.

The identity element of $\mathrm{Pic}(R)$ is $\mathrm{cl}(R)$ and $\mathrm{cl}(M)^{-1} = \mathrm{cl}(M^*)$.

Remark 9.24. The proposition holds more generally for M finitely generated. The equivalent conditions then imply that M is projective and finitely presented. Invertible R-modules are also called projective R-modules of rank 1. See [B2, II.5].

Remark 9.25. For a local ring R, $\mathrm{Pic}(R) = 0$.

Let R be a domain and let $K = \mathrm{Frac}(R)$.

Lemma 9.26. *Every invertible R-module M is an R-submodule of K.*

Definition 9.27. An R-submodule of K is called a *fractional ideal* of R if there exists $x \in R$, $x \neq 0$ such that $xI \subseteq R$.

Example 9.28. (1) Every ideal of R is a fractional ideal of R.
(2) For every $x \in K$, xR is a fractional ideal of R. Such fractional ideals are said to be *principal*. A principal fractional ideal is free of rank ≤ 1. Conversely, any free fractional ideal is principal.

Remark 9.29. Every finitely generated R-submodule of K is a fractional ideal. Conversely, if R is Noetherian, then every fractional ideal if finitely generated.

R-submodules of K form a commutative monoid, with identity element R and $IJ = \{\sum_{i=1}^n a_i b_i \mid a_i \in I,\ b_i \in J\}$. We write $I^{-1} = \{x \in K \mid xI \subseteq R\}$.

Proposition 9.30. *Let I be an R-submodule of K. The following are equivalent:*

(1) I is finitely generated and for every prime ideal $\mathfrak{p}$, $I_\mathfrak{p} \simeq R_\mathfrak{p}$.
(2) I is finitely generated and for every maximal ideal $\mathfrak{m}$, $I_\mathfrak{m} \simeq R_\mathfrak{m}$.
(1′) I is finitely generated and for every prime ideal $\mathfrak{p}$, $I_\mathfrak{p}(I_\mathfrak{p})^{-1} \simeq R_\mathfrak{p}$.
(2′) I is finitely generated and for every maximal ideal $\mathfrak{m}$, $I_\mathfrak{m}(I_\mathfrak{m})^{-1} \simeq R_\mathfrak{m}$.
(3) $II^{-1} = R$.
(4) There exists an R-submodule J of K such that $IJ = R$.

Definition 9.31. An *invertible* (fractional) ideal is a fractional ideal satisfying the above conditions. We let $\mathrm{CaDiv}(R)$ (for Cartier divisors) denote the abelian group of invertible ideals.

Proposition 9.32. *We have a exact sequence* $1 \to R^\times \to K^\times \xrightarrow{\cdot R} \mathrm{CaDiv}(R) \xrightarrow{\mathrm{cl}} \mathrm{Pic}(R) \to 1$.

Remark 9.33. For K a number field, $\mathcal{O}_K^\times$ is a finitely generated abelian group and $\mathrm{Pic}(\mathcal{O}_K^\times)$ is a finite abelian group, called the *class group* of K.

Proposition 9.34. *Let R be a UFD. Then* $\mathrm{Pic}(R) = 1$.

Lemma 9.35. *Let R be a Noetherian domain and let $\mathfrak{p}$ be an invertible prime ideal. Then $R_\mathfrak{p}$ is a DVR.*

Theorem 9.36. *Let R be a domain that is not a field. Then the following are equivalent:*
(1) R is a Dedekind domain.
(2) Every nonzero ideal of R is invertible.
(3) Every nonzero fractional ideal of R is invertible.

Corollary 9.37. *A Dedekind UFD is a PID.*

Proposition 9.38. *Let R be a Dedekind domain. Then* $\mathrm{CaDiv}(R)$ *is a free abelian group with maximal ideals forming a basis.*

Definition 9.39. The *height* of a prime ideal $\mathfrak{p}$ of a ring R is the supremum of the length n of chains $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ of prime ideals.

Remark 9.40. We have $\mathrm{ht}(\mathfrak{p}) = \dim(R_\mathfrak{p})$, $\dim(R) \leq \mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p})$, and $\dim(R) = \sup_\mathfrak{p} \mathrm{ht}(\mathfrak{p}) = \sup_\mathfrak{p} \dim(R/\mathfrak{p})$.

Remark 9.41. (1) For any ring R, one can define an abelian group $\mathrm{CaDiv}(R)$ and there is an exact sequence $1 \to R^\times \to K^\times \to \mathrm{CaDiv}(R) \to \mathrm{Pic}(R)$.
(2) For any ring R, the group $\mathrm{Div}(R)$ of *Weil divisors* is defined to the free abelian group generated by the primes ideals of $\mathfrak{p}$ of height 1. For R a Noetherian domain, there is a homomorphism $\mathrm{CaDiv}(R) \to \mathrm{Div}(R)$, which is injective for R normal and an isomorphism for R locally factorial (namely, $R_\mathfrak{p}$ is a UFD for all $\mathfrak{p}$).

We end this chapter with a criterion of normality.

Proposition 9.42. *Let R be a Noetherian domain. The following conditions are equivalent:*
(1) R is normal.
(2) For every prime ideal $\mathfrak{p}$ associated to a nonzero principal ideal of R, $R_\mathfrak{p}$ is a DVR.
(3) $R = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} R_\mathfrak{p}$ and for each $\mathfrak{p}$ of height 1, $R_\mathfrak{p}$ is a DVR.

For (2)⇒(3), we use the following.

Lemma 9.43. *Let R be a Noetherian domain. Then $R = \bigcap_\mathfrak{p} R_\mathfrak{p}$, where $\mathfrak{p}$ runs through prime ideals associated to nonzero principal ideals of R.*

Chapter 10

Completions

Topologies and completions

Definition 10.1. A *topological group* is a group G equipped with a topology such that the maps

$$\begin{aligned} G \times G \to G &\quad (x,y) \mapsto xy \text{ (multiplication)} \\ G \to G &\quad x \mapsto x^{-1} \text{ (inversion)} \end{aligned}$$

are continuous. A *topological ring* is a ring R equipped with a topology such that the maps

$$\begin{aligned} R \times R \to R &\quad (x,y) \mapsto x+y \\ R \times R \to R &\quad (x,y) \mapsto xy \end{aligned}$$

are continuous. A *topological R-module* is an R-module M equipped with a topology such that the maps

$$\begin{aligned} M \times M \to M &\quad (x,y) \mapsto x+y \\ R \times M \to M &\quad (r,x) \mapsto rx \end{aligned}$$

are continuous.

Let G be a topological abelian group. In the sequel we write the group law additively. For any $a \in G$, the translation map $T_a \colon G \to G$, $g \mapsto g+a$ is a homeomorphism.

Lemma 10.2. *Let H be the intersection of neighborhoods of 0 in G. Then H is a subgroup of G, closure of $\{0\}$. Moreover, the following conditions are equivalent:*
(1) G is Hausdorff.
(2) Every point of G is closed.
(3) $H = 0$.

Proof. That H is a subgroup of G follows from the continuity of group operations. For $x \in G$, $x \in H$ if and only if $0 \in x - U$ for all neighborhoods U of 0, which is equivalent to $x \in \overline{\{0\}}$. Then (3) is equivalent to 0 being a closed point of G. Thus (1)⇒(2)⇒(3). Conversely, if 0 is a closed point of G, then the diagonal $\Delta \subseteq G \times G$ is a closed subset, namely G is Hausdorff. Indeed, $\Delta = d^{-1}(0)$, where $d \colon G \times G \to G$ is the continuous map defined by $d(x,y) = x - y$. □

To define the completion of a topological abelian group in full generality, we need the following generalization of sequences. We will soon restrict to cases where the topology is first-countable, for which sequences suffices.

Definition 10.3. A *directed set* is a set I equipped with a preorder $\leq$ (a reflexive and transitive binary relation: $i \leq i$; $i \leq j$ and $j \leq k$ implies $i \leq k$) such that for each pair $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A *net* in a set X is a collection $(x_i)_{i\in I}$ of elements of X, where I is a directed set. Given a subset $U \subseteq X$, we say that $(x_i)_{i\in I}$ *eventually belongs to* U if there exists $i_0 \in I$ such that $x_i \in U$ for all $i \geq i_0$. A net $(x_i)_{i\in I}$ in a topological spaces X *converges to* $x \in X$ if for it eventually belongs to every neighborhood U of x in X.

Limits of nets in X are unique (whenever they exists) if and only if every point of X is closed.

Definition 10.4. Let G be a topological abelian group. A net $(x_i)_{i\in I}$ in G is called a *Cauchy* net if for every neighborhood U of $0 \in G$, there exists $i_0 \in I$ such that for all $i, j \geq i_0$, $x_i - x_j \in U$ (in other words, the net $(x_i - x_j)_{(i,j)\in I\times I}$ converges to 0). We say that G is *complete* if every Cauchy net converges to a unique point of G.

As usual, convergent nets are Cauchy nets. Our definition of complete includes Hausdorff (unlike in Bourbaki).

Proposition 10.5. *Let G be a topological abelian group. There exists a topological abelian group $\hat{G}$ and a continuous homomorphism $\phi\colon G \to \hat{G}$ such that for every continuous homomorphism $f\colon G \to H$ of topological abelian groups with H complete, there exists a unique continuous homomorphism $g\colon \hat{G} \to H$ such that $f = g\phi$.*

$\hat{G}$ is called the *completion* of G.

Remark 10.6. (1) Similar results hold for topological rings and modules, but not for topological groups [B1, Exercice X.3.16].

(2) By the universal property, $\hat{G}$ is unique up to unique isomorphism (of topological groups). The image of ϕ is dense. Moreover, the assignment $G \mapsto \hat{G}$ is functorial.
(3) By the proof, $\ker(\phi)$ is the intersection of the neighborhoods of $0 \in G$. Thus G is Hausdorff if and only if ϕ is injective.
(4) By the proof, if H is a subgroup of G equipped with the subspace topology, then $\hat{H}$ can be identified with a topological subgroup of $\hat{G}$. The subgroup $\hat{H} < \hat{G}$ is closed (since $\hat{H}$ is complete and $\hat{G}$ is Hausdorff) and is the closure of the image of $H \to \hat{G}$.

The following is an immediate consequence of the universal property.

Corollary 10.7. *$\phi_{\hat{G}}\colon \hat{G} \to \hat{\hat{G}}$ is an isomorphism (of topological groups).*

Assume that $0 \in G$ admits a fundamental system of neighborhoods consisting of subgroups $(G_\lambda)_{\lambda\in\Lambda}$ of G (indexed by inverse inclusion). Note that G_λ is open. The projection $G \to G/G_\lambda$ induces $\hat{G} \to G/G_\lambda$, with G/G_λ equipped with the discrete topology.

Proposition 10.8. *The map $\hat{G} \to \lim_\lambda G/G_\lambda$ is an isomorphism of topological groups.*

Here $\lim_\lambda G/G_\lambda \subseteq \prod_\lambda G/G_\lambda$ is equipped with the subspace topology.

Proposition 10.9. *Let $0 \to G' \to G \xrightarrow{\pi} G'' \to 0$ be an exact sequence of abelian groups. Equip G' with the subspace topology and G'' with the quotient topology. Then we have a an exact sequence of groups $0 \to \widehat{G'} \to \hat{G} \xrightarrow{\hat{\pi}} \widehat{G''}$. Moreover, $\hat{\pi}$ is surjective if $\Lambda = \mathbb{Z}_{\geq 0}$.*

Assume $\Lambda = \mathbb{Z}_{\geq 0}$ in the sequel.

Corollary 10.10. *We have $G/G_n \simeq \hat{G}/\hat{G}_n$.*

Let R be a ring and let M be an R-module. The *I-adic topology* on R is given by $R \supseteq I \supseteq I^2 \supseteq \dots$, and the I-adic topology on M is given by $M \supseteq IM \supseteq I^2M \supseteq \dots$.

Example 10.11. (1) For $R = \mathbb{Z}$ and $I = p\mathbb{Z}$, $\hat{R} = \mathbb{Z}_p$ is the ring of p-adic integers.

(2) For $R = R_0[X_1, \dots, X_n]$ and $I = (X_1, \dots, X_n)$, $\hat{R} = R_0[[X_1, \dots, X_n]]$ is the ring of formal power series.
(3) For $R = \mathbb{Z}_p[X_1, \dots, X_n]$ and $I = (p)$, $\hat{R} = \mathbb{Z}_p\langle X_1, \dots, X_n\rangle$ is the ring of convergent power series $\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n_{\geq 0}} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}$ satisfying $a_{i_1,\dots,i_n} \to 0$ for $i_1 + \dots + i_n \to \infty$.
(4) Let k be field of characteristic $\neq 2$ and $R = k[x,y]/(y^2 - (1+x))$, $I = (x)$. Then $\hat{R} \simeq k[[x]] \oplus k[[x]]$, carrying y to $(\sqrt{1+x}, -\sqrt{1+x})$, where $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots \in k[[x]]$. An important generalization of this is Hensel's Lemma (exercise).
(5) Let k be as above and $R = k[x,y]/(y^2 - x^2(1+x))$, $I = (x)$. Then $\hat{R} \simeq k[[x]][y]/(y - x\sqrt{1+x})(y + x\sqrt{1+x})$.

Proposition 10.12. *Let R be a ring and let $I \subseteq R$ be an ideal such that R is I-adically complete. Then $I \subseteq \mathrm{rad}(R)$.*

Filtrations

Let M be an R-module and let $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ be a decreasing filtration by R-submodules.

Definition 10.13. Let $I \subseteq R$ be a ideal. We say that the filtration (M_i) is an *I-filtration* if $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. We say that the filtration (M_i) is a *stable I-filtration* if moreover there exists N such that $IM_n = M_{n+1}$ for all $n \geq N$.

Lemma 10.14. *Let (M_n) and (M'_n) be stable I-filtrations of M. Then they have bounded difference: There exists an integer $N \geq 0$ such that $M_{n+N} \subseteq M'_n$ and $M'_{n+N} \subseteq M_n$ for all $n \geq 0$. Hence stable I-filtrations determine the same topology on M as the I-adic topology.*

Proof. We may assume $M'_n = I^nM$. Then $I^nM = I^nM_0 \subseteq M_n$ for all $n \geq 0$. If $IM_n = M_{n+1}$ for all $n \geq N$, then $M_{n+N} = I^nM_N \subseteq I^nM$ for all $n \geq N$. □

Proposition 10.15 (Artin-Rees lemma)**.** *Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finitely generated R-module, (M_n) a stable I-filtration of M, $M' \subseteq M$ an R-submodule. Then $(M' \cap M_n)$ is a stable I-filtration of M'. In particular, there exists an integer $N \geq 0$ such that $(I^{N+n}M) \cap M' = I^n((I^N M) \cap M')$ for all $n \geq 0$.*

Corollary 10.16. *The I-adic topology on M' coincides with the subspace topology induced from the I-adic topology of M.*

We will prove the Artin-Rees lemma after some constructions.

Definition 10.17. A *graded ring* is a ring R together with an isomorphism of abelian groups $R \simeq \bigoplus_{n=0}^{\infty} R_n$ such that $R_m R_n \subseteq R_{m+n}$ for all $m, n \geq 0$. A graded R-module is an R-module together with an isomorphism of abelian groups $M \simeq \bigoplus_{n=0}^{\infty} M_n$ such that $R_m M_n \subseteq M_{m+n}$.

It follows that R_0 is a ring and each R_n and each M_n are R_0-modules.

An element $x \in M$ is said to be *homogeneous* if $x \in M_n$ for some n. For $x = \sum_n x_n$ with $x_n \in M_n$, the x_n's are called the *homogeneous components* of x.

Definition 10.18. Let R be a ring and let $I \subseteq R$ be an ideal. The *blowup algebra* is the graded ring (in fact a graded R-algebra) $B_I R = \bigoplus_{n=0}^{\infty} I^n$. For an R-module and an I-filtration $\mathcal{F} = (M_n)$, we define the graded $B_I R$-module $B_{\mathcal{F}} M = \bigoplus_{n=0}^{\infty} M_n$.

Proposition 10.19.

(1) Assume M_n is finitely generated R-module for all $n \geq 0$. Then $B_{\mathcal{F}} M$ is a finitely generated $B_I R$-module if and only if $\mathcal{F}$ is I-stable.

(2) Assume that R is a Noetherian ring. Then $B_I R$ is a Noetherian ring.

The Artin-Rees lemma has many consequences.

Proposition 10.20. *Let R be a Noetherian ring and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. For any ideal $I \subseteq R$, taking I-adic completion gives an exact sequence $0 \to \widehat{M'} \to \hat{M} \to \widehat{M''} \to 0$.*

Proposition 10.21. *Let R be a ring, $I \subseteq R$ an ideal, M a finitely generated R-module. The homomorphism $\phi\colon M \to \hat{M}$ induces a surjection $\psi\colon \hat{R} \otimes_R M \to \hat{M}$, where $\hat{}$ denotes the I-adic completion. In particular, $\hat{M} = \hat{R}\phi(M)$. For M Noetherian, ψ is an isomorphism.*

Corollary 10.22. *Let R be a ring and $I \subseteq R$ a finitely generated ideal. Then $\hat{I}^n = \widehat{I^n}$. Moreover, for any finitely generated R-module M, $\hat{M}$ has the I-adic topology as an R-module and the $\hat{I}$-adic topology as an $\hat{R}$-module.*

Corollary 10.23. *Let R be a ring and let $\mathfrak{m} \subseteq R$ be a finitely generated maximal ideal. Then the $\mathfrak{m}$-adic completion $\hat{R}$ is a local ring of maximal ideal $\mathfrak{m}$. Moreover, $R \to \hat{R}$ factorizes through $R_{\mathfrak{m}} \to \hat{R}$ that identifies $\hat{R}$ as the $\mathfrak{m}R_{\mathfrak{m}}$-adic completion of $R_{\mathfrak{m}}$.*

Corollary 10.24. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. Then the I-adic completion $\hat{R}$ is a flat R-algebra.*

Theorem 10.25 (Krull)**.** *Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finitely generated R-module. Then $\mathrm{Ker}(M \to \hat{M}) = \bigcap_{n=0}^{\infty} I^n M$ consists of those $x \in M$ killed by some $r \in 1 + I$.*

Corollary 10.26. *Let R be a Noetherian domain and let $I \subsetneq R$ be a proper ideal. Then $\bigcap_{n=0}^{\infty} I^n = 0$.*

Corollary 10.27. *Let R be a Noetherian ring, $I \subseteq \mathrm{rad}(R)$ and ideal of R, M a finitely generated R-module. Then $\bigcap_{n=0} I^n M = 0$. In other words, the I-adic topology on M is Hausdorff. Moreover, every R-submodule M' of M is closed.*

Corollary 10.28. *Let R be a Noetherian local ring and let $\mathfrak{m}$ be the maximal ideal of R. Then $\bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0$.*

This allows to extend Proposition 9.7 to Noetherian local rings.

Proposition 10.29. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. The following conditions are equivalent:*
(1) $I \subseteq \mathrm{rad}(R)$.
(2) Every ideal of R is closed for the I-adic topology.
(3) The I-adic completion $\hat{R}$ of R is a faithfully flat R-algebra.

We conclude this chapter with the following.

Theorem 10.30. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. The I-adic completion $\hat{R}$ of R is a Noetherian ring.*

Corollary 10.31. *Let R be a Noetherian ring. Then the ring $R[[X_1, \ldots, X_n]]$ of formal power series is Noetherian.*

Associated graded rings

Definition 10.32. Let R be a ring, $I \subseteq R$ an ideal. The *associated graded ring* $\mathrm{gr}_I R = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. For a graded R-module M with a I-filtration $\mathcal{F} = (M_n)$, the associated graded $\mathrm{gr}_I R$-module $\mathrm{gr}_{\mathcal{F}} M = \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$.

Remark 10.33. The associated graded ring is related to the blowup algebra by $\mathrm{gr}_I R \simeq B_I R \otimes_R R/I$. Similarly, $\mathrm{gr}_{\mathcal{F}} M \simeq B_{\mathcal{F}} M \otimes_R R/I$.

Proposition 10.34. *(1) If every M_n is a finitely generated R-module and $\mathcal{F}$ is I-stable, then $\mathrm{gr}_{\mathcal{F}} M$ is a finitely generated $\mathrm{gr}_I R$-module.*
(2) If R is a Noetherian ring, then $\mathrm{gr}_I R$ is a Noetherian ring and $\mathrm{gr}_I R \simeq \mathrm{gr}_{\hat{I}} \hat{R}$.

The proof of Theorem 10.30 relies on a partial converse of the implication R Noetherian $\Rightarrow$ $\mathrm{gr}_I R$ Noetherian.

Lemma 10.35. *Let $\phi\colon A \to B$ be a homomorphism of filtered abelian groups.*
(1) If $\mathrm{gr}(\phi)$ is injective, then $\hat{\phi}$ is injective.
(2) If $\mathrm{gr}(\phi)$ is surjective, then $\hat{\phi}$ is surjective.

Proposition 10.36. *Let R be a ring, $I \subseteq R$ an ideal such that R is I-adically complete, M an R-module, $\mathcal{F} = (M_n)$ an I-filtration of M such that $\bigcap_n M_n = 0$. If $\mathrm{gr}_{\mathcal{F}} M$ is a finitely generated $\mathrm{gr}_I R$-module, then M is a finitely generated R-module.*

Corollary 10.37. *If $\mathrm{gr}_{\mathcal{F}} M$ is a Noetherian $\mathrm{gr}_I R$-module, then M is a Noetherian R-module.*

Chapter 11

Dimension theory

Hilbert functions

Proposition 11.1. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.*

Corollary 11.2. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a finitely generated graded R-module. Then M_n is a finitely generated R_0 module for each $n \geq 0$.*

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n=0}^{\infty} M_n$ a finitely generated graded R-module.

Definition 11.3. Let λ be an additive function (namely $\lambda(N) = \lambda(N') + \lambda(N'')$ for every exact sequence $0 \to N' \to N \to N'' \to 0$) on the class of finitely generated R_0-modules with values in $\mathbb{Z}$. The Poincaré series of M with respect to λ is $P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n)t^n \in \mathbb{Z}[[t]]$.

It follows from additivity that $\lambda(0) = 0$.

Theorem 11.4 (Hilbert, Serre). *We have $P(M,t) \in \mathbb{Q}(t)$. More precisely, if $R = R_0[x_1, \ldots, x_r]$ with $x_i \in R_{k_i}$, then $P(M,t) = f(t)/\prod_{i=1}^{r}(1 - t^{k_i})$ with $f(t) \in \mathbb{Z}[t]$.*

We let $D(M)$ denote the order of pole of $P(M,t)$ at $t = 1$ (we put $D(M) = 0$ if $P(M,t)$ has no pole at $t = 1$). It is a measurement of the size of M. In the sequel we assume $k_i = 1$ for all i.

Definition 11.5. A *numerical polynomial* is a polynomial $\phi(z) \in \mathbb{Q}[z]$ such that $\phi(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}$, $n \gg 0$.

Example 11.6. For $d \in \mathbb{Z}_{\geq 0}$, $\binom{z}{d} = \frac{1}{d!}z(z-1)\cdots(z-d+1)$ is a numerical polynomial.

Remark 11.7. One can show that every numerical polynomial ϕ is a $\mathbb{Z}$-linear combination of the above. It follows that $\phi(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

Corollary 11.8. *Assume $R = R_0[R_1]$. Let $D = D(M)$. Then there exists a unique numerical polynomial ϕ_M of degree $D-1$ such that $\phi_M(z) = \lambda(M_n)$ for $n \geq N+1-D$, where $N = \deg(1-t)^D P(M,t)$. We adopt the convention that* $\deg(0) = -1$.

Definition 11.9. The function $n \mapsto \lambda(M_n)$ is called the *Hilbert function* and ϕ_M is called the *Hilbert polynomial.*

In the sequel we assume R_0 is Artinian and we take $\lambda(N) = \lg(N)$ to be the length of N.

Remark 11.10. In this case, for $M \neq 0$, $P(M,t)$ is not zero and $t = 1$ is not a zero of $P(M,t)$. In fact, if $D(M) = 0$, $P(M,1) = \sum_{n=0}^{\infty} \lg(M_n) > 0$.

Example 11.11. Let R_0 be an Artinian ring and let $R = R_0[X_0, \dots, X_r]$, graded by degree. Then $\lg(R_n) = \lg(R_0)\binom{n+r}{r}'$ (where $\binom{a}{b}' = \binom{a}{b}$ for $a \geq b$ and $\binom{a}{b}' = 0$ for $a < b$.) We have $\phi_R(z) = \binom{z+r}{r}$ with leading term $\frac{1}{r!}z^r$.

Example 11.12. Let k be a field and $F \in k[X_0, \dots, X_r]$ a homogeneous polynomial of degree s. Let $R = k[X_0, \dots, X_r]/(F)$, graded by degree. Then $\lg(R_n) = \binom{n+r}{r}' - \binom{n-s+r}{r}'$, so that $\phi_R(z) = \binom{z+r}{r} - \binom{z-s+r}{r} = \sum_{i=1}^{s} \binom{z-i+r}{r-1}$. The leading term is $\frac{s}{(r-1)!}z^{r-1}$.

Proposition 11.13. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring with R_0 Artinian and $R = R_0[R_1]$ and $M \neq 0$ a finitely generated graded module. Let $k > 0$ and $x \in R_k$ an M-regular element ($xm = 0$ implies $m = 0$). Then $D(M/xM) = D(M) - 1$.*

Remark 11.14. In algebraic geometry, to a Noetherian graded ring $R = \bigoplus_{n=0}^{\infty} R_n$ with R_0 Artinian and $R = R_0[R_1]$ and a finitely generated graded R-module M, one associates a scheme $\mathrm{Proj}(R)$ and a coherent sheaf $\tilde{M}$ on $\mathrm{Proj}(R)$. Then $M_n \simeq H^0(\mathrm{Proj}(R), \widetilde{M(n)})$, where $M(n)_k = M_{n+k}$, and

$$\phi_M(n) = \sum_i (-1)^i \lg(H^i(\mathrm{Proj}(R), \widetilde{M(n)}))$$

is the Euler characteristic of $\widetilde{M(n)}$.

Dimension theory of Noetherian local rings

Proposition 11.15. *Let R be a Noetherian local ring of maximal ideal $\mathfrak{m}$, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of R, M a finitely generated R-module, $\mathcal{F} = (M_n)$ a stable $\mathfrak{q}$-filtration of M.*

1. *M/M_n has finite length for all $n \geq 0$.*
2. *There exists a unique numerical polynomial $\chi_{\mathcal{F}}^M$ such that $\lg(M/M_n) = \chi_{\mathcal{F}}^M(n)$ for $n \gg 0$. Moreover, $\deg(\chi_{\mathcal{F}}^M) = D(\mathrm{gr}_{\mathcal{F}} M) \leq r$, where r denotes the least number of generators of $\mathfrak{q}$.*
3. *The degree and leading coefficient of $\chi_{\mathcal{F}}^M$ depend only M and $\mathfrak{q}$, not on $\mathcal{F}$.*

For $\mathcal{F} = (\mathfrak{q}^n M)$, we write $\chi_{\mathfrak{q}}^M$ for $\chi_{\mathcal{F}}^M$. For $M = R$, we write $\chi_{\mathfrak{q}}$ for $\chi_{\mathfrak{q}}^R$.

Corollary 11.16. *There exists a unique polynomial of degree $D(\mathrm{gr}_{\mathfrak{q}} R) \leq r$ such that $\lg(R/\mathfrak{q}^n) = \chi_{\mathfrak{q}}(n)$ for $n \gg 0$.*

Proposition 11.17. $\deg(\chi_{\mathfrak{q}}) = \deg(\chi_{\mathfrak{m}})$.

Notation 11.18. We write $d(R)$ for $\deg(\chi_{\mathfrak{m}}) = D(\mathrm{gr}_{\mathfrak{m}} R)$. We denote by $\delta(R)$ the least number of generators of $\mathfrak{m}$-primary ideals of R.

Theorem 11.19. *Let R be a Noetherian local ring. We have $d(R) = \dim(R) = \delta(R)$.*

We will show $\dim(R) \le d(R)$ and $\delta(R) \le \dim(R)$. We start with an analogue of Proposition 11.13 for Noetherian local rings.

Proposition 11.20. *Let M be a finitely generated R-module, $x \in R$ such that $\mathrm{Ker}(M \xrightarrow{\times x} M) = 0$, $M' = M/xM$. Then $\deg(\chi_{\mathfrak{q}}^{M'}) \le \deg(\chi_{\mathfrak{q}}^{M}) - 1$.*

Corollary 11.21. *Let $x \in R$ that is not a unit or zero-divisor. Then $d(R/xR) \le d(R) - 1$.*

We will show later that equality holds in this case (Corollary 11.33).

Proposition 11.22. $\dim(R) \le d(R)$.

Proposition 11.23. $\delta(R) \le \dim(R)$.

This finishes the proof of Theorem 11.19. The dimension theorem has many consequences.

Example 11.24. Let R_0 be a nonzero Artinian ring and $R = R_0[X_1, \dots, X_d]$, $\mathfrak{m} = (X_1, \dots, X_d)$. We have $\mathrm{gr}_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}}) \simeq \mathrm{gr}_{\mathfrak{m}}(R_{\mathfrak{m}}) \simeq R$. We have seen $\phi_R(z) = \mathrm{lg}(R_0)\binom{z+d-1}{d-1}$. Thus $\dim(R_{\mathfrak{m}}) = d(R_{\mathfrak{m}}) = D(R) = d$.

Corollary 11.25. *Let R be a Noetherian local ring of maximal ideal $\mathfrak{m}$, $\hat{R}$ its $\mathfrak{m}$-adic completion. Then $\dim(R) = \dim(\hat{R})$.*

Corollary 11.26. *Every prime ideal in a Noetherian ring R has finite height. In other words, R satisfies the descending chain condition for prime ideals. In particular, if R is a Noetherian local ring, then $\dim(R) < \infty$.*

Remark 11.27. Nagata constructed a Noetherian ring R with $\dim(R) = \infty$.

Definition 11.28. The embedding dimension $\mathrm{emb.dim}(R)$ of a Noetherian local ring R of maximal ideal $\mathfrak{m}$ is $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$, where $k = R/\mathfrak{m}$.

By Nakayama's lemma, $\mathrm{emb.dim}(R)$ is the least number of generators of $\mathfrak{m}$.

Corollary 11.29. $\dim(R) \le \mathrm{emb.dim}(R)$.

Corollary 11.30. *Let R be a Noetherian ring, $x_1, \dots, x_r \in R$. Every isolated prime ideal $\mathfrak{p}$ belonging to $(x_1, \dots, x_r)$ has height $\le r$.*

The case $r = 1$ is called Krull's principal ideal theorem.

Corollary 11.31. *Let R be a Noetherian ring, $x \in R$ not a zero-divisor. Then every isolated prime ideal $\mathfrak{p}$ belonging to (x) has height 1.*

Systems of parameters

Let R be a Noetherian local ring of dimension d, $\mathfrak{m}$ the maximal ideal of R.

Definition 11.32. $x_1, \dots, x_d \in R$ is called a *system of parameters* if $(x_1, \dots, x_d)$ is $\mathfrak{m}$-primary.

Corollary 11.33. *Let $x_1, \dots, x_r \in \mathfrak{m}$. We have $\dim(R/(x_1, \dots, x_r)) \geq \dim(R) - r$. Equality holds if $x_1, \dots, x_r$ is part of a system of parameters of R.*

Proposition 11.34. *Let $x_1, \dots, x_d \in R$ be a system of parameters and $\mathfrak{q} = (x_1, \dots, x_d)$. Let $f \in R[X_1, \dots, X_d]$ be a homogeneous polynomial of degree s. Assume $f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}$. Then $f \in \mathfrak{m}R[X_1, \dots, X_d]$.*

Let $A = R/\mathfrak{q}$ and $\alpha\colon A[X_1, \dots, X_d] \to \mathrm{gr}_{\mathfrak{q}}(R)$ the homomorphism carrying X_i to x_i mod $\mathfrak{q}$. The proposition says $\ker(\alpha) \subseteq \mathfrak{m}A[X_1, \dots, X_d]$.

Corollary 11.35. *Assume that R has a subfield k. Then any system of parameters $x_1, \dots, x_d$ is algebraically independent over k.*

Theorem 11.36. *Let k be a field, R a finitely generated k-algebra that is a domain, $K = \mathrm{Frac}(R)$. Then for every maximal ideal $\mathfrak{m}$ of R, $\dim(R) = \dim(R_{\mathfrak{m}}) = \mathrm{tr.deg}(K/k)$, where $\mathrm{tr.deg}$ denotes the transcendence degree.*

Lemma 11.37. *Let $A \subseteq B$ be an extension of integral domains with A integrally closed and B integral over A. Let $\mathfrak{m}$ be a maximal ideal of B and $\mathfrak{n} = \mathfrak{m} \cap A$. Then $\mathfrak{n}$ is maximal and $\dim(B_{\mathfrak{n}}) = \dim(A_{\mathfrak{m}})$.*

Regular local rings

Theorem 11.38. *Let R be a Noetherian local ring of dimension d, $\mathfrak{m}$ its maximal ideal, $k = R/\mathfrak{m}$. The following conditions are equivalent:*

(1) We have an isomorphism $\mathrm{gr}_{\mathfrak{m}}(R) \simeq k[X_1, \dots, X_d]$ of k-algebras.
(2) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$.
(3) $\mathfrak{m}$ is generated by d elements.

Definition 11.39. A *regular local ring* is a Noetherian local ring R satisfying the above conditions. A regular system of generators for R is $x_1, \dots, x_d$ such that $(x_1, \dots, x_d) = \mathfrak{m}$ (where $d = \dim(R)$).

Example 11.40.

(1) Regular local rings of dimension 0 are precisely fields. Regular local rings of dimension 1 are precisely DVRs.
(2) Let k be a field, $R = k[X_1, \dots, X_d]$, $\mathfrak{m} = (X_1, \dots, X_d)$. Then $R_{\mathfrak{m}}$ is a regular local ring. Indeed, $\mathrm{gr}_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}}) \simeq R$.
(3) Let R be a regular local ring of dimension d and $x_1, \dots, x_d$ a regular system of parameters. Then $A = R/(x_1, \dots, x_r)$ is a regular ring of dimension $d - r$. Indeed, $\bar{x}_{r+1}, \dots, \bar{x}_d$ is a regular system of parameters for A.

Proposition 11.41. *Let R be a ring, I an ideal satisfying $\bigcap_{n=0}^{\infty} I^n = 0$. Assume that $\mathrm{gr}_I(R)$ is a domain. Then R is a domain.*

Corollary 11.42. *A regular local ring is a domain.*

Proposition 11.43. *Let R be a Noetherian local ring of maximal ideal $\mathfrak{m}$. Then R is regular if and only if the $\mathfrak{m}$-adic completion $\hat{R}$ is regular.*

CM rings

Definition 11.44. Let R be a ring, M an R-module. A sequence $x_1, \dots, x_n \in R$ is called *M-regular* if it satisfies the following conditions:

(1) Multiplication by x_i is an injection on $M/\sum_{j=1}^{i-1} x_j M$ for all $1 \leq i \leq n$.
(2) $M/\sum_{j=1}^{n} x_j M \neq 0$.

The *depth* of M is the supremum of the lengths of M-regular sequences.

We will only use M-regularity when R is a Noetherian local ring and $M \neq 0$ is a finitely generated R-module. In this case, by Nakayama's lemma, condition (2) is equivalent to $x_1, \dots, x_n \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{m}$.

Proposition 11.45. *Let R be a Noetherian local ring, $x_1, \dots, x_n$ an R-regular sequence. Then* $\dim(R/(x_1, \dots, x_n)) = \dim(R) - n$. *In particular,* $\operatorname{depth}(R) \leq \dim(R)$.

Definition 11.46. A *Cohen-Macaulay (CM) local ring* is a Noetherian local ring satisfying $\operatorname{depth}(R) = \dim(R)$.

Example 11.47. (1) Artinian local rings are CM local rings.
(2) Regular local rings are CM local rings. Indeed, any regular system of parameters is an R-regular sequence.

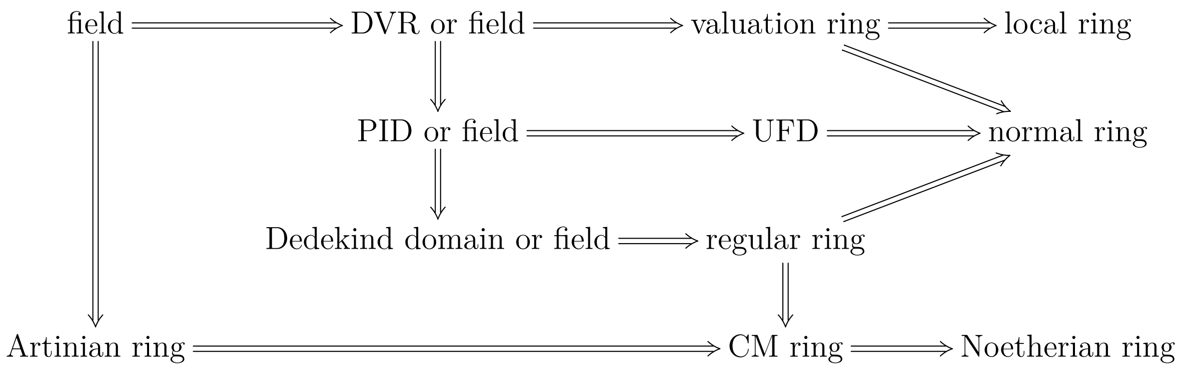
Remark 11.48. One can show that if R is a regular (resp. CM) local ring, then $R_\mathfrak{p}$ is a regular (resp. CM) local ring for every prime ideal $\mathfrak{p}$.

Definition 11.49. A regular (resp. CM) ring is a Noetherian ring such that $R_\mathfrak{p}$ is a regular (resp. CM) local ring for every prime ideal $\mathfrak{p}$.

Remark 11.50. A regular ring is normal. More generally, Serre proved the following criterion of normality: A Noetherian ring R is normal if and only if the following conditions are satisfied:

(R1) For every prime ideal $\mathfrak{p}$ of height ≤ 1, $R_\mathfrak{p}$ is regular.
(S2) For every prime ideal $\mathfrak{p}$ of height ≥ 2, $\operatorname{depth}(R_\mathfrak{p}) \geq 2$.

Summary of properties of rings



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