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Chapter 1

Categories and functors

Very rough historical sketch

Homological algebra studies derived functors between

- categories of modules (since the 1940s, culminating in the 1956 book by Cartan and Eilenberg [CE]);
- abelian categories (Grothendieck’s 1957 Tôhoku article [G]); and
- derived categories (Verdier’s 1963 notes [V1] and 1967 thesis of *doctorat d’État* [V2] following ideas of Grothendieck).

1.1 Categories

Definition 1.1.1. A category $\mathcal{C}$ consists of a set of objects $\text{Ob}(\mathcal{C})$, a set of morphisms $\text{Hom}(X,Y)$ for every pair of objects $(X,Y)$ of $\mathcal{C}$, and a composition law, namely a map

$$\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z),$$

denoted by $(f,g) \mapsto gf$ (or $g \circ f$), for every triple of objects $(X,Y,Z)$ of $\mathcal{C}$. These data are subject to the following axioms:

- (associativity) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, we have $h(gf) = (hg)f$.
- (unit law) For every object $X$ of $\mathcal{C}$, there exists an identity morphism $\text{id}_X \in \text{End}(X) := \text{Hom}(X,X)$ such that $f\text{id}_X = f$, $\text{id}_X g = g$ for all $f \in \text{Hom}(X,Y)$, $g \in \text{Hom}(Y,X)$.

The morphism $\text{id}_X$ is clearly unique.

Remark 1.1.2. For convenience we usually assume that the $\text{Hom}$ sets are disjoint. In other words, every morphism $f \in \text{Hom}(X,Y)$ has a unique source $X$ and a unique target $Y$.

Remark 1.1.3. Russell’s paradox shows that not every collection is a set. Indeed, the collection $R$ of sets $S$ such that $S \not\in S$ cannot be a set, for otherwise $R \in R$ if and only if $R \not\in R$. To avoid the paradox, the conventional ZFC (Zermelo–Fraenkel + axiom of choice) set theory does not allow the existence of a set containing all sets or unrestricted comprehension. In category theory, however, it is convenient to introduce a collection of all sets in some sense. In NBG (von Neumann–Bernays–Gödel)
set theory, which is an extension of ZFC set theory, one distinguishes between sets and proper classes. Another approach, which we adopt, is to assume the existence of an uncountable Grothendieck universe \( U \). Elements of \( U \) are called small sets.\footnote{A Grothendieck universe \( U \) is a set satisfying the following conditions: \( y \in x \in U \) implies \( y \in U \); \( x, y \in U \) implies \( \{x, y\} \in U \); \( x \in U \) implies \( P(x) \in U \) where \( P(x) \) is the power set of \( x \); \( x_i \in U \), \( i \in I \in U \) implies \( \bigcup_{i \in I} x_i \in U \). TG (Tarski–Grothendieck) set theory is obtained from ZFC by adding Tarski’s axiom, which states that for every set \( x \), there exists a Grothendieck universe \( U \ni x \).}

The following table loosely summarizes the basic terminological differences of the two approaches.

<table>
<thead>
<tr>
<th></th>
<th>NBG</th>
<th>class</th>
<th>set</th>
<th>proper class</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZFC + ( U )</td>
<td>set</td>
<td>small set</td>
<td>large set</td>
<td></td>
</tr>
</tbody>
</table>

We will mostly be interested in categories whose Hom sets are small, which are sometimes called locally small categories. A category \( C \) is called small if it is locally small and \( \text{Ob}(C) \) is small.

**Example 1.1.4.** (1) Let \( R \) be a ring.

<table>
<thead>
<tr>
<th>category</th>
<th>objects</th>
<th>morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Set} )</td>
<td>small sets</td>
<td>maps</td>
</tr>
<tr>
<td>( \text{Top} )</td>
<td>small topological spaces</td>
<td>continuous maps</td>
</tr>
<tr>
<td>( \text{Grp} )</td>
<td>small groups</td>
<td>homomorphisms of groups</td>
</tr>
<tr>
<td>( \text{R-Mod} ) (e.g. ( \text{Ab} ))</td>
<td>small (left) ( R )-modules</td>
<td>homomorphisms of ( R )-modules</td>
</tr>
</tbody>
</table>

In all of the above examples, composition is given by composition of maps.

(2) Any partially ordered set \((S, \leq)\) can be regarded as a category by

\[
\text{Hom}(x, y) = \begin{cases} \{*\} & x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}
\]

(3) Any monoid \( M \) can be regarded as a category with one object \(*\) and \( \text{End}(*) = M \). Conversely, given any object \( X \) of a category \( C \), \( \text{End}(X) \) is a monoid.

**Definition 1.1.5.** A morphism \( f : X \to Y \) in \( C \) is called an isomorphism if there exists a morphism \( g : Y \to X \) such that \( gf = \text{id}_X \) and \( fg = \text{id}_Y \). The morphism \( g \) is unique and is called the inverse of \( f \), denoted by \( f^{-1} \).

**Remark 1.1.6.** The identity map \( \text{id}_X \) is an isomorphism. Isomorphisms are stable under composition. In particular, \( \text{Aut}(X) \) is a group.

**Definition 1.1.7.** Let \( f : X \to Y \) be a morphism in a category \( C \). We say that \( f \) is a monomorphism if for every pair of morphisms \((g_1, g_2) : W \rightrightarrows X\) satisfying \( fg_1 = fg_2 \), we have \( g_1 = g_2 \). We say that \( f \) is an epimorphism if for every pair of morphisms \((h_1, h_2) : Y \rightrightarrows Z\) satisfying \( h_1 f = h_2 f \), we have \( h_1 = h_2 \). In other words, \( f \) is a monomorphism if and only if the map \( \text{Hom}(W, X) \to \text{Hom}(W, Y) \) carrying \( g \) to \( fg \) is an injection; \( f \) is an epimorphism if and only if the map \( \text{Hom}(Y, Z) \to \text{Hom}(X, Z) \) carrying \( h \) to \( hf \) is an injection.

**Remark 1.1.8.** One can show that a morphism in \( \text{Set}, \text{Top}, \text{Grp}, \) or \( \text{R-Mod} \) is a monomorphism (resp. epimorphism) if and only if it is an injection (resp. surjection). See Problems for \( \text{Grp} \). There are epimorphisms that are not surjective in the category \( \text{Ring} \) of small rings and in the category \( \text{HausTop} \) of small Hausdorff topological spaces.
Remark 1.1.9. An isomorphism is necessarily a monomorphism and an epimorphism. The converse does not hold in general. For example, in \( \text{Top} \), the continuous map \( \mathbb{R}_{\text{disc}} \to \mathbb{R} \) carrying \( x \) to \( x \), where \( \mathbb{R}_{\text{disc}} \) denotes the set \( \mathbb{R} \) equipped with the discrete topology, is a monomorphism and an epimorphism, but not an isomorphism.

We leave the proof of the following lemma as an exercise.

Lemma 1.1.10. Consider morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \). Then
1. If \( f \) and \( g \) are monomorphisms, then \( gf \) is a monomorphism.
2. If \( f \) and \( g \) are epimorphisms, then \( gf \) is an epimorphism.
3. If \( gf \) is a monomorphism, then \( f \) is a monomorphism.
4. If \( gf \) is an epimorphism, then \( g \) is an epimorphism.

Definition 1.1.11. The opposite category \( C^\text{op} \) of a category \( C \) is defined by \( \text{Ob}(C^\text{op}) = \text{Ob}(C) \) and \( \text{Hom}_{C^\text{op}}(X,Y) = \text{Hom}_C(Y,X) \).

A morphism \( f \) of a category \( C \) is a monomorphism in \( C \) if and only if it is an epimorphism in \( C^\text{op} \).

### 1.2 Functors

#### Functors

Definition 1.2.1. Let \( C \) and \( D \) be categories. A functor \( F: C \to D \) consists of a map \( \text{Ob}(C) \to \text{Ob}(D) \) and, for every pair of objects \( (X,Y) \) of \( C \), a map \( \text{Hom}_C(X,Y) \to \text{Hom}_D(FX, FY) \), compatible with composition and identity: \( F(\text{id}_X) = \text{id}_{FX} \) for all \( X \in \text{Ob}(C) \) and \( F(gf) = F(g)F(f) \) for all morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \).

Remark 1.2.2. Given functors \( F: C \to D \) and \( G: D \to E \), we have the composite functor \( GF: C \to E \). For any category \( C \), we have the identity functor \( \text{id}_C \). We can thus organize small categories and functors into a category \( \text{Cat} \).

Example 1.2.3. (1) We have forgetful functors \( \text{Top} \to \text{Set} \) and
\[
R\text{-Mod} \to \text{Ab} \to \text{Grp} \to \text{Set}.
\]
(2) We have a functor \( \text{Set} \to R\text{-Mod} \) carrying a set \( S \) to the free \( R \)-module \( R^{(S)} = \bigoplus_{s \in S} Rs \).
(3) We have a functor \( H_n: \text{Top} \to \text{Ab} \) carrying a topological space \( X \) to its \( n \)-th singular homology group \( H_n^{\text{sing}}(X) \).
(4) For any object \( X \) in a category \( C \) with small Hom sets, we have functors \( \text{Hom}(X,\_): C \to \text{Set} \) and \( h_C(X) = \text{Hom}(\_,X): C^\text{op} \to \text{Set} \).

Definition 1.2.4. A contravariant functor from \( C \) to \( D \) is a functor \( C^\text{op} \to D \).

Definition 1.2.5. Let \( (C_i)_{i \in I} \) be a family of categories. The product category \( C = \prod_{i \in I} C_i \) is defined by \( \text{Ob}(C) = \prod_{i \in I} \text{Ob}(C_i) \) and \( \text{Hom}_C((X_i), (Y_i)) = \prod_{i \in I} \text{Hom}_{C_i}(X_i, Y_i) \).

A functor \( C \times D \to E \) is sometimes called a bifunctor.
Example 1.2.6. (1) For any category $C$ with small Hom sets, we have a functor $\text{Hom}(-,-): C^{\text{op}} \times C \to \text{Set}$.

(2) Let $R$, $S$, and $T$ be rings. We have functors $- \otimes_S -: (R,S)-\text{Mod} \times (S,T)-\text{Mod} \to (R,T)-\text{Mod}$, $\text{Hom}_{R-\text{Mod}}(-,-): ((R,S)-\text{Mod})^{\text{op}} \times (R,T)-\text{Mod} \to (S,T)-\text{Mod}$, $\text{Hom}_{\text{Mod}-T}(-,-): ((S,T)-\text{Mod})^{\text{op}} \times (R,T)-\text{Mod} \to (R,S)-\text{Mod}$.

Here $(R,S)-\text{Mod}$ denotes the category of small $(R,S)$-bimodules, which can be identified with $(R \otimes_{\mathbb{Z}} S^{\text{op}})-\text{Mod}$.

Natural transformations

Definition 1.2.7. Let $F,G: C \to D$ be functors. A natural transformation $\alpha: F \to G$ consists of morphisms $\alpha_X: FX \to GX$ in $D$ for all objects $X$ of $C$, such that for every morphism $f: X \to Y$ of $C$, the following diagram commutes

\[
\begin{array}{ccc}
FX & \xrightarrow{Ff} & FY \\
\downarrow{\alpha_X} & & \downarrow{\alpha_Y} \\
GX & \xrightarrow{Gf} &GY
\end{array}
\]

Example 1.2.8. Let $U: R-\text{Mod} \to \text{Set}$ be the forgetful functor carrying an $R$-module to its underlying set. Let $F: \text{Set} \to R-\text{Mod}$ be the free module functor. Then the injection $S \to R^{(S)}$ carrying $s$ to $1 \cdot s$ defines a natural transformation $\alpha: \text{id}_{\text{Set}} \to UF$.

Remark 1.2.9. Given functors $F,G,H: C \to D$ and natural transformations $\alpha: F \to G$ and $\beta: G \to H$, we have the (vertically) composite natural transformation $\beta \alpha: F \to H$. Functors $C \to D$ and natural transformations form a category $\text{Fun}(C,D)$. Isomorphisms in this category are called natural isomorphisms. A natural transformation $\alpha$ is a natural isomorphism if and only if $\alpha_X$ is an isomorphism for every object $X$ of $C$.

There is also a horizontal composition of natural transformations: Given a natural transformation $\alpha: F \to G$ between functors $C \to D$ and a natural transformation $\beta': F' \to G'$ between functors $D \to E$, we have $\alpha' \beta: F'F \to G'G$ between functors $C \to E$. This composition satisfies various compatibilities. Small categories, functors, and natural transformations, together with horizontal and vertical compositions, form a “2-category”.

There is an obvious notion of isomorphism of categories. A more useful notion is the following.

Definition 1.2.10. An equivalence of categories is a functor $F: C \to D$ such that there exists a functor $G: D \to C$ and natural isomorphisms $\text{id}_C \simeq GF$ and $FG \simeq \text{id}_D$. The functors $F$ and $G$ are then called quasi-inverses of each other.

\[\text{Some authors call them natural equivalences.}\]

\[\text{Some authors write } \simeq \text{ for equivalences and } \cong \text{ for isomorphisms. We will write } \simeq \text{ for isomorphisms and state equivalences verbally.}\]
1.2. FUNCTORS

Quasi-inverses of a functor $F$ are unique up to natural isomorphisms.

**Example 1.2.11.** Let $X$ be a (path-connected) simply-connected space. The fundamental groupoid $\Pi_1(X)$ is equivalent to $\{\ast\}$, but not isomorphic to $\{\ast\}$ unless $X$ is a singleton.

**Remark 1.2.12.** Given a composable pair of functors $C \xrightarrow{F} D \xrightarrow{G} E$, if two of $F$, $G$, and $GF$ are equivalences of categories, then so is the third one. If $F \rightarrow F'$ is a natural isomorphism of functors, then $F$ is an equivalence of categories if and only if $F'$ is.

### Faithful functors, full functors

**Definition 1.2.13.** A functor $F: C \rightarrow D$ is faithful (resp. full, resp. fully faithful) if the map $\text{Hom}_C(X,Y) \rightarrow \text{Hom}_D(FX,FY)$ is an injection (resp. surjection, resp. bijection) for all $X,Y \in \text{Ob}(C)$.

**Lemma 1.2.14.** Let $F: C \rightarrow D$ be a fully faithful functor.

1. Let $f: X \rightarrow Y$ be a morphism of $C$ such that $Ff$ is an isomorphism. Then $f$ is an isomorphism.
2. Let $X$ and $Y$ be objects of $C$ such that $FX \simeq FY$. Then $X \simeq Y$.

**Proof.** (1) Let $g'$ be an inverse of $Ff$ and let $g: Y \rightarrow X$ be such that $Fg = g'$. Then $g$ is an inverse of $f$.

(2) Let $f': FX \rightarrow FY$ be an isomorphism and let $f: X \rightarrow Y$ be such that $Ff = f'$. By (1), $f$ is an isomorphism. 

**Remark 1.2.15.** There is an obvious notion of subcategory. For a subcategory of a category, the inclusion functor is faithful. A full subcategory is a subcategory such that the inclusion functor is fully faithful.

**Example 1.2.16.** The category $\text{Ab}$ is a full subcategory of $\text{Grp}$. The forgetful functor $\text{Grp} \rightarrow \text{Set}$ is faithful, but not fully faithful.

**Definition 1.2.17.** A functor $F: C \rightarrow D$ is essentially surjective if for every object $Y$ of $D$, there exists an object $X$ of $C$ and an isomorphism $FX \simeq Y$.

**Proposition 1.2.18.** A functor $F: C \rightarrow D$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

**Corollary 1.2.19.** Let $F: C \rightarrow D$ be a fully faithful functor. Then $F$ induces an equivalence of categories $C \rightarrow D_0$, where $D_0$ is the full subcategory of $D$ spanned by the image of $F$.

**Lemma 1.2.20.** For any category $C$, there exists a full subcategory $C_0$ such that the inclusion functor $C_0 \rightarrow C$ is an equivalence of categories and isomorphic objects in $C_0$ are equal.
Proof. By the axiom of choice, we can pick a representative in each isomorphism class of objects \( \mathcal{C} \). For every object \( X \), pick an isomorphism \( \alpha_X: X \cong R_X \), where \( R_X \) is the representative of the class of \( X \). Let \( \mathcal{C}_0 \) be the full subcategory of \( \mathcal{C} \) spanned by the representatives. Consider the functor \( F: \mathcal{C} \to \mathcal{C}_0 \) carrying \( X \) to \( R_X \) and \( f: X \to Y \) to \( \alpha_Y f \alpha_X^{-1}: R_X \to R_Y \). Let \( i: \mathcal{C}_0 \to \mathcal{C} \) be the inclusion functor. Then \( \alpha_X \) defines natural isomorphisms \( i_{\mathcal{C}_0} \cong Fi \) and \( i_{\mathcal{C}} \cong IF \).

**Proof of Proposition [1.2.13]** Let \( F \) be an equivalence of categories and let \( G \) be a quasi-inverse. Since \( FG \cong \text{id} \), \( F \) is essentially surjective. Since \( GF \cong \text{id} \), \( G \) is essentially surjective. For objects \( X, X' \) of \( \mathcal{C} \), since the composition

\[
(GF)_{X,X'}: \text{Hom}_\mathcal{C}(X, X') \xrightarrow{F_{X,X'}} \text{Hom}_\mathcal{D}(FX, FX') \xrightarrow{G_{FX,FX'}} \text{Hom}_\mathcal{C}(GFX, GFX')
\]

is an isomorphism, \( F_{X,X'} \) is an injection. Similarly, for objects \( Y, Y' \) of \( \mathcal{D} \), \( F_{GY,GY'} \) is a surjection. Since \( G \) is essentially surjective, it follows that \( F_{X,X'} \) is a surjection.

Now let \( F \) be a fully faithful and essentially surjective functor. We apply the lemma to \( \mathcal{C} \) and to \( \mathcal{D} \). Let \( i: \mathcal{C}_0 \to \mathcal{C} \) be the inclusion functor and choose a quasi-inverse \( j' \) of the inclusion functor \( i' : \mathcal{D}_0 \to \mathcal{D} \). Then \( j'Fi: \mathcal{C}_0 \to \mathcal{D}_0 \) is fully faithful and essentially surjective, thus a surjection on objects. By Lemma [1.2.14] (2), \( j'Fi \) is also an injection an objects. Thus \( j'Fi \) is an isomorphism of categories. Since \( j' \) and \( i \) are equivalences of categories, it follows that \( F \) is an equivalence of categories. \( \square \)

**Yoneda’s lemma and representable functors**

Let \( \mathcal{C} \) be a category with small Hom sets. For every object \( X \) of \( \mathcal{C} \), consider the functor \( h_\mathcal{C}(X): \mathcal{C}^{\text{op}} \to \text{Set} \).

**Lemma 1.2.21** (Yoneda). For every functor \( F: \mathcal{C}^{\text{op}} \to \text{Set} \), the map

\[
\phi: \text{Nat}(h_\mathcal{C}(X), F) \to F(X)
\]

given by \( \phi(\alpha) = \alpha_X(\text{id}_X) \) is a bijection.

We leave it as an exercise to state a dual of Yoneda’s lemma.

**Proof.** We construct the inverse \( \psi: F(X) \to \text{Nat}(h_\mathcal{C}(X), F) \) by \( \psi(y)(f) = F(f)(x) \) for \( f: Y \to X \). We have \( (\phi \psi)(x) = \psi(x)(\text{id}_X) = F(\text{id}_X)(x) = x \). Moreover, \( (\psi \phi)(\alpha)_Y = F(f)(\phi(\alpha)) = F(f)(\alpha_X(\text{id}_X)) = \alpha_Y(h_\mathcal{C}(f)(\text{id}_X)) = \alpha_Y(f) \).

Note that \( h_\mathcal{C}(X) \) is functorial in \( X \), in the sense that we have a functor \( h_\mathcal{C}: \mathcal{C} \to \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set}) \).

**Corollary 1.2.22.** The functor \( h_\mathcal{C} \) is fully faithful.

This functor is called the **Yoneda embedding**.

**Proof.** Indeed, the map \( \text{Hom}_\mathcal{C}(X,Y) \to \text{Nat}(h_\mathcal{C}(X), h_\mathcal{C}(Y)) \) given by \( h_\mathcal{C} \) coincides with the bijection \( \psi \) constructed in the proof of the lemma for \( F = h_\mathcal{C}(Y) \). \( \square \)

Applying Lemma [1.2.14] we get the following.
Corollary 1.2.23. (1) Let \( f: X \to Y \) be a morphism such that \( h_C(f): h_C(X) \to h_C(Y) \) is a natural isomorphism. Then \( f \) is an isomorphism.

(2) Let \( X, Y \) be objects such that \( h_C(X) \cong h_C(Y) \). Then \( X \cong Y \).

Definition 1.2.24. We say that a functor \( F: C^{\text{op}} \to \text{Set} \) is represented by an object \( X \) of \( C \) if there exists a natural isomorphism \( F \cong h_C(X) \).

1.3 Universal constructions

Initial objects, final objects, zero objects

Definition 1.3.1. Let \( C \) be a category. An object \( X \) of \( C \) is called an initial object if, for every object \( Y \) of \( C \), there exists precisely one morphism \( X \to Y \). An object \( Y \) of \( C \) is called a final (or terminal) object if, for every object \( X \) of \( C \), there exists precisely one morphism \( X \to Y \).

Remark 1.3.2. An initial object of \( C \) is a final object of \( C^{\text{op}} \) and a final object of \( C \) is an initial object of \( C^{\text{op}} \).

Proposition 1.3.3. If \( X_1 \) and \( X_2 \) are initial objects of \( C \), then there exists a unique isomorphism between them. If \( Y_1 \) and \( Y_2 \) are final objects of \( C \), then there exists a unique isomorphism between them.

Proof. By definition there exists a unique morphism \( f: X_1 \to X_2 \) and a unique morphism \( f': X_2 \to X_1 \). By the uniqueness of morphisms \( X_1 \to X_1 \), we have \( f'f = \text{id}_{X_1} \), and similarly \( ff' = \text{id}_{X_2} \). Thus \( f \) is an isomorphism. The case of final objects follows by duality. \( \square \)

Example 1.3.4. Let \( R \) be a ring.

<table>
<thead>
<tr>
<th>category</th>
<th>initial object</th>
<th>final object</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Set} )</td>
<td>( \emptyset )</td>
<td>( {*} )</td>
</tr>
<tr>
<td>( \text{Top} )</td>
<td>( {} )</td>
<td>( {1} )</td>
</tr>
<tr>
<td>( \text{Grp} )</td>
<td>( {0} )</td>
<td>( {0} )</td>
</tr>
<tr>
<td>( R-\text{Mod} )</td>
<td>( \mathbb{Z} )</td>
<td>( {0} )</td>
</tr>
<tr>
<td>( \text{Ring} )</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>( (S, \leq) )</td>
<td>least element (if any)</td>
<td>greatest element (if any)</td>
</tr>
</tbody>
</table>

Definition 1.3.5. If an object is both initial and final, it is called a zero object.

Remark 1.3.6. If \( C \) admits a zero object, then for every pair of objects \( X, Y \), there exists a unique morphism \( X \to Y \), called the zero morphism, that factors through a zero object. Zero objects and zero morphisms are often denoted by \( 0 \).

Products, coproducts

Definition 1.3.7. Let \( (X_i)_{i \in I} \) be a family of objects in \( C \). A product of \( (X_i)_{i \in I} \) is an object \( P \) of \( C \) equipped with morphisms \( p_i: P \to X_i, \ i \in I \), satisfying the following
universal property: for each object $Q$ of $C$ equipped with morphisms $q_i: Q \to X_i$, $i \in I$, there exists a unique morphism $q: Q \to P$ such that $q_i = p_i q$. A coproduct of $(X_i)_{i \in I}$ is an object $U$ of $C$ equipped with morphisms $u_i: X_i \to C$, $i \in I$, satisfying the following universal property: for each object $V$ of $C$ equipped with morphisms $v_i: X_i \to V$, $i \in I$, there exists a unique morphism $v: U \to V$ such that $v_i = v u_i$.

**Remark 1.3.8.** A product in $C$ is a coproduct in $C^{\text{op}}$ and a coproduct in $C$ is a product in $C^{\text{op}}$.

**Remark 1.3.9.** For $I = \emptyset$, a product is a final object and a coproduct is an initial object.

**Proposition 1.3.10.** The product of $(X_i)_{i \in I}$, if it exists, is unique up to unique isomorphism. More precisely, if $(P, (p_i))$ and $(P', (p'_i))$ are products of $(X_i)$, then there exists a unique morphism $f: P \to P'$ such that $p_i = fp'_i$ for all $i \in I$.

**Proof.** Indeed, by the universal property of product, we have unique morphisms $f: P \to P'$ and $f': P' \to P$ such that $p_i = fp'_i$, $p'_i = f'p_i$ for all $i \in I$. It follows that $p_i = f'fp_i$ for all $i \in I$, so that $f'f = \text{id}_P$ by the universal property (applied to $Q = P$). Similarly, $ff' = \text{id}_{P'}$. Therefore, $f$ is an isomorphism. 

**Notation 1.3.11.** We let $\prod_{i \in I} X_i$ denote the underlying object of a product of $(X_i)_{i \in I}$ if it exists. We let $\coprod_{i \in I} X_i$ denote the underlying object of a coproduct of $(X_i)_{i \in I}$ if it exists.

We say that $C$ admits finite (resp. small, etc.) products if $C$ admits limits indexed by all finite (resp. small, etc.) set $I$.

**Example 1.3.12.** Let $R$ be a ring.

<table>
<thead>
<tr>
<th>category</th>
<th>small coproduct</th>
<th>small product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Set}$</td>
<td>disjoint union</td>
<td>usual product</td>
</tr>
<tr>
<td>$\text{Top}$</td>
<td></td>
<td>direct sum</td>
</tr>
<tr>
<td>$R\text{-Mod}$ (e.g. $\text{Ab}$)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the case of $\text{Top}$, the product $P = \prod_{i \in I} X_i$ is equipped with coarsest topology (sometimes called the Tychonoff topology) for which the projections $P \to X_i$ are continuous. The disjoint union $U = \coprod_{i \in I} X_i$ is equipped with finest topology for which the inclusions $X_i \to U$ are continuous.

In the categories $\text{Grp}$, $\text{Ring}$, $\text{CRing}$, small products are usual products. In the category $\text{CRing}$ of small commutative rings, the coproduct of a pair of rings is tensor product.

**Example 1.3.13.** In the category associated to a partially ordered set $(S, \leq)$, product means infimum and coproduct means supremum. In particular, if $\leq$ is a total order, then $(S, \leq)$ admits products of pairs of objects and coproducts of pairs of objects.

**Remark 1.3.14.** Let $(X_i)_{i \in I}$ be a family of objects of $C$ and let $I = \coprod_{j \in J} I_j$ be a partition. If $P_j = \prod_{i \in I_j} X_i$ exists for each $j$, and $P = \prod_{j \in J} P_j$ exists, then $P$ is a project of $(X_i)_{i \in I}$. In particular, a category admitting products of pairs of objects admits finite nonempty products.
Remark 1.3.15. The universal property for product can be summarized as a bijection
\[ \text{Hom}_C(Q, \prod_{i \in I} X_i) \cong \prod_{i \in I} \text{Hom}_C(Q, X_i). \]

We defined \( \prod_{i \in I} X_i \) by spelling out the functor \( \text{Hom}(\cdot, \prod_{i \in I} X_i) \).

\section*{Comma categories}

Let \( F: C \to D \) be a functor. For an object \( Y \) of \( D \), we let \( (F \downarrow Y) \) denote the category defined as follows. An object of \( (F \downarrow Y) \) is a pair \((X, f)\), where \( X \) is an object of \( C \) and \( f: FX \to Y \) is a morphism of \( D \). A morphism \((X, f) \to (X', f')\) is a morphism \( h: X \to X' \) such that \( f = f'(Fh) \), i.e., the following diagram commutes

\[
\begin{array}{ccc}
FX & \xrightarrow{Fh} & FX' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{i} & Y
\end{array}
\]

Dually, we let \((Y \downarrow F)\) denote the category of pairs \((X, f)\), where \( X \) is an object of \( C \) and \( f: Y \to FX \) is a morphism of \( D \). A morphism \((X, f) \to (X', f')\) is a morphism \( h: X \to X' \) such that \( f' = (Fh)f \). We have an isomorphism of categories \((F \downarrow Y)^\text{op} \cong (Y \downarrow F)^\text{op}\), where \( F^\text{op} : C^\text{op} \to D^\text{op}\).

We may regard a family of objects \((X_i)_{i \in I}\) of \( C \) as one object \( X \) of the category \( C^I \cong \text{Fun}(I, C) \). Here \( I \) denotes both the index set and the associated discrete category. Let \( \Delta: C \to C^I \) be the diagonal functor sending \( A \) to the constant functor of value \( A \). Then a product of \((X_i)_{i \in I}\) is the same as a final object of \((\Delta \downarrow X)\) and a coproduct is the same as an initial object of \((X \downarrow \Delta)\).

\section*{Limits, colimits}

The above construction generalizes to the case where \( I \) is a category. Let \( C \) and \( I \) be categories and let \( \Delta: C \to C^I := \text{Fun}(I, C) \) be the “diagonal” functor carrying \( X \) to the constant functor of value \( X \).

Definition 1.3.16. Let \( F: I \to C \) be a functor. A \textit{limit} (also called projective limit) of \( F \) is a final object of \((\Delta \downarrow F)\) and a \textit{colimit} (also called inductive limit) of \( F \) is an initial object of \((F \downarrow \Delta)\).

We sometimes refer to \((\Delta \downarrow F)\) as the category of objects of \( C \) over \( F \) (or cones to the base \( F \)) and denote it by \( C_{/F} \). Dually, we sometimes refer to \((F \downarrow \Delta)\) as the category of objects of \( C \) under \( F \) (or cones from the base \( F \)) and denote it by \( C_{F/-} \).

Limits and colimits are unique up to unique isomorphisms.

Remark 1.3.17. Let us spell out the definition of limit. A limit of \( F \) is an object \( L \) of \( C \) equipped with morphisms \( p_i: L \to F(i), \ i \in \text{Ob}(I) \) such that \( p_j = F(f)p_i \) for all morphisms \( f: i \to j \) and satisfying the following universal property: for every object \( M \) of \( C \) equipped with morphisms \( q_i: M \to F(i), \ i \in \text{Ob}(I) \) satisfying \( q_j = F(f)q_i \) for all morphisms \( f: i \to j \), there exists a unique morphism \( a: M \to L \) such that \( q_i = p_ia \) for all \( i \). We leave it as an exercise to spell out the definition of colimit.
Notation 1.3.18. We let $\lim F$ or $\lim_{i \in I} F(i)$ denote the underlying object of a limit of $F$ if it exists. We let $\text{colim} F$ or $\text{colim}_{i \in I} F(i)$ denote the underlying object of a colimit of $F$ if it exists. (Other notation: $\varprojlim$ for limit and $\varinjlim$ for colimit.)

Example 1.3.19. For $I = (\bullet \Rightarrow \bullet)$, $F: I \to \mathcal{C}$ is represented by a pair of morphisms $f, g: X \to Y$ with the same source and target. The limit of $F$ is called the equalizer of the pair and denoted by $\text{Eq}(f, g)$. The colimit of $F$ is called the coequalizer and denoted by $\text{Coeq}(f, g)$. If $\mathcal{C}$ admits a zero object and $g$ is the zero morphism, these are called respectively the kernel and the cokernel of $f$: $\ker(f) = \text{Eq}(f, 0)$, $\text{coker}(f) = \text{Coeq}(f, 0)$.

Example 1.3.20. For $I = (\bullet \Rightarrow \bullet \Rightarrow \bullet)$, $F: I \to \mathcal{C}$ is given by a pair of morphisms $f: A \to B$, $g: C \to B$ with the same target in $\mathcal{C}$. A limit of $F$ is called a pullback of $(f, g)$. Let us spell out the definition. A pullback of $(f, g)$ is a pair of morphisms $a: D \to A$, $c: D \to C$ such that $fa = gc$ and satisfying the following universal property: for every pair of morphisms $a': E \to A$, $c': E \to C$, there exists a unique morphism $e: E \to D$ such that $a' = ae$ and $c' = ce$. The universal property can be visualized as follows

\[
\begin{array}{ccc}
E & \xrightarrow{e} & D \\
\downarrow & & \downarrow f \\
C & \xrightarrow{g} & B
\end{array}
\]

We write $A \times_B C$ for the underlying object of the limit.

Dually, for $I = (\bullet \Leftarrow \bullet \Rightarrow \bullet)$, a colimit is called a pushout.

We say that $\mathcal{C}$ admits finite (resp. small, etc.) limits if $\mathcal{C}$ admits limits indexed by all finite (resp. small, etc.) category $I$.

Example 1.3.21. The category $\text{Set}$ admits small limits. For a functor $F: I \to \text{Set}$, $\lim F$ is represented by the subset $L \subseteq \prod_{i \in \text{Ob}(I)} F(i)$ consisting of elements $(x_i)$ such that $F(f)(x_i) = x_j$ for every morphism $f: i \to j$ is small, whenever $L$. The same holds for limits in $\text{Grp}$, $\text{R-Mod}$, and $\text{Ring}$.

Example 1.3.22. The category $\text{Set}$ admits small colimits. For a functor $F: I \to \text{Set}$ with $I$ a small category, $\text{colim} F$ is represented by the quotient

$$Q = \coprod_{i \in \text{Ob}(I)} F(i)/\sim$$

by the equivalence relation $\sim$ generated by $x \sim F(f)(x)$ for $f: i \to j$ and $x \in F(i)$, whenever $Q$ is small.

Similarly, the category $\text{R-Mod}$ admits small colimits. For a functor $F: I \to \text{Set}$ with $I$ a small category, $\text{colim} F$ is represented by the quotient

$$Q = \bigoplus_{i \in \text{Ob}(I)} F(i)/M$$

by the $R$-submodule $M$ generated by $x - F(f)(x)$ for $f: i \to j$ and $x \in F(i)$, whenever $Q$ is small.
Definition 1.3.23. We say that a functor \( F : C \to D \) preserves limits if for every functor \( S : I \to C \), and every limit \( a : \Delta X \to S \) of \( S \), \( \Delta(FY) = F \circ \Delta Y \to FS \) is a limit of \( GS \).

Proposition 1.3.24. Let \( C \) be a category with small Hom sets. For every object \( X \) of \( C \), the functor \( \text{Hom}_C(X, _) : C \to \text{Set} \) preserves limits, and, in particular, small limits.

Proof. Let \( F : I \to C \) be a functor. Then
\[
\text{Hom}_C(X, \lim F) \simeq \text{Hom}_{C^\text{op}}(\Delta X, F) \simeq \lim \text{Hom}_C(X, F-) .
\]
(We leave it to the reader to give a more rigorous proof.)

Remark 1.3.25. None of the forgetful functors
\[
\text{Grp} \to \text{Set}, \quad \text{Ab} \to \text{Set}, \quad \text{Ab} \to \text{Grp}
\]
preserve finite coproducts.

Universal constructions

Let \( F : C \to D \) be a functor and let \( Y \) be an object of \( D \). A universal arrow from \( U \) to \( Y \) is a final object of \( (F \downarrow Y) \). A universal arrow from \( Y \) to \( U \) is an initial object of \( (Y \downarrow F) \).

Remark 1.3.26. Note that \( (X_0, \epsilon : FX_0 \to Y) \) is a final object of \( (F \downarrow Y) \) if and only if the map \( \text{Hom}_C(X, X_0) \to \text{Hom}_D(FX, Y) \) carrying \( f \) to the composite \( FX \xrightarrow{F\epsilon} FX_0 \xrightarrow{\epsilon} Y \) is a bijection.

Example 1.3.27. Let \( U : \text{Grp} \to \text{Set} \) be the forgetful functor and let \( S \) be a small set. The free group \( FS \) with basis \( S \), equipped with the map \( i : S \to UFS \) satisfies the following property: for every small group \( G \) equipped with a map \( f : S \to UG \), there exists a unique homomorphism \( h : FS \to G \) such that \( f = (Uh)i \). Thus \( (FS, i) \) is an initial object of \( (S \downarrow U) \).

1.4 Adjunction

Adjunction

Definition 1.4.1 (Kan). Let \( C \) and \( D \) be categories. An adjunction is a triple \( (F, G, \phi) \), where \( F : C \to D \) and \( G : D \to C \) are functors, and \( \phi \) is a natural isomorphism \( \phi_{XY} : \text{Hom}_D(FX, Y) \cong \text{Hom}_C(X, GY) \). We then say that \( F \) is left adjoint to \( G \) and \( G \) is right adjoint to \( F \) and we sometimes write \( \phi : F \dashv G \).

If \( C \) and \( D \) have small Hom sets, then \( \phi \) is a natural isomorphism of functors \( C^{\text{op}} \times D \to \text{Set} \).

Example 1.4.2. The free group functor \( F : \text{Set} \to \text{Grp} \) is left adjoint to the forgetful functor \( U : \text{Grp} \to \text{Set} \).
Example 1.4.3. Let $I$ be a category. If $\mathcal{C}$ admits limits indexed by $I$, then the limit functor $\lim: \mathcal{C}^I \to \mathcal{C}$ is right adjoint to the diagonal functor $\Delta: \mathcal{C} \to \mathcal{C}^I$. If $\mathcal{C}$ admits colimits indexed by $I$, then the colimit functor $\text{colim}: \mathcal{C}^I \to \mathcal{C}$ is left adjoint to the diagonal functor $\Delta: \mathcal{C} \to \mathcal{C}^I$.

Example 1.4.4. Let $X$, $Y$, $Z$ be small sets. Then we have a bijection
\[ \text{Hom}_{\text{Set}}(X \times Y, Z) = \text{Hom}_{\text{Set}}(X, \text{Hom}_{\text{Set}}(Y, Z)) \]
natural in $X$, $Y$, $Z$. This exhibits $- \times Y$ and $\text{Hom}_{\text{Set}}(Y, -)$ as adjoint functors.

Example 1.4.5. Let $R$, $S$, $T$ be rings and consider small bimodules $R M_S$, $S N_T$, and $R P_T$. We have an isomorphism of abelian groups
\begin{align*}
\text{Hom}_{(R,T)\text{-Mod}}(M \otimes_S N, P) &= \text{Hom}_{(R,S)\text{-Mod}}(M, \text{Hom}_{\text{Mod}-T}(N, P)), \\
\text{Hom}_{(R,T)\text{-Mod}}(M \otimes_S N, P) &= \text{Hom}_{(S,T)\text{-Mod}}(N, \text{Hom}_{\text{Mod-R}}(M, P)),
\end{align*}
natural in $M$, $N$, $P$. The first one exhibits $- \otimes_S N$ and $\text{Hom}_{\text{Mod-T}}(N, -)$ as adjoint functors, and the second one exhibits $M \otimes_S -$ and $\text{Hom}_{\text{Mod-R}}(M, -)$ as adjoint functors.

Remark 1.4.6. Let $\phi: F \dashv G$ be an adjunction. Then $\phi$ induces $G^{\text{op}} \dashv F^{\text{op}}$.

Proposition 1.4.7. Let $\phi: F \dashv G$. Then $G$ is determined by $F$ up to natural isomorphism.

Proof. Let $\phi': F \dashv G'$. Consider the natural isomorphism $\phi'^{-1} \circ \phi: \text{Hom}_{\mathcal{D}}(X, GY) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(X, G'Y)$. By Yoneda’s lemma, this is given by an isomorphism $GY \to G'Y$, which is natural in $Y$ by the naturality of $\phi$ and $\phi'$.

The following proposition shows that the functoriality of an adjoint is automatic.

Proposition 1.4.8. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ have small Hom sets. Then $F$ admits a right adjoint if and only if for every object $Y$ of $\mathcal{D}$, the functor $h_{\mathcal{D}}(Y) \circ F = \text{Hom}_\mathcal{C}(F-, Y)$ is representable.

Proof. We can construct an adjunction $\phi: F \dashv G$ as follows. For every object $Y$ of $\mathcal{D}$, choose an object $GY$ of $\mathcal{C}$ and an isomorphism $\phi: h_{\mathcal{D}}(Y) \circ F \xrightarrow{\sim} h_{\mathcal{C}}(GY)$. For every morphism $Y \to Y'$, we get a morphism $GY \to G'Y$ in $\mathcal{C}$ by Yoneda’s lemma.

We leave it to the reader to state duals of the preceding propositions.

Proposition 1.4.9. Let $F: \mathcal{C} \to \mathcal{D}$, $F': \mathcal{D} \to \mathcal{E}$, $G: \mathcal{D} \to \mathcal{C}$, $G': \mathcal{E} \to \mathcal{D}$ be functors and let $\phi: F \dashv G$, $\phi': F' \dashv G'$ be adjunctions. Then $\phi \phi': F'F \dashv GG'$.

Proof. Indeed we have
\[ \text{Hom}_{\mathcal{E}}(F'FX, Y) \xrightarrow{\phi_{F'F}^X Y} \text{Hom}_{\mathcal{D}}(FX, G'Y) \xrightarrow{\phi_{X,G'Y}} \text{Hom}_{\mathcal{C}}(X, GG'Y). \]
Unit, counit

The naturality of \( \phi \) means

\[
\phi(b \circ f \circ Fa) = Gb \circ \phi(f) \circ a,
\]

for all \( a: X' \to X, \ f: FX \to Y, \ b: Y \to Y' \). Let \( \eta_X = \phi(id_{FX}) : X \to GFX \) and let \( \epsilon_Y = \phi^{-1}(id_{GY}) : FGY \to Y \). By (1.4.1), \( \eta : id_C \to GF \) and \( \epsilon : FG \to id_D \) are natural transformations. We call \( \eta \) the unit and \( \epsilon \) the counit. Note that (1.4.1) implies that

\[
F \xrightarrow{F\eta} FGF \xrightarrow{id_F} F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G
\]

are identity transformations: \( \epsilon F \circ F\eta = id_F, \ G\epsilon \circ \eta G = id_G \). Indeed, \( \phi(\epsilon_{FX} \circ F\eta_X) = \phi(\epsilon_{FX}) \circ \eta_X = \eta_X = \phi(id_{FX}) \) and the second relation is proved similarly. Note that (1.4.1) also implies that \( \phi \) is determined by \( \eta \) by the rule \( \phi(f) = Gf \circ \eta_X \) for \( f : FX \to Y \). Moreover, \( \phi^{-1} \) is determined by \( \epsilon \) by the rule \( \phi^{-1}(g) = \epsilon_Y \circ Fg \) for \( g : X \to GY \).

**Example 1.4.10.** Let \( U : \text{Ab} \to \text{Set} \) be the forgetful functor and let \( F : \text{Set} \to \text{Ab} \) be the functor carrying \( S \to \mathbb{Z}^{(S)} = \bigoplus_{s \in S} \mathbb{Z}a_s \). The unit \( S \to UFS \) carries \( s \) to \( a_s \). The counit \( FDA \to A \) carries \( \sum_{s \in A} n_s a_s \) to \( \sum_{s \in A} n_s s \).

**Proposition 1.4.11.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. An adjunction \( (F,G,\phi) \) is uniquely determined by each of the following data:

1. Functors \( F : \mathcal{C} \to \mathcal{D}, \ G : \mathcal{D} \to \mathcal{C} \) and natural transformations \( \eta : id_C \to GF, \ \epsilon : FG \to id_D \) be such that \( \epsilon F \circ F\eta = id_F, \ G\epsilon \circ \eta G = id_G \).
2. A functor \( F : \mathcal{C} \to \mathcal{D} \), and for every object \( Y \) of \( \mathcal{D} \), a final object \( (GY, \epsilon_Y) \) of \( (F \downarrow Y) \);
3. A functor \( G : \mathcal{D} \to \mathcal{C} \), and for every object \( X \) of \( \mathcal{C} \), an initial object \( (FX, \eta_X) \) of \( (X \downarrow G) \).

Part (2) is in some sense a restatement of Proposition 1.4.8

**Proof.** For (1), note we have seen that \( \phi(f) = Gf \circ \eta_X \) is uniquely determined. Clearly \( \phi \) a natural transformation. Put \( \psi(g) = \epsilon_Y \circ Fg \). Then

\[
\phi \psi(g) = \phi(\epsilon_Y \circ Fg) = G\epsilon_Y \circ GFG \circ \eta_X = G\epsilon_Y \circ \eta_{GY} \circ g = g.
\]

Similarly, \( \psi \phi(f) = f \). Thus \( \phi \) is a natural isomorphism.

(2) and (3) are dual to each other. We only treat (3). For any morphism \( a : X \to X' \) in \( \mathcal{C} \), there exists a unique morphism \( Fa : FX \to FX' \) in \( \mathcal{D} \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & GFX \\
\downarrow{a} & & \downarrow{GFa} \\
X' & \xrightarrow{\eta_{X'}} & GFX'
\end{array}
\]

commutes. It is easy to check that \( F : \mathcal{C} \to \mathcal{D} \) is a functor. The above commutativity means that \( \eta : id_C \to GF \) is a natural transformation. Then \( \phi(f) = Gf \circ \eta_X \) is uniquely determined. Clearly \( \phi \) is a natural transformation, and is a natural isomorphism by the universal property. \( \square \)
Chapter 1. Categories and Functors

Proposition 1.4.12. Let \( \phi: F \dashv G \). Then

1. \( F \) is fully faithful if and only if the unit \( \eta: \text{id}_C \to GF \) is a natural isomorphism.
2. \( G \) is fully faithful if and only if the counit \( \epsilon: FG \to \text{id}_D \) is a natural isomorphism.

Proof. By [1.4.1], for \( f: X \to X' \), we have \( \phi(Ff) = \phi(\text{id}_{FX'} \circ f) = \phi(\text{id}_{FX'}) \circ f = \eta_{X'} \circ f \). In other words, the composite \( \text{Hom}_C(X, X') \xrightarrow{F} \text{Hom}_D(FX, FX') \xrightarrow{\phi} \text{Hom}_C(X, GFX') \) is induced by \( \eta_{X'} \). Then (1) follows from Yoneda’s lemma. We obtain (2) by duality.

Corollary 1.4.13. Let \( \phi: F \dashv G \). Then the following conditions are equivalent:

1. \( F \) is an equivalence of categories.
2. \( G \) is an equivalence of categories.
3. \( F \) and \( G \) are fully faithful.
4. The unit \( \eta: \text{id}_C \to GF \) and counit \( \epsilon: FG \to \text{id}_D \) are natural isomorphisms.

Under these conditions, \( F \) and \( G \) are quasi-inverse to each other.

Proof. If \( F \) is an equivalence, then, by the proposition, \( \text{id}_C \simeq GF \), so that \( G \) is also an equivalence. By duality, (1) \( \iff \) (2). It is clear that (4) \( \Rightarrow \) (1)+(2) \( \Rightarrow \) (3). By the proposition, (3) \( \Rightarrow \) (4).

Remark 1.4.14. If \( F \) is an equivalence of categories and \( G \) is a quasi-inverse to \( F \), then \( G \) is both right adjoint to \( F \) and left adjoint to \( F \).

Adjunction and (co)limits

Proposition 1.4.15. Let \( F: C \to D \) be left adjoint to \( G: D \to C \). Then

1. \( F^J: C^J \to D^J \) is left adjoint to \( G^J: D^J \to C^J \) for any category \( J \);
2. \( G \) preserves limits (that exist in \( D \)) and \( F \) preserves colimits (that exist in \( C \)).

Proof. (1) follows from the determination of adjunction by unit and counit (Proposition 1.4.11 (1)). For (2), by duality it suffices to show the first assertion. Let \( S: J \to D \) be a functor such that \( \text{lim} \ S \) exists. Consider the commutative square

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C^J \\
F \downarrow & & \downarrow F^J \\
D & \xrightarrow{\Delta} & D^J.
\end{array}
\]

We have

\[
\text{Hom}(X, \text{lim} G^J S) \simeq \text{Hom}(\Delta X, G^J S) \\
\simeq \text{Hom}(F^J \Delta X, S) = \text{Hom}(\Delta FX, S) \simeq \text{Hom}(FX, \text{lim} S) \simeq \text{Hom}(X, G \text{lim} S),
\]

so that \( G \text{lim} S \) is a limit of \( GS \) (again we leave it to the reader to give a more rigorous proof).
1.4. ADJUNCTION

In the case where \( C \) and \( D \) admit limits indexed by \( J \), part (2) of the proposition can be paraphrased as follows: (1.4.2) induces by taking right adjoints a square

\[
\begin{array}{ccc}
C & \xleftarrow{\lim} & C^J \\
G & \downarrow & \downarrow G^J \\
D & \xleftarrow{\lim} & D^J
\end{array}
\]

which commutes up to natural isomorphism.

The rest of this section is not used elsewhere in these notes. In favorable cases one can give necessary and sufficient conditions for the existence of adjoints. Let us first give a criterion for the existence of an initial object.

**Theorem 1.4.16.** Let \( D \) be a category with small \( \text{Hom} \) sets and admitting small limits. Then \( D \) has an initial object if and only if it satisfies the following Solution Set Condition: there exists a small set \( S \) of objects of \( D \) that is weakly initial in the sense that for every object \( x \) of \( D \), there exist \( s \in S \) and a morphism \( s \to x \).

Note that the condition is a set-theoretic one that is automatically satisfied if \( D \) is small (by taking \( S \) to be \( \text{Ob}(D) \)).

**Proof.** (Copied from [ML2, Theorem X.2.1]) The “only if” part is clear: if \( i \) is an initial object, then \( S = \{ i \} \) satisfies the Solution Set Condition. Let us show the “if” part. We let \( S \) also denote the full subcategory of \( D \) spanned by \( S \). Let \( F: S \to D \) be the inclusion. We claim that \( i = \lim F \) is an initial object of \( D \). Choose, for every object \( x \) of \( D \), a morphism \( f_x: i \to x \) that factorizes through the projection \( p: i \to s \) to some \( s \in S \). We get a cone \( \Delta i \to \text{id}_D \). Indeed, for every morphism \( x \to x' \), we have a commutative diagram

![Diagram](image_url)

where \( x'' \) is the fiber product of \( s \) and \( s' \) over \( x' \). We may assume that for all \( s \in S \), \( f_s: i \to s \) is the projection. We conclude by the following lemma. \( \square \)

**Lemma 1.4.17.** Let \( F: S \to D \) be a functor and let \( c: \Delta i \to \text{id}_D \) be a cone such that \( cF: \Delta i \to F \) is a limiting cone. Then \( i \) is initial.

**Proof.** Since \( c \) is a cone, the diagram

\[
\begin{array}{ccc}
i & \xrightarrow{c_{i}} & F_i \\
& \xleftarrow{c_{FS}} & \downarrow \\
i & \xrightarrow{cF} & FS
\end{array}
\]
commutes for every $s \in \text{Ob}(S)$. Since $cF$ is limiting, this implies $c_i = \text{id}_i$. Since $c$ is a cone, for every morphism $i \to x$, the diagram

\[
\begin{array}{ccc}
 i & \longrightarrow & x \\
 \downarrow c_i & & \downarrow c_x \\
 i & \longrightarrow & x
\end{array}
\]

commutes, so that $c_x$ is the unique morphism $i \to x$.

**Theorem 1.4.18** (Freyd Adjoint Functor Theorem). Let $\mathcal{D}$ be a category with small Hom sets and admitting small limits. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then $G$ admits a left adjoint if and only if $G$ preserves small limits and satisfies the Solution Set Condition: for each object $X$ of $\mathcal{C}$, there exists a small set of objects $S$ of $(X \downarrow G)$ that is weakly initial (namely, for every object $x$ of $(X \downarrow G)$, there exist $s \in S$ and a morphism $s \to x$ in $(X \downarrow G)$).

**Proof.** Recall from Proposition 1.4.11 that $G$ admits a left adjoint if and only if $(X \downarrow G)$ admits an initial object for every object $X$ of $\mathcal{C}$. The “only if” part then follows from Proposition 1.4.15. To show the “if” part, we apply Theorem 1.4.16. It suffices to check, under the assumption that $G$ preserves small limits, that $(X \downarrow G)$ admits small limits. Let $I$ be a small category and let $F : I \to (X \downarrow G)$ be a functor. We write $Fi$ as $(Y_i, f_i : X \to GY_i)$. Let $Y$ be the limit of $Y_i$ in $\mathcal{D}$. Since $G$ preserves small limits, $(f_i)$ determines a morphism $f : X \to GY$. It is easy to check that $(X, f)$ is the limit of $F$. \qed

### 1.5 Additive categories

**Additive categories**

**Proposition 1.5.1.** Let $\mathcal{A}$ be a category with each $\text{Hom}_\mathcal{A}(X, Y)$ equipped with a structure of unital magma such that composition is bilinear. Let $A_1$ and $A_2$ be objects of $\mathcal{A}$. Then the following conditions are equivalent.

1. $A_1 \times A_2$ exists.
2. $A_1 \coprod A_2$ exists.

Under these assumptions, the morphism $\phi_{A_1, A_2} : A_1 \coprod A_2 \to A_1 \times A_2$ described by the matrix $\begin{pmatrix} \text{id}_{A_1} & 0 \\ 0 & \text{id}_{A_2} \end{pmatrix}$ is an isomorphism. Moreover, if $Y \times Y$ and $Y \times Y \times Y$ exist, then $\text{Hom}_\mathcal{A}(X, Y)$ is a commutative monoid, and for $f, f' : X \to Y$, $f + f'$ is given by the composition

\[
X \xrightarrow{(f, f')} Y \xrightarrow{\phi_{Y, Y}^{-1}} Y \coprod Y \xrightarrow{(\text{id}_Y, \text{id}_Y)} Y.
\]

We denote the operation of $\text{Hom}_\mathcal{A}(X, Y)$ by $+$ and the unit by $0 = 0_{XY}$. The bilinearity of composition means the following: for $f, f' : X \to Y$ and $g, g' : Y \to Z$, we have

(a) $g(f + f') = gf + gf'$, $(g + g')f = gf + g'f$; (b) $g0_{XY} = 0_{XZ}, 0_{YZ}f = 0_{XZ}$.

Condition (b) follows from (a) if $\text{Hom}_\mathcal{A}(X, Z)$ is a group. If $\mathcal{A}$ admits a zero object, it follows from (b) that the zero morphism $z_{XY} : X \to Y$ (that factors through every zero object) is the unit of $\text{Hom}_\mathcal{A}(X, Y)$. Indeed, $z_{XY} = 0_{YY}z_{XY} = 0_{XY}$.  

Proof. By duality we may assume that (1) holds. Let \( i_1 = (\text{id}_{A_1}, 0) : A_1 \to A_1 \times A_2 \) and let \( i_2 = (0, \text{id}_{A_2}) : A_2 \to A_1 \times A_2 \). We show that \( (A_1 \times A_2, i_1, i_2) \) exhibits \( A_1 \times A_2 \) as a coproduct of \( A_1 \) and \( A_2 \). Let \( B \) be an object of \( \mathcal{A} \) equipped with \( h_1 : A_1 \to B \) and \( h_2 : A_2 \to B \). We put \( h = (h_1, h_2) = h_1 p_1 + h_2 p_2 : A_1 \times A_2 \to B \). Then \( h i_1 = h_1 p_1 i_1 + h_2 p_2 i_1 = h_1 + 0 = h_1 \) and similarly \( h i_2 = h_2 \). If \( h' : A_1 \times A_2 \to B \) is a morphism such that \( h' i_1 = h_1, h' i_2 = h_2 \). Then

\[
h' = h'(i_1 p_1 + i_2 p_2) = h_1 p_1 + h_2 p_2 = h
\]

by Lemma 1.5.2 below. Therefore, (2) and the second assertion hold.

Assume that \( Y \times Y \) exists. Then in \((1.5.1), (\text{id}_Y, \text{id}_Y) \circ \phi_{Y} = p_1 + p_2 \). Thus \( f + f' = (p_1 + p_2)(f, f') \) is given by \((1.5.1)\). The diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(f', f)} & Y \\
\downarrow (f', f) & \searrow (p_1, p_2) \quad (i_2, i_1) & \swarrow (id, id) \\
Y & \xrightarrow{\phi_{Y}^{-1}} & Y \parallel Y
\end{array}
\]

commutes, which implies \( f + f' = f' + f \). Assume moreover that \( Y \times Y \times Y \) exists. For \( f, f', f'' : X \to Y \), \( f + (f' + f'') \) is given by the composition

\[
\begin{array}{cccc}
X & \xrightarrow{(f', f'')} & Y \parallel Y & \xrightarrow{\phi_{Y}^{-1}} Y \\
\downarrow (f, (f', f'')) & \searrow \xrightarrow{id_Y} (Y \parallel Y) & \swarrow \xrightarrow{\phi_{Y}^{-1}} Y \parallel (Y \parallel Y)
\end{array}
\]

The diagram

\[
\begin{array}{c}
Y \times (Y \times Y) \xrightarrow{id_Y \times \phi_{Y}^{-1}} (Y \parallel Y) \xrightarrow{\phi_{Y}^{-1}} Y \parallel (Y \parallel Y) \\
\downarrow \approx \\
(Y \times Y) \times Y \xrightarrow{\phi_{Y}^{-1} \times id_Y} (Y \parallel Y) \times (Y \parallel Y) \xrightarrow{\phi_{Y}^{-1} \parallel \phi_{Y}^{-1}} (Y \parallel Y) \parallel Y
\end{array}
\]

commutes, because the inverse of the composition of the upper horizontal arrows and the inverse of the composition of the lower horizontal arrows are both given by the matrix \[
\begin{pmatrix}
0 & 0 \\
0 & id_Y \\
0 & 0
\end{pmatrix}
\]. Therefore, \( f + (f' + f'') = (f + f') + f'' \). \(\square\)

Lemma 1.5.2. Under the above notation, \( i_1 p_1 + i_2 p_2 = \text{id}_{A_1 \times A_2} \).

Proof. We have \( p_1(i_1 p_1 + i_2 p_2) = p_1 i_1 p_1 + p_1 i_2 p_2 = p_1 + 0 = p_1 \) and similarly \( p_2(i_1 p_1 + i_2 p_2) = p_2 \). Therefore, \( i_1 p_1 + i_2 p_2 = \text{id}_{A_1 \times A_2} \). \(\square\)

Remark 1.5.3. Under the assumptions (1) and (2) of Proposition 1.5.1, the composition of \( C \xrightarrow{(a, b)} A_1 \times A_2 \xrightarrow{(c, d)} B \) is \((c, d)(a, b) = ca + db\). In other words, composition is given by matrix multiplication, with \((a, b)\) considered as a column vector and \((c, d)\) considered as a row vector.
Proposition 1.5.4. Let $\mathcal{A}$ be a category admitting a zero object, finite products, and finite coproducts satisfying

(*) The morphism $\phi_{YY}: Y \amalg Y \to Y \times Y$ described by the matrix $\begin{pmatrix} \text{id}_Y & 0 \\ 0 & \text{id}_Y \end{pmatrix}$ is an isomorphism for every object $Y$ of $\mathcal{A}$.

Then there exists a unique way to equip every $\text{Hom}_{\mathcal{A}}(X,Y)$ with the structure of a unital magma such that composition is bilinear. Moreover, $\text{Hom}_{\mathcal{A}}(X,Y)$ is a commutative monoid.

Here $0$ in the the description of $\phi_{YY}$ denotes the zero morphism (that factors through every zero object).

Proof. For morphisms $f,f': X \to Y$, we define $f + f'$ to be the composition

$$X \xrightarrow{(f,f')} Y \times Y \xrightarrow{\phi_{YY}^{-1}} Y \amalg Y \xrightarrow{\text{id}_Y, \text{id}_Y} Y.$$ 

The diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(f,0)} & Y \times Y \\
\downarrow{f} & & \downarrow{\phi_{YY}^{-1}} \\
Y & \xrightarrow{(\text{id}_Y,0)} & Y \amalg Y
\end{array}
$$

commutes, so that $f + 0 = f$. Similarly $0 + f = f$. This construction equips $\text{Hom}_{\mathcal{A}}(X,Y)$ with the structure of a unital magma. It is clear from the construction that $(g + g')f = gf + gf'$. The diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(gf,gf')} & Y \times Y \\
\downarrow{(gf,g)} & & \downarrow{\phi_{YY}^{-1}} \\
Z \times Z & \xrightarrow{g \times g} & Z \amalg Z
\end{array}
$$

commutes, so that $g(f + f') = gf + gf'$. Moreover, $0f = 0$ and $g0 = 0$. Thus the composition law on $\mathcal{A}$ is bilinear.

The uniqueness and the fact that $\text{Hom}_{\mathcal{A}}(X,Y)$ is a commutative monoid follows from Proposition 1.5.1.

Definition 1.5.5. An additive category is a category $\mathcal{A}$ admitting a zero object, finite products, finite coproducts, satisfying (*) above, and such that the commutative monoids $\text{Hom}_{\mathcal{A}}(X,Y)$ are abelian groups.

Remark 1.5.6. By Proposition 1.5.1 if $\mathcal{A}$ is a category with each $\text{Hom}_{\mathcal{A}}(X,Y)$ equipped with a structure of abelian group such that composition is bilinear and such that $\mathcal{A}$ admits a zero object, finite products (or finite coproducts), then $\mathcal{A}$ is an additive category.

In an additive category, coproducts are also called direct sums and we often write $\oplus$ instead of $\amalg$.

\footnote{A category with each $\text{Hom}_{\mathcal{A}}(X,Y)$ equipped with a structure of abelian group such that composition is bilinear is called a preadditive category, or a category enriched over $(\text{Ab}, \otimes)$.}
Example 1.5.7. Let $R$ be a ring. Then $R\text{-Mod}$ is an additive category. Indeed, $R\text{-Mod}$ admits finite products, the Hom sets are naturally equipped with structures of abelian groups and composition is bilinear.

Example 1.5.8. Let $B$ be an additive category. Let $A \subseteq B$ be a full subcategory such that for $A$ and $A'$ in $A$, the direct sum $A \oplus A'$ in $B$ is isomorphic to an object of $A$. Then $A$ is an additive category by Lemma 1.5.9 below. In particular, the full subcategory $R\text{-mod}$ of $R\text{-Mod}$ spanned by finitely generated $R$-modules is an additive category. Similarly, the full subcategory of $R\text{-Mod}$ spanned by free left $R$-modules is also an additive category.

Lemma 1.5.9. Let $B$ be a full subcategory of $C$ and let $F: I \to B$ be a functor. If $p: \Delta X \to F$ exhibits $X$ as the limit of $F$ in $C$ with $X$ in $B$, then $p$ exhibits $X$ as the limit of $F$ in $B$.

Proof. This follows easily from the universal properties of limits. We leave the details to the reader.

Example 1.5.10. Let $A$ be an additive category. Then $A^{op}$ is an additive category.

For the next example, we need the general fact that limits can be computed pointwise in a functor category. More precisely, we have the following.

Proposition 1.5.11. Let $C$, $I$, $P$ be categories and let $F: I \to C^P$ be a functor such that for each object $p$ of $P$, $F_p = E_p \circ F: I \to C$ admits a limit $\tau_p: \Delta L_p \to F_p$. Here $E_p: C^P \to C$ denotes the evaluation functor at $p$ carrying $G$ to $G(p)$. Then there exists a unique functor $L: P \to C$ such that $L(p) = L_p$ and $p \mapsto \tau_p$ gives a natural transformation $\tau: \Delta L \to F$. Moreover, this $\tau$ exhibits $L$ as a limit of $F$.

Proof. This follows easily from the universal properties of limits. We leave the details to the reader.

Corollary 1.5.12. Let $C$, $P$, $I$ be categories. If $C$ admits limits indexed by $I$, then $C^P$ admits limits indexed by $I$ and for each object $p$ of $P$, the evaluation functor at $p$, $E_p: C^P \to C$, preserves limits indexed by $I$.

Example 1.5.13. Let $A$ be an additive category and let $P$ be a category. Then the functor category $A^P$ is an additive category. For $X, Y: P \to A$, $\Hom_{A^P}(X, Y)$ is a subgroup of $\prod_{p \in \Ob(P)} \Hom_A(X_p, Y_p)$.

Additive functors

Proposition 1.5.14. Let $F: A \to B$ be a functor between additive categories. Then the following conditions are equivalent:

1. $F$ preserves products of pairs of objects;
2. $F$ preserves coproducts of pairs of objects;
3. for every pair of objects $A, A'$ of $A$, the map $\Hom_A(A, A') \to \Hom_B(FA, FA')$ induced by $F$ is a homomorphism.
The proposition holds more generally for functors between categories satisfying Proposition 1.5.4 (admitting zero objects, finite products, finite coproducts, and satisfying (*)).

**Proof.** (1) \(\Rightarrow\) (2). By the proof of Proposition 1.5.1, it suffices to show that \(F(\text{id}, 0) = (\text{id}, 0)\) and \(F(0, \text{id}) = (0, \text{id})\). For this, it suffices to show that \(F\) carries zero morphisms to zero morphisms, or that \(F\) carries zero objects to zero objects. Note that any zero object 0 is the product of 0 and 0, with \(\text{id}_0\) as the projections. By (1), \(F(0)\) is the product of \(F(0)\) and \(F(0)\), with \(\text{id}_{F(0)}\) as the projections, so that \(F(0)\) is a zero object by Lemma 1.5.15 below.

By duality, we have (2) \(\Rightarrow\) (1).

(1) \(\Rightarrow\) (3). Let \(f, g: A \rightarrow B\). Then \(f + g = (\text{id}, \text{id})(f, g)\), so that \(F(f + g) = \langle \text{id}, \text{id}\rangle(Ff, Fg) = Ff + Fg\).

(3) \(\Rightarrow\) (1). We must show that \((Fp_1, Fp_2): F(A_1 \times A_2) \rightarrow FA_1 \times FA_2\) is an isomorphism, where \(p_j: A_1 \times A_2 \rightarrow A_j\) is the projection, \(j = 1, 2\). It is easy to check that \(F(i_1)q_1 + F(i_2)q_2\) is an inverse to \((Fp_1, Fp_2)\), where \(q_j: FA_1 \times FA_2 \rightarrow FA_j\) is the projection and \(i_j: A_j \rightarrow A_1 \times A_2\) is the canonical morphism. \(\square\)

**Lemma 1.5.15.** Let \(A\) and \(B\) be objects of a category \(C\) with a zero object. If \(A \times B\) exists, with the projection \(p: A \times B \rightarrow A\) being an isomorphism, then \(B\) is a zero object.

**Proof.** Let \(q: A \times B \rightarrow B\) be the projection. Consider the morphisms \(0: B \rightarrow A\) and \(\text{id}_B: B \rightarrow B\). There exists a unique morphism \(f = (0, \text{id}_B): B \rightarrow A \times B\) such that \(pf = 0, qf = \text{id}_B\). It follows that \(f = 0, \text{id}_B = 0\), so that \(B\) is a zero object. \(\square\)

**Remark 1.5.16.** Even in an additive category, \(A \times B \simeq A\) does not imply \(B \simeq 0\) in general. For example, if \(A = \mathbb{Z}^{(S)}\) is the free abelian group with an infinite basis \(S\), then any bijection \(S \coprod S \simeq S\) induces an isomorphism \(A \times A \simeq A\) in the category \(\text{Ab}\).

**Definition 1.5.17.** We say that a functor \(F: \mathcal{A} \rightarrow \mathcal{B}\) between additive categories is **additive** if it satisfies the conditions of the above proposition. We say that a subcategory \(\mathcal{A}\) of an additive category \(\mathcal{B}\) is an **additive subcategory** if \(\mathcal{A}\) is additive and the inclusion functor is additive.

If \(F\) is a functor between additive categories admitting a left or right adjoint, then \(F\) is additive. A composition of additive functors is additive. The term “additive subcategory” needs to be used with caution, as a subcategory of an additive category can be an additive category without being an additive subcategory.

**Remark 1.5.18.** Let \(\mathcal{B}\) be an abelian category. A full subcategory \(\mathcal{A}\) of \(\mathcal{B}\) is an additive subcategory if and only if \(\mathcal{A}\) admits a zero object 0 of \(\mathcal{B}\) and for \(A\) and \(A'\) in \(\mathcal{A}\), the direct sum \(A \oplus A'\) in \(\mathcal{B}\) is isomorphic to an object of \(\mathcal{A}\).

**Example 1.5.19.** Let \(\mathcal{A}\) be an additive category and let \(F: P \rightarrow Q\) be a functor. Then the functor \(\mathcal{A}^P \rightarrow \mathcal{A}^P\) induced by \(F\) is additive.

**Example 1.5.20.** Let \(\mathcal{A}\) be an additive category. The functor \(- \times -: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}\) is additive. It follows that the functor \(\mathcal{A} \rightarrow \mathcal{A}\) given by \(A \mapsto A \times A\) is additive. Let \(B\) be an object of \(\mathcal{A}\). The functor \(- \times B: \mathcal{A} \rightarrow \mathcal{A}\) is not additive unless \(B = 0\).
Example 1.5.21. Let $R$, $S$, $T$ be rings. The functor $- \otimes_S - : (R,S)\text{-Mod} \times (S,T)\text{-Mod} \rightarrow (R,T)\text{-Mod}$ not additive in general. By contrast, it is additive in each variable. That is, given $R M_S$ and $S N_T$, $M \otimes_S : (S,T)\text{-Mod} \rightarrow (R,T)\text{-Mod}$ and $- \otimes_S N : (R,S)\text{-Mod} \rightarrow (R,T)\text{-Mod}$ are additive functors. The functor $(S,S)\text{-Mod} \rightarrow (S,S)\text{-Mod}$, $A \mapsto A \otimes_S A$ is not additive unless $S = 0$ (assuming $S$ small).

Example 1.5.22. Let $A$ be an additive category with small Hom sets. The functor $\text{Hom}_{\text{A}}: \text{A}^{\text{op}} \times \text{A} \rightarrow \text{Ab}$ is additive in each variable.

1.6 Abelian categories

Kernels and cokernels

In any category, the equalizer $e: E \rightarrow X$ of a pair of morphisms $(f,g): X \rightrightarrows Y$, whenever it exists, is always a monomorphism. Indeed, if $(a,b): A \rightrightarrows E$ are morphisms such that $ea = eb$, then $a = b$ by the universal property of the equalizer. Dually, a cokernel, whenever it exists, is an epimorphism.

Lemma 1.6.1.

1. In a category with a zero object, every monomorphism $f: X \rightarrow Y$ has zero kernel, and every epimorphism has zero cokernel.

2. In an additive category, every morphism of zero kernel $f: X \rightarrow Y$ is a monomorphism and every morphism of zero cokernel is an epimorphism.

Proof. We prove the assertions on monomorphisms and those on epimorphisms follow by duality.

1. We check that $0 \rightarrow X$ satisfies the universal property of a kernel of $f$. We have $f0 = 0$. Let $g: Z \rightarrow X$ be a morphism such that $fg = 0$. Since $g$ is a monomorphism, we have $g = 0$.

2. Let $(g,h): Z \rightarrow X$ be morphisms such that $fg = fh$. Then $f(g - h) = 0$. It follows that $g - h = 0$, so that $g = h$.

In an additive category, the equalizer of $(f,g)$ is the kernel of $f - g$ and the coequalizer of $(f,g)$ is the cokernel of $f - g$.

Remark 1.6.2. Let $F: I \rightarrow C$ be a functor. Assume that the products $A = \prod_{i \in \text{Obj}(I)} F(i)$ and $B = \prod_{f: i \rightarrow j} F(j)$ (f running through morphisms of $I$) exist. Then $\lim F$ can be identified with $\text{Eq}(a,b)$, where $a, b: A \rightarrow B$ are such that $a f: A \rightarrow F(j)$ is the projection and $b f: A \rightarrow F(i) \xrightarrow{F(f)} F(j)$ is the composition of the projection with $F(f)$.

It follows that a category $C$ admits small (resp. finite) limits if and only if $C$ admits equalizers and small (resp. finite) products. Dually, $C$ admits small (resp. finite) colimits if and only if $C$ admits coequalizers and small (resp. finite) coproducts. Similar statements hold for preservation of limits and colimits.

An additive category admitting kernels and cokernels admits all finite products and finite coproducts.
Example 1.6.3. Let $R$ be a ring. The additive category $R$-$\text{Mod}$ admits finite kernels and cokernels. Indeed, for a morphism $f: A \to B$, $\text{Ker}(f) = f^{-1}(0)$ and $\text{Coker}(f) = B/\text{Im}(f)$, where $\text{Im}(f)$ denotes the image of $f$.

### Abelian categories

**Definition 1.6.4.** Let $\mathcal{A}$ be an additive category admitting kernels and cokernels and let $f: A \to B$ be a morphism. We define the *coimage* and *image* of $f$ to be $\text{Coim}(f) = \text{Coker}(g)$, $\text{Im}(f) = \text{Ker}(h)$, where $g: \text{Ker}(f) \to A$ and $h: B \to \text{Coker}(f)$ are the canonical morphisms.

In the above situation, every morphism $f: A \to B$ factors uniquely into $A \twoheadrightarrow \text{Coim}(f) \to \text{Im}(f) \hookrightarrow B$.

**Definition 1.6.5.** An *abelian category* is an additive category $\mathcal{A}$ satisfying the following axioms:

(AB1) $\mathcal{A}$ admits kernels and cokernels.

(AB2) For each morphism $f: A \to B$, the morphism $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism.

The axioms were introduced in Grothendieck’s seminal 1957 Tôhoku paper [G]. The notion was introduced independently in Buchsbaum’s 1954 thesis (under the name of “exact category”\(^5\)) (see also [B] and [CE, Appendix]).

**Example 1.6.6.** By the first isomorphism theorem, $R$-$\text{Mod}$ is an abelian category for every ring $R$. The full subcategory of $R$-$\text{Mod}$ consisting of Noetherian (resp. Artinian) $R$-modules is stable under subobjects and quotients, hence is an abelian category. The category $R$-$\text{mod}$ of finitely-generated $R$-modules is an abelian category if and only if $R$ is left Noetherian. Indeed, if $I$ is a left ideal that is not finitely generated, then the morphism $A \to A/I$ has no kernel in $R$-$\text{mod}$.

**Example 1.6.7.** Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}^{\text{op}}$ is an abelian category.

**Example 1.6.8.** Let $\mathcal{A}$ be an abelian category and let $P$ be a category. Then the functor category $\mathcal{A}^P$ is an abelian category.

**Example 1.6.9.** The category of topological abelian groups is an additive category admitting kernels and cokernels, but does not satisfy (AB2). For example, let $f$ be the map $\mathbb{R}_{\text{disc}} \to \mathbb{R}$ carrying $x$ to $x$. Then $\text{Coim}(f) = \mathbb{R}_{\text{disc}}$ and $\text{Im}(f) = \mathbb{R}$.

**Remark 1.6.10.** The following properties follow from (AB2) and Lemma [1.6.1]:

1. If a morphism is both a monomorphism and an epimorphism, then it is an isomorphism.
2. Every monomorphism is the kernel of its cokernel.
3. Every epimorphism is the cokernel of its kernel.

---

\(^5\)This terminology is no longer in use. In modern usage, *exact category* refers to a more general notion introduced by Quillen.
(4) Every morphism \( f: A \to B \) can be decomposed into
\[ A \xrightarrow{g} \text{Im}(f) \xrightarrow{h} B, \]
where \( g \) is an epimorphism and \( h \) is a monomorphism.

Let \( A \xrightarrow{f} B \xrightarrow{g} C \) be a sequence such that \( gf = 0 \). Then the sequence decomposes uniquely into
\[ A \twoheadrightarrow \text{Coim}(f) \hookrightarrow \text{Ker}(g) \hookrightarrow B \twoheadrightarrow \text{Coker}(f) \hookrightarrow \text{Im}(g) \hookrightarrow C. \]

**Definition 1.6.11.** We say that a sequence \( A \xrightarrow{f} B \xrightarrow{g} C \) satisfying \( gf = 0 \) in an abelian category is exact at \( B \) if the morphism \( \text{Coim}(f) \to \text{Ker}(g) \) is an isomorphism (or equivalently, \( \text{Coker}(f) \to \text{Im}(g) \) is an isomorphism). We say that a sequence \( A^0 \to A^1 \to \cdots \to A^n \) is exact if it is exact at each \( A^i \), \( 1 \leq i \leq n-1 \).

**Example 1.6.12.**
(1) A sequence \( 0 \to A \to 0 \) is exact if and only if \( A \) is a zero object.
(2) A sequence \( 0 \to A \xrightarrow{f} B \) is exact if and only if \( f \) is a monomorphism. Dually, a sequence \( A \xrightarrow{f} B \to 0 \) is exact if and only if \( f \) is an epimorphism.
(3) A sequence \( 0 \to A \xrightarrow{f} B \to 0 \) is exact if and only if \( f \) is an isomorphism.
(4) A sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \) is exact if and only if \( f \) is the kernel of \( g \).
Dually, a sequence \( A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is exact if and only if \( g \) is the cokernel of \( f \).
(5) A sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is an exact sequence if and only if \( g \) is the cokernel of \( f \) and \( f \) is the kernel of \( g \). Such a sequence is called a short exact sequence.

**Yoneda embedding for additive categories**

Let \( \mathcal{A} \) and \( \mathcal{B} \) be additive categories. We let \( \text{Fun}^{\text{add}}(\mathcal{A}, \mathcal{B}) \subseteq \text{Fun}(\mathcal{A}, \mathcal{B}) \) denote the full subcategory spanned by additive functors. Note that \( \text{Fun}^{\text{add}}(\mathcal{A}, \mathcal{B}) \) is an abelian category if \( \mathcal{B} \) is an abelian category.

**Lemma 1.6.13.** The forgetful functor induces a fully faithful functor \( \text{Fun}^{\text{add}}(\mathcal{A}, \text{Ab}) \to \text{Fun}(\mathcal{A}, \text{Set}) \).

**Proof.** The functor is faithful because the forgetful functor \( U: \text{Ab} \to \text{Set} \) is faithful. It remains to show that for additive functors \( F \) and \( F' \), every natural transformation \( UF \to UF' \) lifts to a natural transformation \( F \to F' \). This follows from the fact that the group structure of \( FX \) is induced by the map \( \langle \text{id}_{FX}, \text{id}_{FX} \rangle: FX \times FX \to FX \), which is the composite
\[ FX \times FX \simeq F(X \times X) \xrightarrow{F(\text{id}_X, \text{id}_X)} FX. \]

\[ \square \]

**Remark 1.6.14.** Let \( \mathcal{A} \) be an additive category with small Hom sets. Then the Yoneda embedding can be lifted to an additive functor \( \mathcal{A} \to \text{Fun}^{\text{add}}(\mathcal{A}^{\text{op}}, \text{Ab}) \) carrying \( X \) to \( \text{Hom}_\mathcal{A}(-, X) \), which is fully faithful by the above lemma.
Exact functors

Definition 1.6.15. Let $\mathcal{C}$ be a category admitting finite limits (resp. finite colimits). We say that a functor $F: \mathcal{C} \to \mathcal{D}$ is left exact (resp. right exact) if it preserves finite limits (resp. finite colimits). For $\mathcal{C}$ admitting finite limits and finite colimits, we say that $F$ is exact if it is both left exact and right exact.

A left exact functor between abelian categories is additive. The same holds for right exact functor.

Proposition 1.6.16. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:

1. $F$ is left exact.
2. $F$ preserves kernels (or equivalently, for every exact sequence $0 \to X \to Y \to Z$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$).
3. For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(2) $\Rightarrow$ (1). This follows from Remark 1.6.2 and the assumption that $F$ preserves finite products.

(3) $\Rightarrow$ (2). We decompose the sequence into a short exact sequence $0 \to X \to Y \to Z' \to 0$ and a monomorphism $Z' \to Z$. By (3), $0 \to FX \to FY \to FZ'$ is exact and $FZ' \to FZ$ is a monomorphism. It follows that $0 \to FX \to FY \to FZ$ is exact.

Corollary 1.6.17. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:

1. $F$ is exact.
2. For every exact sequence $X \to Y \to Z$ in $\mathcal{A}$, $FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$.
3. For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ \to 0$ is a short exact sequence in $\mathcal{B}$.
4. $F$ is left exact and preserves epimorphisms.
5. $F$ is right exact and preserves monomorphisms.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). This follows from the proposition.

(1) $\Rightarrow$ (4) $\Rightarrow$ (3). Obvious.

Example 1.6.18. Let $\mathcal{A}$ be an abelian category and let $F: P \to Q$ be a functor. Then the functor $\mathcal{A}^Q \to \mathcal{A}^P$ induced by $F$ is exact.

Example 1.6.19. Let $\mathcal{A}$ be an abelian category with small Hom sets. Then the functor $\text{Hom}_\mathcal{A}: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Ab}$ is left exact in each variable.

Example 1.6.20. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. If a functor $F: \mathcal{A} \to \mathcal{B}$ admits a left (resp. right) adjoint, then $F$ is left (resp. right) exact.
1.6. ABELIAN CATEGORIES

(1) If an abelian category \( \mathcal{A} \) admits limits (resp. colimits) indexed by \( I \), then 
\[
\lim_{I} : \mathcal{A}^I \to \mathcal{A} \quad \text{(resp. colim: } \mathcal{A}^I \to \mathcal{A})
\]
is left (resp. right) exact. In particular, for \( I = (\bullet \to \bullet) \), \( \text{Ker: } \mathcal{A}^I \to \mathcal{A} \) is left exact and \( \text{Coker: } \mathcal{A}^I \to \mathcal{A} \) is right exact. 
For \( I \) finite and discrete, the product functor \( \mathcal{A}^I \to \mathcal{A} \) is exact.

(2) Let \( R, S, T \) be rings and consider small bimodules \( R M_S, S N_T \). Then \( - \otimes S N \) and \( M \otimes_S - \) are right exact.

**Theorem 1.6.21** (Freyd–Mitchell). Let \( \mathcal{A} \) be a small abelian category. Then there exists a small ring and a fully faithful exact functor \( F : \mathcal{A} \to R\text{-Mod} \).

We refer the reader to [KS2, Theorem 9.6.10] for a proof of the theorem, which is beyond the scope of these lectures. Let us briefly indicate some ingredients used in the proof. Applying the Yoneda embedding to \( \mathcal{A}^{\text{op}} \), we get a fully faithful functor
\[
i : \mathcal{A} \to \text{Pro}(\mathcal{A})
\]
carrying \( \text{Fun}(\mathcal{A}, \mathcal{B}) \) into \( \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}) \). One shows that \( \text{Pro}(\mathcal{A}) \) is an abelian category and \( i \) is an exact functor. We take \( R = \text{End}_{\mathcal{A}}(G)^{\text{op}} \) for a suitable projective (see the next section) object \( G \) of \( \text{Pro}(\mathcal{A}) \) and we take \( F \) to be the composite of \( i \) and the exact functor \( \text{Pro}(\mathcal{A}) \to R\text{-Mod} \) carrying \( H \) to \( \text{Hom}_{\text{Pro}(\mathcal{A})}(G, H) \).

**Sheaves**

**Definition 1.6.22.** A nonempty category \( I \) is said to be **filtered** if

(1) For objects \( i, j \) in \( I \), there exist morphisms \( i \to k, j \to k \) in \( I \); and

(2) For morphisms \( u, v : i \Rightarrow j \) in \( I \), there exists a morphism \( w : j \to k \) such that \( wu = vw \).

**Remark 1.6.23.** Recall from Example 1.3.22 and small colimits exist in the category \( \text{Set} \). Filtered colimits can be described more explicitly. Let \( I \) be a filtered category and let \( F : I \to \text{Set} \) be a functor. Then colim \( F \) is represented by
\[
Q = \prod_{i \in \text{Ob}(I)} F(i)/\sim
\]
whenever \( Q \) is small. Here for \( x \in F(i) \) and \( y \in F(j) \), \( x \sim y \) if and only if there exist morphisms \( u : i \to k \) and \( v : j \to k \) in \( I \) such that \( F(u)x = F(v)y \).

Filtered colimits in \( \text{Grp} \), \( R\text{-Mod} \), \( \text{Ring} \) admit similar descriptions (with disjoint union). For example, in the case of \( R\text{-Mod} \), for \( x \in F(i) \) and \( y \in F(j) \), \( [x] + [y] \) is defined to be \( F(u)x + F(v)y \), where \( u : i \to k \) and \( v : j \to k \). The forgetful functors \( \text{Ring} \to \text{Ab}, R\text{-Mod} \to \text{Grp}, \text{Grp} \to \text{Set} \) commute with small filtered colimits.

**Proposition 1.6.24.** Small filtered colimits in \( R\text{-Mod} \) are exact. In other words, for any small filtered category \( I \), the functor colim: \( R\text{-Mod}^I \to R\text{-Mod} \) is exact.

**Proof.** Since colim is a left adjoint functor, it is right exact. It remains to check that colim preserves monomorphisms. Let \( f : F \to G \) be a monomorphism in \( R\text{-Mod}^I \).

---

6More generally, for any category \( \mathcal{C} \), the category \( \text{Pro}(\mathcal{C}) \) of pro-objects of \( \mathcal{C} \) is the full subcategory of \( \text{Fun}(\mathcal{C}, \text{Set})^{\text{op}} \) spanned by small filtered limits of the image of the Yoneda embedding \( \mathcal{C} \to \text{Fun}(\mathcal{C}, \text{Set})^{\text{op}} \).
Let \([x] \in \ker(\colim f)\) be the equivalence class of \(x \in F(i)\). Then \([f_i(x)] = 0\), so that there exists \(u: i \to j\) such that \(f_jF(u)(x) = G(u)f_i(x) = 0\). Since \(f_j\) is a monomorphism, we have \(F(u)(x) = 0\), so that \([x] = 0\).

One can also deduce the above from the following.

**Proposition 1.6.25.** Small filtered colimits in \(\mathsf{Set}\) are exact. In other words, for any small filtered category \(I\), any finite category \(J\), and any functor \(F: I \times J \to \mathsf{Set}\), the map

\[
\colim_{i \in I} \lim_{j \in J} F(i, j) \to \lim_{j \in J} \colim_{i \in I} F(i, j)
\]

is a bijection.

It follows from the proposition that the same holds for \(\mathsf{Grp}\) and \(\mathsf{Ring}\).

**Proof.** Since any finite limit is an equalizer of finite products (Remark 1.6.2), it suffices to show that \(\colim I\) preserves equalizers and finite products.

Let \((f, g): G \Rightarrow H\) be morphisms in \(\mathsf{Set}^I\). We show that the map

\[
\phi: \colim \text{Eq}(f(i), g(i)) \to \text{Eq}(\colim f, \colim g)
\]

is bijective. Let \([x]\) and \([y]\) be elements of the left-hand side, \(x \in \text{Eq}(f(i), g(i))\), \(y \in \text{Eq}(f(j), g(j))\) such that \(\phi([x]) = \phi([y])\). Then there exist \(u: i \to k\), \(v: j \to k\) such that \(G(u)(x) = G(v)(y)\). Thus \([x] = [y]\) in \(\colim_{i \in I} \text{Eq}(f(i), g(i))\). This proves that \(\phi\) is injective (in fact this is equivalent to the fact that \(\colim I\) preserves monomorphisms, which can be proved similarly to Proposition 1.6.24).

Consider an element \([x]\) of the right-hand side, equivalence class of an element \(x\) of \(G(i)\) such that \(f(i)(x) \sim g(i)(x)\). In other words, there exists \(u, v: i \Rightarrow j\), such that \(H(u)(f(i)(x)) = H(v)(g(i)(x))\). Since \(I\) is filtered, there exists \(w: j \to k\) such that \(wu = wv\). Since

\[
f(i)G(wu)(x) = H(wu)f(i)(x) = H(wv)g(j)(x) = g(k)H(wv)(x),
\]

we have \(G(wu)(x) \in \text{Eq}(f(k), g(k))\). Then \(\phi([G(wu)(x)]) = [x]\). This proves that \(\phi\) is surjective.

Similarly, one proves that \(\colim I\) preserves finite products.

Let \(X\) be a small topological space. Let \(\text{Open}(X)\) be the set of open subsets of \(X\), ordered by inclusion.

**Definition 1.6.26.** A **sheaf of abelian groups** on \(X\) is a functor \(\mathcal{F}: \text{Open}(X)^{\text{op}} \to \mathsf{Ab}\) satisfying the following gluing condition: for every open covering \((U_i)_{i \in I}\) of an open subset \(U\), the restriction maps \(\mathcal{F}(U) \to \mathcal{F}(U_i)\) induce a bijection from \(\mathcal{F}(U)\) to the equalizer of the maps

\[
\prod_{i \in I} \mathcal{F}(U_i) \cong \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j),
\]

induced by the restriction maps \(\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)\) and \(\mathcal{F}(U_j) \to \mathcal{F}(U_i \cap U_j)\). The category \(\text{Fun}(\text{Open}(X)^{\text{op}}, \mathsf{Ab})\) is called the category of **presheaves**. The category \(\text{Shv}(X)\) of sheaves of abelian groups on \(X\) is the full subcategory spanned by sheaves of abelian groups on \(X\).
The category \( \text{Shv}(X) \) is stable under small limits in \( \text{Fun}(\text{Open}(X)^{\text{op}}, \text{Ab}) \). In particular, \( \text{Shv}(X) \) admits small limits. The inclusion functor \( \iota \) admits a left adjoint \( a \), called the sheafification functor, with \( (a\mathcal{F})(U) \) given by the colimit indexed by \( \text{Cov}(U) \) of the equalizer of (1.6.1). Here \( \text{Cov}(U) \) denotes the set of open coverings of \( U \), with \( U \leq U' \) if \( U' \) refines \( U \). Since \( \text{Cov}(U) \) is filtered, \( a \) is exact. The category \( \text{Shv}(X) \) also admits small colimits, given by \( \text{colim} F_i = a \text{colim} \iota F_i \). It is clear that \( \text{Shv}(X) \) is an abelian category (the exactness of \( a \) is used in (AB2)).

**Example 1.6.27.** Let \( X \) be a complex manifold. We have an epimorphism \( \exp: \mathcal{O} \to \mathcal{O}^\times \) of sheaves on \( X \), where \( \mathcal{O}(U) \) is the additive group of holomorphic functions on \( U \) and \( \mathcal{O}^\times \) is the multiplicative group of nowhere-vanishing holomorphic functions on \( U \). As a morphism of presheaves, \( \exp \) is not an epimorphism in general (for example if \( X = \mathbb{C} - \{0\} \), then the function \( z \mapsto z \) is not in the image of \( \exp \)). However, for any \( f \in \mathcal{O}^\times(U) \), and any point \( x \in U \), there exists an open neighborhood \( x \in V \subseteq U \) such that \( f|_V \) is in the image of \( \exp \).

**Diagram lemmas**

Let \( \mathcal{A} \) be an abelian category.

**Proposition 1.6.28** (Snake lemma). Consider a commutative diagram in \( \mathcal{A} \) with exact rows

\[
\begin{array}{c}
X' \rightarrow X \rightarrow X'' \rightarrow 0 \\
\uparrow u' \downarrow u \downarrow u'' \\
0 \rightarrow Y' \rightarrow Y \rightarrow Y''.
\end{array}
\]

There exists a unique morphism \( v: X \times_{X''} \text{Ker}(u'') \to Y' \) making the diagram

\[
\begin{array}{c}
X \times_{X''} \text{Ker}(u'') \xrightarrow{p_1} X \\
\downarrow v \downarrow u \\
Y' \xrightarrow{\delta} Y
\end{array}
\]

commute and a unique morphism \( \delta: \text{Ker}(u'') \to \text{Coker}(u') \) making the diagram

\[
\begin{array}{c}
X \times_{X''} \text{Ker}(u'') \xrightarrow{p_2} \text{Ker}(u'') \\
\downarrow v \downarrow \delta \\
Y' \xrightarrow{} \text{Coker}(u')
\end{array}
\]

commute. Here \( p_1 \) and \( p_2 \) are the projections. Moreover, the sequence

\[
\text{Ker}(u') \to \text{Ker}(u) \to \text{Ker}(u'') \xrightarrow{\delta} \text{Coker}(u') \to \text{Coker}(u) \to \text{Coker}(u'').
\]

is exact.
Proof. By the Freyd–Mitchell theorem, we may work in a module category and take elements. Let \( a \in \text{Ker}(u'') \) and let \( b \in X \) be a preimage. Then the image of \( u(b) \) in \( Y'' \) is \( u''(a) = 0 \), so that \( u(b) \) is the image of \( c \in Y' \). We define \( \delta(a) \) to be the class of \( c \) in \( \text{Coker}(u') \), as shown by the diagram:

\[
\begin{array}{ccc}
    & & a \\
   & \downarrow & \\
   b & \rightarrow & a \\
   & \downarrow & \\
   c & \rightarrow & u(b) \\
   & \downarrow & \\
   & \delta(a) & \rightarrow & 0
\end{array}
\]

It is easy to check that the assertions of the proposition.

For a direct (but tedious) proof of the snake lemma without diagram-chasing, we refer to [KS2, Lemma 12.1.1]. It is also possible to give a proof by diagram-chasing yet not relying on the Freyd–Mitchell theorem, by introducing a notion of members as substitutes for elements [ML2, Section VIII.4].

Remark 1.6.29. If the upper row of (1.6.2) is a short exact sequence, then the sequence (1.6.3) extends to an exact sequence \( 0 \rightarrow \text{Ker}(u') \rightarrow \text{Ker}(u) \rightarrow 0 \). Dually, if the lower row of (1.6.2) is a short exact sequence, then the sequence (1.6.3) extends to an exact sequence \( \text{Coker}(u) \rightarrow \text{Coker}(u'') \rightarrow 0 \).

Corollary 1.6.30. Under the assumptions of Proposition 1.6.28,

1. if \( u' \) and \( u'' \) are monomorphisms, then \( u \) is a monomorphism;
2. if \( u' \) and \( u'' \) are epimorphisms, then \( u \) is an epimorphism;
3. if \( u' \) and \( u'' \) are isomorphisms, then \( u \) is an isomorphism;
4. if \( u' \) is an epimorphism and \( u \) is a monomorphism, then \( u'' \) is a monomorphism.
5. if \( u'' \) is a monomorphism and \( u \) is an epimorphism, then \( u' \) is an epimorphism.

Remark 1.6.31. In case (3) of the corollary, the two rows of (1.6.2) are short exact sequences. Note that given short exact sequences \( 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \), \( 0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0 \), with \( X' \cong Y' \) and \( X'' \cong Y'' \), it is in general not true that \( X \cong Y \). The existence of a morphism \( X \rightarrow Y \) compatible with the isomorphisms \( X' \cong Y' \) and \( X'' \cong Y'' \) is crucial to the conclusion of case (3).

Corollary 1.6.32. Let \( 0 \rightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \rightarrow 0 \) be a short exact sequence in \( \mathcal{A} \). Then the following conditions are equivalent:

1. \( f \) admits a retraction: there exists \( r : X \rightarrow X' \) such that \( rf = \text{id}_{X'} \).
2. \( g \) admits a section: there exists \( s : X'' \rightarrow X \) such that \( gs = \text{id}_{X''} \).
3. The sequence is isomorphic (as an object of \( \mathcal{A}^{\cdots \rightarrow \cdots} \)) to the short exact sequence \( 0 \rightarrow X' \xrightarrow{i} X' \times X'' \xrightarrow{\pi} X'' \rightarrow 0 \), where \( i \) and \( \pi \) are the canonical morphisms.
Proof. It is clear that (3) implies (1). Assuming (1), we obtain a commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & X' \\
\downarrow & (r,g) & \downarrow \\
0 & \rightarrow & X' \times X'' \\
\end{array}
\]
By Corollary 1.6.30, \((r,g)\) is an isomorphism.

Definition 1.6.33. A short exact sequence satisfying the above equivalent conditions is said to be **split**.

Remark 1.6.34. Any additive functor preserves split short exact sequences.

Corollary 1.6.35. Consider a commutative diagram in \(\mathcal{A}\)
\[
\begin{array}{ccc}
X^0 & \rightarrow & X^1 \\
\downarrow & u^0 & \downarrow \\
Y^0 & \rightarrow & Y^1 \\
\end{array}
\]
with exact rows.
(1) If \(u^0\) is an epimorphism, \(u^1\) and \(u^3\) are monomorphisms, then \(u^2\) is a monomorphism.
(2) If \(u^3\) is a monomorphism, \(u^0\) and \(u^2\) are epimorphisms, then \(u^1\) is an epimorphism.

Proof. Indeed, it suffices to apply Corollary 1.6.30 to the diagrams of short exact sequences associated to the long exact sequences.

Corollary 1.6.36 (Five lemma). Consider a commutative diagram in \(\mathcal{A}\)
\[
\begin{array}{ccc}
X^0 & \rightarrow & X^1 \\
\downarrow & u^0 & \downarrow \\
Y^0 & \rightarrow & Y^1 \\
\end{array}
\]
with exact rows. If \(u^0\) is an epimorphism, \(u^4\) is a monomorphism, and \(u^1, u^3\) are isomorphisms, then \(u^2\) is an isomorphism.

Corollary 1.6.37 (Nine lemma). Consider a commutative diagram in \(\mathcal{A}\)
\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & X' & X \\
\downarrow & \downarrow & \downarrow \\
0 & Y' & Y \\
\downarrow & \downarrow & \downarrow \\
0 & Z' & Z \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]
with exact columns. If the top two rows or the bottom two rows are exacts, then all the three rows are exact.

The diagram in the nine lemma (commutative with exact rows and columns) is called a nine-diagram.

**Proof.** It suffices to apply the snake lemma to the top two rows in the first case, and the bottom two rows in the second case. \(\square\)

**Remark 1.6.38.** If the top and bottom rows are exact, then the middle row is not exact in general. (The claim of [ML2, Section VIII.4, Exercise 5 (c)] is mistaken.) Indeed, if \(f: Y \to X\) is a nonzero morphism, then

\begin{align*}
0 & \to 0 \to X \to id_X \to 0 \\
0 & \to Y \to X \times Y \to p_X \to 0 \\
0 & \to Y \to Y \to 0 \to 0
\end{align*}

provides a counterexample. By contrast, if, moreover, the composition of \(Y' \to Y \to Y''\) is zero, then the middle row is exact.

### 1.7 Projective and injective objects

**Definition 1.7.1.** Let \(\mathcal{C}\) be a category. An object \(P\) of \(\mathcal{C}\) is said to be **projective** if given a morphism \(f\) and an epimorphism \(u\) in \(\mathcal{C}\) as shown in the diagram

\begin{equation}
X \xrightarrow{u} Y, \quad P \xrightarrow{f} \end{equation}

there exists \(g\) rendering the diagram commutative. Dually, an object \(I\) of \(\mathcal{C}\) is said to be **injective** if given a morphism \(f\) and a monomorphism \(u\) in \(\mathcal{C}\) as shown in the diagram

\begin{equation}
X \xrightarrow{u} Y, \quad I \xrightarrow{g} \end{equation}

there exists \(g\) rendering the diagram commutative.

**Remark 1.7.2.** An object \(I\) is injective in \(\mathcal{C}\) if and only if it is projective in \(\mathcal{C}^{\text{op}}\).
Note that we do not require uniqueness of the dotted arrow. By definition, an object \( P \) of \( C \) is projective if and only if for every epimorphism \( u: X \to Y \) in \( C \), the induced map \( \text{Hom}_C(P, X) \to \text{Hom}_C(P, Y) \) is a surjection. Dually, an object \( I \) of \( C \) is injective if and only if for every monomorphism \( u: X \to Y \) in \( C \), the induced map \( \text{Hom}_C(Y, I) \to \text{Hom}_C(X, I) \) is a surjection. We obtain the following.

**Proposition 1.7.3.** Let \( A \) be an abelian category with small \( \text{Hom} \) sets. An object \( P \) is projective if and only if the functor \( \text{Hom}_A(P, -): A \to \text{Ab} \) is exact. An object \( I \) is injective if and only if the functor \( \text{Hom}_A(-, I): A^{\text{op}} \to \text{Ab} \) is exact.

**Proposition 1.7.4.** Let \( A \) be an abelian category. An object \( P \) is projective if and only if every epimorphism \( f: M \to P \) admits a section. An object \( I \) is injective if and only if every monomorphism \( g: I \to M \) admits a retraction.

The “only if” parts hold in any category.

**Proof.** By duality, it suffices to prove the first assertion. If \( P \) is projective, applying the definition to the diagram

\[
\begin{array}{ccc}
P & \to & P \\
\downarrow & & \downarrow \\
M & \to & P,
\end{array}
\]

we obtain a section of \( f \). Conversely, given the diagram \([1.7.1]\), we form the pullback square

\[
\begin{array}{ccc}
M & \to & P \\
\downarrow & & \downarrow \\
X & \to & Y.
\end{array}
\]

Since \( v \) is an epimorphism (Exercise), it admits a section \( s: P \to M \) and we take \( g = ps \).

**Corollary 1.7.5.** Let \( A \) be an abelian category. The following conditions are equivalent:

1. Every object of \( A \) is projective.
2. Every object of \( A \) is injective.
3. Every short exact sequence in \( A \) is split.

Note that \( A \) satisfies the above conditions if and only if \( A^{\text{op}} \) does. Compare with Example \([1.7.10]\) below.

**Remark 1.7.6.** If \( A \) is a category satisfying the conditions of Corollary \([1.7.5]\) then any additive functor \( F: A \to B \) is exact.

**Example 1.7.7.** Recall that a ring \( R \) is called semisimple if it satisfies the following equivalent conditions:

1. The (left) \( R \)-module \( R \) is semisimple;
2. Every (left) \( R \)-module is semisimple;
3. (Artin–Wedderburn) \( R \) is isomorphic to a finite product of matrix rings \( M_n(D) \) over division rings \( D \).
Note that by Condition (3), $R$ is semisimple if and only if $R^{\text{op}}$ is semisimple. Recall that an $R$-module $M$ is called semisimple if every submodule is a direct summand. Thus, by Condition (2), that $R$ is a semisimple ring is further equivalent to the conditions of Corollary 1.7.5 for $\mathcal{A} = R\text{-Mod}$.

**Proposition 1.7.8.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. If $F$ admits a right adjoint $G$ that carries epimorphisms to epimorphisms, then $F$ carries projective objects to projective objects.

2. If $F$ admits a left adjoint $G$ that carries monomorphisms to monomorphisms, then $F$ carries injective objects to injective objects.

In particular, if $F$ is a functor between abelian categories admitting an exact right (resp. left) adjoint, then $F$ carries projective (resp. injective) objects to projective (resp. injective) objects.

**Proof.** By duality it suffices to prove (1). Let $P$ be a projective object of $\mathcal{C}$ and let $u : X \to Y$ be an epimorphism in $\mathcal{D}$. We have a commutative diagram

$$
\begin{array}{ccc}
\Hom_{\mathcal{C}}(FP, X) & \xrightarrow{u \circ -} & \Hom_{\mathcal{C}}(FP, Y) \\
\cong & & \cong \\
\Hom_{\mathcal{D}}(P, GX) & \xrightarrow{G u \circ -} & \Hom_{\mathcal{D}}(P, GY).
\end{array}
$$

By assumption, $Gu$ is an epimorphism, so that the lower row is a surjection. It follows that the upper row is also a surjection. 

**Remark 1.7.9.** Consider a family of objects $(X_i)_{i \in I}$ of a category $\mathcal{C}$ admitting a zero object. Assuming that the coproduct $\coprod_{i \in I} X_i$ exists, then the coproduct is projective if and only if each $X_i$ is projective. Dually, assuming that product $\prod_{i \in I} X_i$ exists, then the product is injective if and only if each $X_i$ is injective.

**Example 1.7.10.** In $\text{Set}$, every object is projective. Every nonempty set is injective. The empty set is not injective. ([HS, Proposition II.10.1] is inaccurate.)

**Example 1.7.11.** Let $R$ be a ring. The free module functor $F : \text{Set} \to R\text{-Mod}$ is a right adjoint to the forgetful functor $U : R\text{-Mod} \to \text{Set}$. Since $U$ carries epimorphisms to epimorphisms, $F$ carries projective objects to projective objects. Thus every free module is projective.

Note that every $R$-module is the quotient of a free $R$-module. Indeed, the adjunction map $FUM \to M$ is clearly surjective. Thus every $R$-module is the quotient of a projective $R$-module. We will see later that the dual of this statement also holds.

### 1.8 Projective and injective modules

Let $R$ be a ring. As before, we will concentrate of left $R$-modules. Right $R$-modules can be identified with left $R^{\text{op}}$-modules. This duality is not to be confused with the duality between a category and its opposite: The opposite of $R\text{-Mod}$ is not equivalent to a module category in general.\(^7\)

\(^7\)We have seen that in $R\text{-Mod}$ small filtered colimits are exact, but limits indexed by $I^{\text{op}}$ with $I$ small and filtered are not exact in general.
1.8. PROJECTIVE AND INJECTIVE MODULES

Projective modules

Proposition 1.8.1. Let $P$ be an $R$-module. The following conditions are equivalent:
(1) $P$ is projective.
(2) $P$ is a direct summand of some free $R$-module.

Proof. (1) $\Rightarrow$ (2). Let $F$ be a free $R$-module equipped with a surjective homomorphism $f: F \to P$. By Proposition 1.7.4, the short exact sequence $0 \to \ker(f) \to F \to P \to 0$ splits.

(2) $\Rightarrow$ (1). This follows from Remarks 1.7.9 and 1.7.11. $\square$

The same proof gives the following.

Proposition 1.8.2. Let $P$ be an $R$-module. The following conditions are equivalent:
(1) $P$ is finitely-generated and projective.
(2) $P$ is a direct summand of some $R^n$.

Example 1.8.3. Let $R = \mathbb{Z}/6$. The $R$-module $\mathbb{Z}/2$ is projective, but not free.

In these lectures a domain is a nonzero ring with no zero-divisors, not necessarily commutative.

Definition 1.8.4. (1) A ring $R$ is said to be left hereditary if every left ideal of $R$ is projective as an $R$-module.
(2) A PLID (principal left ideal domain) is a domain in which every left ideal is principal.

Similarly one defines right hereditary rings and PRIDs using right $R$-modules.

Remark 1.8.5. A principal left ideal of a domain is free. Thus a PLID is left hereditary. An ideal $I$ of a commutative domain is principal if and only if it is free.

Theorem 1.8.6 (Kaplansky). Let $R$ be a left hereditary ring. Then any submodule $P$ of a free $R$-module $F = \bigoplus_{\alpha \in I} Re_\alpha$ is isomorphic to a direct sum of left ideals of $R$; in particular, $P$ is a projective module.

Proof. Choose a well-order on $I$. For each $\alpha \in I$, let

$$F_{<\alpha} = \sum_{\beta < \alpha} Re_\beta, \quad F_{\leq \alpha} = \sum_{\beta \leq \alpha} Re_\beta.$$ 

Let $P_{<\alpha} = P \cap F_{<\alpha}$ and $P_{\leq \alpha} = P \cap F_{\leq \alpha}$. Consider the homomorphism

$$f_\alpha: P_{\leq \alpha} \subseteq F_{\leq \alpha} \to R$$

carrying $a + re_\alpha$ with $a \in F_{<\alpha}$ to $r$. We have $\ker(f_\alpha) = P_{<\alpha}$. Since $\text{Im}(f_\alpha)$ is a left ideal of $R$, it is projective, so that we have

$$P_{\leq \alpha} = P_{\leq \alpha} \oplus Q_\alpha,$$

where $Q_\alpha$ is an $R$-submodule of $P_{\leq \alpha}$ isomorphic to $\text{Im}(f_\alpha)$. 
Let us show $P = \bigoplus_{\alpha \in I} Q_\alpha$. Suppose we have $a_{\alpha_1} + \cdots + a_{\alpha_n} = 0$ with $a_{\alpha_i} \in Q_{\alpha_i}$ for $i = 1, \ldots, n$. We may assume that $\alpha_1 < \cdots < \alpha_n$. Then $a_{\alpha_1}, \ldots, a_{\alpha_{n-1}} \in P_{<\alpha_n}$, so that $a_{\alpha_n} \in P_{<\alpha_n} \cap Q_{\alpha_n} = 0$. By induction we have $a_{\alpha_i} = 0$ for all $i$.

It remains to show that $P = \sum_{\alpha \in I} Q_\alpha$. Assume the contrary. Since $P = \bigcup_{\alpha \in I} P_{\leq \alpha}$, there exists a smallest $\beta \in I$ and $a \in P_{\leq \beta}$ such that $a \notin Q = \sum_{\alpha \in I} Q_\alpha$. Write $a = b + c$, where $b \in P_{< \beta}$ and $c \in Q_\beta$. We have $b \in P_{< \gamma}$ for some $\gamma < \beta$. By the minimality of $\beta$, we have $b \in Q$. Then $a = b + c \in Q$, contradiction. \qed

**Corollary 1.8.7.** Let $R$ be a left hereditary ring. An $R$-module is projective if and only if it is a submodule of a free $R$-module.

**Corollary 1.8.8.** A ring $R$ is left hereditary if and only if submodules of projective $R$-modules are projective.

**Corollary 1.8.9.** Let $R$ be PLID. Then any submodule of a free $R$-module is free. Moreover, an $R$-module is free if and only if it is projective.

**Definition 1.8.10.** (1) A hereditary commutative domain is called a *Dedekind domain* (or Dedekind ring).

(2) A commutative PLID is called a *PID* (principal ideal domain).

Some authors exclude fields from the definition of Dedekind domain.

**Definition 1.8.11.** An ideal $I$ of a commutative domain $R$ is called *invertible* if there exists an $R$-submodule of $M$ of the quotient field $K$ of $R$ such that $IM = R$.

The condition implies that $1 = \sum_{i=1}^{n} a_i q_i$ with $a_i \in I$, $q_i \in M$. Then any $b \in I$ satisfies $b = \sum_{i=1}^{n} a_i q_i b$ with $q_i b \in IM = R$. Thus invertible ideals are finitely generated.

**Proposition 1.8.12.** A nonzero ideal $I$ of a commutative domain $R$ is invertible if and only if it is projective as an $R$-module.

Thus a Dedekind domain is a commutative domain of which every nonzero ideal is invertible.

*Proof.* Let $I$ be an invertible ideal with $IM = R$. Then $1 = \sum_{i=1}^{n} a_i q_i$ with $a_i \in I$, $q_i \in M$. Consider the free $R$-module $F = \bigoplus_{i=1}^{n} Re_i$. Let $f = (a_1, \ldots, a_n): F \to I$. Then $s: I \to F$ given by $s(a) = \sum_{i=1}^{n} a_i e_i$ is a section of $f$. Thus $I$ is a direct summand of $F$, hence projective.

Conversely, let $I$ be an ideal of $R$ that is projective as an $R$-module. Then there exists a free $R$-module $F = \bigoplus_{\alpha \in J} Re_\alpha$ and homomorphisms $f = (a_\alpha)_{\alpha \in J}: F \to I$ and $s: I \to F$ with $fs = \text{id}_I$. Let $a \in I$ with $a \neq 0$. By the following lemma, $s(a) = \sum_{\alpha \in J} q_\alpha a_\alpha e_\alpha$, where $q_\alpha \in K$ (zero for all but finitely many $\alpha \in J$) satisfies $q_\alpha I \subseteq R$. Take $M = \sum_{\alpha \in J} Rq_\alpha$. Then $IM \subseteq R$. Moreover, $a = fs(a) = \sum_{\alpha \in J} a_\alpha a_\alpha$, so that $1 = \sum_{\alpha \in J} a_\alpha a_\alpha \in IM$. It follows that $IM = R$. \qed

**Lemma 1.8.13.** Let $s: I \to R$ be a homomorphism of $R$-modules. Then there exists $q \in K$ such that $s(a) = qa$ for all $a \in I$. 

Proof. For \( a, b \in I \), \( bs(a) = s(ab) = as(b) \). Thus \( q = s(a)/a \) does not depend on the choice of \( a \in I, a \neq 0 \).

Corollary 1.8.14. Dedekind domains are Noetherian.

We refer the reader to standard textbooks on commutative algebra for other characterizations of Dedekind domains.

Example 1.8.15. Every field is a PID. More generally, every division ring is a PLID.

Example 1.8.16. The ring of rational integers \( \mathbb{Z} \) is a PID. The ring of Gaussian integers \( \mathbb{Z}[\sqrt{-1}] \) is a PID. For any field \( k \), the polynomial ring \( k[x] \) is a PID. More generally, for any division ring \( D \), \( D[x] \) is a PLID (and a PRID).

Example 1.8.17. Every semisimple ring \( R \) is left (and right) hereditary.

Example 1.8.18. If \( R \) and \( S \) are Morita equivalent, then \( R \) is left (resp. right) hereditary if and only if \( S \) is left (resp. right) hereditary. In particular, if \( R \) is a Dedekind domain, then \( M_n(R) \) is left (and right) hereditary.

Example 1.8.19. The free algebra \( R = k\langle X_i \rangle_{i \in I} \) over a field \( k \) generated by a set \( I \) of variables is a left (and right) hereditary domain (in fact any left ideal of \( R \) is free), but not left (or right) Noetherian for \( \#I > 1 \).

Example 1.8.20. A ring \( R \) is said to be von Neumann regular if for each \( r \), there exists \( s \in R \) such that \( rsr = r \); Boolean if \( r^2 = r \) for all \( r \in R \). Countable von Neumann regular rings are hereditary \([L1, \text{Example 2.32 (e)}]\). The countable Boolean ring \( R = \{ f : \mathbb{N} \to \mathbb{F}_2 \mid f^{-1}(0) \text{ or } f^{-1}(1) \text{ is finite} \} \) is hereditary, but not Noetherian.

Example 1.8.21. \( R = \mathbb{Z}[\sqrt{-5}] \) is a Dedekind domain, but not a PID: the ideal \( (2, 1 + \sqrt{-5}) \) is not principal. As an \( R \)-module, this ideal is projective but not free.

Example 1.8.22. The commutative domain \( \mathbb{Z}[\sqrt{-3}] \) is not a Dedekind domain. The commutative domain \( \mathbb{Z}[x_1, \ldots, x_n] \) is not a Dedekind domain for \( n \geq 1 \), and, in particular, it is not left hereditary. Indeed the ideal \( (2, x_1) \) is not invertible. The commutative domain \( k[x_1, \ldots, x_n] \) is not a Dedekind domain for \( n \geq 2 \). Indeed, the ideal \( (x_1, x_2) \) is not invertible. However, we have the following deep result.

Theorem 1.8.23 (Quillen, Suslin). Let \( R \) be a PID and let \( S = R[x_1, \ldots, x_n] \). Then every projective \( S \)-module is free.

The theorem was proved independently by Quillen and Suslin in 1976. The question (for \( R \) a field and finitely-generated modules) was first raised by Serre. See \([L2]\) for an exposition.

Remark 1.8.24. Kaplansky and later Small constructed examples of right hereditary rings that are not left hereditary. Small’s example is \( \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \). See \([L1, \text{Section 2F}]\).
Injective modules

**Theorem 1.8.25** (Baer’s test). Let \( R \) be a ring. An \( R \)-module \( I \) is injective if and only if for every left ideal \( A \), every homomorphism \( A \to I \) extends to a homomorphism \( R \to I \):

\[
\begin{array}{ccc}
A & \longrightarrow & R \\
\downarrow & & \downarrow \\
& & I.
\end{array}
\]

**Proof.** The “only if” part follows from the definition. To show the “if” part, consider an injective homomorphism \( u: X \to Z \) and a homomorphism \( f: X \to D \). To simplify notation, we consider \( u \) as the inclusion of a submodule. We look at the set \( S \) of pairs \((Y, g)\), where \( X \subseteq Y \subseteq Z \) and \( g: Y \to I \) extends \( f \). We equip \( S \) with the following order: \((Y, g) \leq (Y', g')\) if and only if \( Y \subseteq Y' \) and \( g' \) extends \( g \). Every chain \( \{(Y_\alpha, g_\alpha)\} \) in \( S \) admits the upper bound \((Y, g)\), where \( Y = \bigcup_\alpha Y_\alpha \) and \( g: Y \to I \) the unique homomorphism extending all the \( g_\alpha \). By Zorn’s lemma, there exists a maximal element \((Y_0, g_0)\) in \( S \). It suffices to show that \( Y_0 = Z \). Assume the contrary. Then there exists \( z \in Z \) such that \( z \not\in Y_0 \). Consider the left ideal \( A \) of \( R \) consisting of \( r \in R \) such that \( rz \in Y_0 \). By assumption, the homomorphism \( h_0: A \to I \) given by \( h_0(r) = g_0(rz) \) extends to a homomorphism \( h_1: R \to I \). Then \( g_0 \) extends to \( g_1: Y_1 = Y_0 + Rz \to I \) by \( g_1(y + rz) = g_0(y) + h_1(r) \). It is easy to check that \( g_1 \) is well-defined. Then \((Y_0, g_0) < (Y_1, g_1)\), contradicting the maximality of \((Y_0, g_0)\). \( \square \)

**Definition 1.8.26.** Let \( R \) be a domain. We say that an \( R \)-module \( D \) is divisible if for every \( d \in D \) and every nonzero \( r \in R \), there exists \( c \in D \) such that \( rc = d \).

Note that we do not require the uniqueness of \( c \).

**Example 1.8.27.** Let \( R = \mathbb{Z} \). Then \( \mathbb{Q} \) and \( \mathbb{Q}/\mathbb{Z} \) are divisible, but \( \mathbb{Z}/n\mathbb{Z} \) is not divisible for \( n \geq 2 \).

**Remark 1.8.28.** Any quotient of a divisible \( R \)-module is divisible. Any direct sum of divisible \( R \)-modules is divisible. Any product of divisible \( R \)-modules is divisible. Any product of injective \( R \)-modules is injective.

**Proposition 1.8.29.** Let \( R \) be a domain. Then every injective \( R \)-module is divisible. Moreover, if \( R \) is a PLID or a Dedekind domain, then an \( R \)-module is injective if and only if it is divisible.

**Proof.** Let \( I \) be an injective \( R \)-module. Let \( d \in I \) and \( r \in R \), \( r \neq 0 \). Since \( R \) is a domain, the homomorphism of \( R \)-modules \( m: R \to R \) defined by \( m(s) = sr \), is injective. Let \( f: R \to I \) be the homomorphism carrying 1 to \( d \). Then there exists \( g: R \to I \) such that \( f = gm \). Then \( d = f(1) = gm(1) = g(r) = rg(1) \).

Assume that \( R \) is a PLID and let \( D \) be a divisible \( R \)-module. We use Baer’s test. Consider a left ideal \( A \) of \( R \) and a homomorphism \( f: A \to D \). Since \( R \) is a PLID, \( A \) is a principal left ideal: \( A = Ra \). Since \( D \) is divisible, there exists \( c \in D \) such that \( ac = f(a) \). Then \( f \) extends to \( g: R \to D \) given by \( g(r) = rc \).
Assume now that $R$ is a Dedekind domain and let $D$ be a divisible $R$-module. Consider a nonzero ideal $A$ of $R$ and a homomorphism $f: A \to D$. Since $R$ is a Dedekind domain, $A$ is invertible: $1 = \sum_{i=1}^{n} a_{i} q_{i}$ with $a_{i} \in A$, $q_{i} \in K = \text{Frac}(R)$, $q_{i}A \subseteq R$. Since $D$ is divisible, there exists $c_{i} \in D$ such that $a_{i} c_{i} = f(a_{i})$. Then

$$f(a) = f(\sum_{i=1}^{n} a_{i} q_{i} a) = \sum_{i=1}^{n} q_{i} a f(a_{i}) = a \sum_{i=1}^{n} q_{i} a c_{i}.$$

Thus $f$ extends to $g: R \to D$ given by $g(r) = rc$ where $c = \sum_{i=1}^{n} q_{i} a_{i} c_{i}$. 

**Corollary 1.8.30.** If $R$ is a PLID or a Dedekind domain, then quotients of injective $R$-modules are injective.

More generally, we have the following dual of Corollary 1.8.8.

**Proposition 1.8.31** (Cartan–Eilenberg). A ring $R$ is left hereditary if and only if quotients of injective $R$-modules are injective.

We will deduce this later from general facts on homological dimensions. It is not hard to give a direct proof. See for example [CE, Theorem I.5.4] or [L1, Theorem 3.22].

**Corollary 1.8.32.** Let $R$ be a domain. If every divisible $R$-module is injective, then $R$ is left hereditary.

**Example 1.8.33.** Let $R = \mathbb{Z}[x]$ and let $K = \mathbb{Q}(x)$ be the fraction field of $R$. Then the $R$-module $M = K/R$ is divisible but not injective. Indeed, the homomorphism $A = 2R + xR \to M$ carrying $2$ to $0$ and $x$ to the class of $1/2$ does not extend to a homomorphism $R \to M$.

**Remark 1.8.34.** A ring $R$ is left Noetherian if and only if direct sums of injective $R$-modules are injective. The “only if” part follows easily from Baer’s test (Exercise 8). The “if” part is a theorem of Bass and Papp (see [L1, Theorem 3.46] for a proof and other equivalent conditions). Thus, if $R$ is a domain such that every divisible $R$-module is injective, then $R$ is left Noetherian.

We refer the reader to [L1, Section 3C] for a more general discussion on the relation between injectivity and divisibility when $R$ is not necessarily a domain.

**Enough injective modules**

**Proposition 1.8.35.** Any $\mathbb{Z}$-module (i.e. abelian group) $M$ can be embedded into a divisible, hence injective, $\mathbb{Z}$-module.

**Proof.** We have $M = F/H$ with $F$ free. Embedding $F$ into a $\mathbb{Q}$-vector space $V$, we get $M \subseteq V/H$. Since $V$ is divisible as a $\mathbb{Z}$-module, $V/H$ is divisible. 

---

*This part was known to Cartan and Eilenberg [CE, Exercise VII.8]. By contrast, for $R$ left Noetherian, a product of projective $R$-modules is not projective in general. For example, $\mathbb{Z}^N$ is not a projective $\mathbb{Z}$-module (Baer, Exercise).*
Remark 1.8.36. We will see later that every \( \mathbb{Z} \)-module can be embedded into a product of \( \mathbb{Q}/\mathbb{Z} \) (such a product is sometimes called “cofree”).

Remark 1.8.37. Let \( R \to S \) be a ring homomorphism. The functor

\[
\text{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod}
\]

is a right adjoint to the restriction of scalars functor \( S\text{-Mod} \to R\text{-Mod} \), which is exact. It follows that \( \text{Hom}_R(S, -) \) carries injective \( R \)-modules to injective \( S \)-modules.

Proposition 1.8.38. Let \( R \) be a ring. Any \( R \)-module \( M \) can be embedded into an injective \( R \)-module.

Proof. We embed the underlying \( \mathbb{Z} \)-module of \( M \) into an injective \( \mathbb{Z} \)-module \( I \). Then \( \text{Hom}_\mathbb{Z}(R, I) \) is an injective \( R \)-module and we have injective homomorphisms

\[
M \cong \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_\mathbb{Z}(R, M) \hookrightarrow \text{Hom}_\mathbb{Z}(R, I).
\]

Self-injective rings

Definition 1.8.39. We say that a ring \( R \) is left self-injective if the left \( R \)-module \( R \) is injective.

Similarly one defines right self-injective rings. There are left self-injective rings that are not right self-injective \([L1, \text{Example } 3.74B]\). However we have the following theorem.

Theorem 1.8.40 (Faith–Walker). Let \( R \) be a ring. The following conditions are equivalent.

1. \( R \) is left Noetherian and left self-injective;
2. Every projective \( R \)-module is injective;
3. Every injective \( R \)-module is projective.

Moreover, \( R \) satisfies the above conditions if and only if \( R^{\text{op}} \) satisfies the above conditions.

By Remark 1.8.34 we have (1) \( \Rightarrow \) (2) and (2) + (3) \( \Rightarrow \) (1) (the latter follows from the theorem of Bass and Pass). The other parts of Theorem \([L.8.40]\) are harder. We refer the reader to \([F, \text{Chapter } 24]\) for a proof.

Definition 1.8.41. A ring \( R \) is called quasi-Frobenius if it satisfies Condition (1) of the above theorem.

Proposition 1.8.42. Let \( R \) be a PLID and let \( a \in R \) such that \( Ra = aR \) and \( a \neq 0 \). Then \( R/aR \) is quasi-Frobenius. In particular, the quotient of any PID by a proper ideal is quasi-Frobenius.
Proof. The ring $R/Ra$ is clearly left Noetherian. We apply Baer’s test to show that it is left self-injective. Let $A = Rb/Ra$ be an ideal of $R/Ra$ and let $h: A \to R/Ra$ be a homomorphism. We need to extend $h$ to $R/Ra$. Set $h(\bar{b}) = \bar{r}$ for some $r \in R$, where $\bar{r}$ denotes the image of $r$ in $R/Ra$. We have $Ra \subseteq Rb$, so that $a = cb$ for some nonzero $c \in R$. We have $0 = h(\bar{a}) = h(\bar{c}\bar{b}) = \bar{c}\bar{r}$. Thus $cr = as$ for some $s \in R$. Canceling $c$, we get $r = bs$, so $h(\bar{b}) = b\bar{s}$. Thus $h$ extends to the homomorphism $R/Ra \to R/Ra$ carrying $\bar{1}$ to $\bar{s}$.  

Example 1.8.43. For $m \neq 0$, the ring $\mathbb{Z}/m\mathbb{Z}$ is quasi-Frobenius. For any field $k$ and any nonzero $f \in k[x]$, the ring $k[x]/(f)$ is quasi-Frobenius.

Example 1.8.44. The group algebra $k[G]$ is a quasi-Frobenius ring for any field $k$ and any finite group $G$ [L1, Proposition 3.14, Example 3.15E]. This fact is especially useful when the characteristic of $k$ divides the order of $G$ (otherwise $k[G]$ is a semisimple ring by Maschke’s theorem).
Chapter 2

Derived categories and derived functors

Introduction

Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and let \( F : \mathcal{A} \to \mathcal{B} \) be a left exact functor. For any short exact sequence
\[
0 \to X \to Y \to Z \to 0
\]
in \( \mathcal{A} \), we have, by the left exactness of \( F \), an exact sequence
\[
0 \to FX \to FY \to FZ
\]
in \( \mathcal{B} \). Under suitable conditions, we can define additive functors \( R^nF : \mathcal{A} \to \mathcal{B} \), \( i \geq 1 \), called the right derived functors of \( F \), such that the exact sequence in \( \mathcal{B} \) extends to a long exact sequence
\[
0 \to FX \to FY \to FZ \to R^1FX \to R^1FY \to R^1FZ \to \cdots
\]
\[
\to R^nFX \to R^nFY \to R^nFZ \to \cdots
\]

Roughly speaking, the right derived functors measure the lack of right exactness of \( F \). The functors can be assembled into one single functor \( RF : D^+(\mathcal{A}) \to D^+(\mathcal{B}) \) between derived categories.

2.1 Complexes

Complexes

Let \( \mathcal{A} \) be an additive category.

Definition 2.1.1. A (cochain) complex in \( \mathcal{A} \) consists of \( X = (X^n, d^n)_{n \in \mathbb{Z}} \), where \( X^n \) is an object of \( \mathcal{A} \), \( d_X^n : X^n \to X^{n+1} \) is a morphism of \( \mathcal{A} \) (called differential) such that for any \( n \), \( d_X^{n+1}d_X^n = 0 \). The index \( n \) in \( X^n \) is called the degree. A (cochain) morphism of complexes \( X \to Y \) is a collection of morphisms \( (f^n)_{n \in \mathbb{Z}} \) of morphisms \( f^n : X^n \to Y^n \) in \( \mathcal{A} \) such that \( d_Y^n f^n = f^{n+1} d_X^n \). We let \( C(\mathcal{A}) \) denote the category of complexes in \( \mathcal{A} \).
Note that \( C(\mathcal{A}) \) is a full subcategory of \( \mathcal{A}^{(\mathbb{Z}, \leq)} \). Furthermore, \( C(\mathcal{A}) \) is an additive category. We have \((X \oplus Y)^n = X^n \oplus Y^n\) and the zero complex \(0^n = 0\) is a zero object of \( C(\mathcal{A})\). If \( \mathcal{A} \) is an abelian category, then \( C(\mathcal{A}) \) is an abelian category as well, with \( \text{Ker}(f)^n = \text{Ker}(f^n) \) and \( \text{Coker}(f)^n = \text{Coker}(f^n) \).

**Definition 2.1.2.** We say that a complex \( X \) is *bounded below* (resp. *bounded above*) if \( X^n = 0 \) for \( n \ll 0 \) (resp. \( n \gg 0 \)). We say that \( X \) is bounded if it is bounded below and bounded above. For an interval \( I \subseteq \mathbb{Z} \), we say that \( X \) is concentrated in degrees in \( I \) if \( X^n = 0 \) for \( n \not\in I \). We let \( C^+ (\mathcal{A}) \), \( C^- (\mathcal{A}) \), \( C^b (\mathcal{A}) \), \( C^I (\mathcal{A}) \) denote the full subcategories of \( C(\mathcal{A}) \) consisting of complexes bounded below, bounded above, bounded, concentrated in \( I \), respectively. These are additive subcategories.

The functor \( C^{[0,0]} (\mathcal{A}) \to \mathcal{A} \) carrying \( X \) to \( X^0 \) is an equivalence of categories. We sometimes use this equivalence to identify \( \mathcal{A} \) with a full subcategory of \( C(\mathcal{A}) \).

**Remark 2.1.3.** The inclusion functor \( C^{\leq n} (\mathcal{A}) \subseteq C(\mathcal{A}) \) admits a left adjoint

\[ \sigma^{\leq n} : C(\mathcal{A}) \to C^{\leq n} (\mathcal{A}) \]

with \((\sigma^{\leq n} X)^m = X^m\) for \( m \leq n \) and \((\sigma^{\leq n} X)^m = 0\) for \( m > n \). Similarly, the inclusion functor \( C^{\geq n} (\mathcal{A}) \subseteq C(\mathcal{A}) \) admits a right adjoint

\[ \sigma^{\geq n} : C(\mathcal{A}) \to C^{\geq n} (\mathcal{A}) \]

with \((\sigma^{\geq n} X)^m = X^m\) for \( m \geq n \) and \((\sigma^{\geq n} X)^m = 0\) for \( m < n \). These functors are called *naive truncation* functors, as opposed to the truncation functors introduced later. If \( \mathcal{A} \) is an abelian category, the naive truncation functors are exact.

**Definition 2.1.4.** Let \( X \) be a complex and let \( k \) be an integer. We define a complex \( X[k] \) by \( X[k]^n = X^{n+k} \) and \( d^k_{X[k]} = (-1)^k d^{n+k}_X \). For a morphism of complexes \( f : X \to Y \), we define \( f[k] : X[k] \to Y[k] \) by \( f[k]^n = f^{n+k} \). The functor \([k] : C(\mathcal{A}) \to C(\mathcal{A})\) is called the translation (or shift) functor of degree \( k \).

The sign in the definition of \( X[k] \) will be explained later, after the definition of mapping cone (Definition 2.1.14). Note that if \( X \) is concentrated in degrees \([a, b]\), then \( X[k] \) is concentrated in degrees \([a-k, b-k]\).

**Remark 2.1.5.** We define an isomorphism of categories \( F : C(\mathcal{A})^{\text{op}} \to C(\mathcal{A}^{\text{op}}) \) as follows. For \( X \) in \( C(\mathcal{A}) \), we define \( Y = FX \) in \( C(\mathcal{A}^{\text{op}}) \) by \( Y^n = X^{-n} \), \( d^n_Y = (-1)^n d^{-n-1}_X \). We have a natural isomorphism \( F(X[1]) \simeq (FX)[-1] \) with \( F(X[1])^n \simeq (FX)^{-1} \) given by \((-1)^{n+1} \text{id}_{X[1-n]} \).

There are several other constructions for complexes in an additive category. We will return to them later. Now we proceed to define cohomology.

**Cohomology**

Let \( \mathcal{A} \) be an abelian category. Consider a sequence

\[ X' \xrightarrow{f} X \xrightarrow{g} X'' \]
with \( gf = 0 \). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Im}(f) & \xrightarrow{\phi} & \text{Ker}(g) \\
\downarrow f & & \downarrow g \\
\text{Coker}(f) & \xleftarrow{\psi} & \text{Im}(g) \\
\end{array}
\]

We have isomorphisms

\[ \text{Im}(\text{Ker}(g) \to \text{Coker}(f)) \cong \text{Coker}(\phi) \cong \text{Ker}(\psi). \]

**Definition 2.1.6.** Let \( X \) be a complex in \( \mathcal{A} \). We define

\[
\begin{align*}
Z^n X &= \text{Ker}(d^n_X : X^n \to X^{n+1}), \\
B^n X &= \text{Im}(d^{n-1}_X : X^{n-1} \to X^n), \\
H^n X &= \text{Coker}(B^n X \hookrightarrow Z^n X),
\end{align*}
\]

and call them the cocycle, coboundary, cohomology objects, of degree \( n \).

The letter \( Z \) stands for German Zyklus, which means cycle. We get additive functors

\[
Z^n, B^n, H^n : C(\mathcal{A}) \to \mathcal{A},
\]

with \( Z^n \) left exact.

**Example 2.1.7.** Let \( M \) be a smooth manifold of dimension \( n \). The de Rham complex \( \Omega^\bullet(M) \) of \( M \) is a complex of \( \mathbb{R} \)-vector spaces:

\[
\cdots \to 0 \to \Omega^0(M) \to \cdots \to \Omega^n(M) \to 0 \cdots,
\]

where \( \Omega^i(M) \) denotes the space of smooth differential \( i \)-forms on \( M \). The \( i \)-th de Rham cohomology of \( M \), \( H^i_{\text{dR}}(M) \), is by definition the cohomology of \( \Omega^\bullet(M) \) of degree \( i \).

**Definition 2.1.8.** A complex \( X \) is said to be acyclic if \( H^n X = 0 \) for all \( n \). A morphism of complexes \( X \to Y \) is called a quasi-isomorphism if \( H^n f : H^n X \to H^n Y \) is an isomorphism for all \( n \).

Later we will define the derived category \( D(\mathcal{A}) \) of \( \mathcal{A} \). Roughly speaking, \( D(\mathcal{A}) \) is \( C(\mathcal{A}) \) modulo quasi-isomorphisms.

We have \( H^n(X[k]) \cong H^{n+k} X \).

Note that the morphisms \( H^n X \to H^n \sigma^{\leq n} X \), \( H^n \sigma^{\leq n} X \to H^n X \) are not isomorphisms in general. Moreover, if \( f : X \to Y \) is a quasi-isomorphism, \( \sigma^{\leq n} f : \sigma^{\leq n} X \to \sigma^{\leq n} Y \) and \( \sigma^{\geq n} f : \sigma^{\geq n} X \to \sigma^{\geq n} Y \) are not quasi-isomorphisms in general. To remedy this problem, we introduce the following truncation functors.
Definition 2.1.9. Let $X$ be a complex. We define
\[
\tau_{\leq n} X = (\cdots \rightarrow X^{n-1} \xrightarrow{d_{X}^{n-1}} Z^{n} X \rightarrow 0 \rightarrow \cdots),
\]
\[
\tau_{\geq n} X = (\cdots \rightarrow 0 \rightarrow X^{n}/B^{n} X \xrightarrow{d_{X}^{n}} X^{n+1} \rightarrow \cdots).
\]
Here $X^{n}/B^{n} X$ denotes $\text{Coker}(d_{X}^{n-1})$.

We obtain functors
\[
\tau_{\leq n}, \tau_{\geq n} : C(A) \rightarrow C(A),
\]
with $\tau_{\leq n}$ left exact and $\tau_{\geq n}$ right exact.

Remark 2.1.10. The morphism $\tau_{\leq n} X \rightarrow X$ induces an isomorphism $H^{m} \tau_{\leq n} X \rightarrow H^{m} X$ for $m \leq n$ and $H^{m} \tau_{\leq n} X = 0$ for $m > n$. The morphism $X \rightarrow \tau_{\geq n} X$ induces an isomorphism $H^{m} X \rightarrow H^{m} \tau_{\geq n} X$ for $m \geq n$ and $H^{m} \tau_{\geq n} X = 0$ for $m < n$. The functors $\tau_{\leq n}$ and $\tau_{\geq n}$ preserve quasi-isomorphisms.

Remark 2.1.11. For $a \leq b$, we have $\tau_{\leq a} \tau_{\geq b} X \simeq \tau_{\geq b} \tau_{\leq a} X$ and we write $\tau_{[a,b]} X$ for either of them. We have $\tau_{[n,n]} X \simeq H^{n} X[-n]$.

The functor $H^{n}$ is neither left exact nor right exact in general. However, it has the following important property.

Proposition 2.1.12. Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a short exact sequence of complexes. Then we have a long exact sequence
\[
\cdots \rightarrow H^{n} L \xrightarrow{H^{n} f} H^{n} M \xrightarrow{H^{n} g} H^{n} N \rightarrow H^{n+1} L \xrightarrow{H^{n+1} f} H^{n+1} M \xrightarrow{H^{n+1} g} H^{n+1} N \rightarrow \cdots,
\]
which is functorial with respect to the short exact sequence.

This generalizes the case of the snake lemma where the exact rows are short exact.

Proof. The sequence $\tau_{[n,n+1]} L \rightarrow \tau_{[n,n+1]} M \rightarrow \tau_{[n,n+1]} N$ provides a commutative diagram
\[
\begin{array}{ccc}
L^{n}/B^{n} L & \rightarrow & M^{n}/B^{n} M \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z^{n+1} L \\
\end{array}
\begin{array}{ccc}
M^{n}/B^{n} M & \rightarrow & N^{n}/B^{n} N \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z^{n+1} M \\
\end{array}
\begin{array}{ccc}
N^{n}/B^{n} N & \rightarrow & Z^{n+1} N \\
\end{array}
\]
with exact rows. Applying the snake lemma, we obtain the desired exact sequence.

Corollary 2.1.13. Let
\[
0 \rightarrow L \xrightarrow{u} M \xrightarrow{v} N \rightarrow 0
\]
\[
0 \rightarrow L' \xrightarrow{v} M' \xrightarrow{w} N' \rightarrow 0
\]
be a commutative diagram of complexes with exact rows. If two of the three morphisms $u, v, w$ are quasi-isomorphisms, then the third one is a quasi-isomorphism too.
The short exact sequence induces a long exact sequence

\[ H^{n-1}N \longrightarrow H^nL \longrightarrow H^nM \longrightarrow H^nN \longrightarrow H^{n+1}L \]

with exact rows. By the five lemma, \( H^nv \) is an isomorphism.

For any morphism of complexes \( L \to M \), not necessarily monomorphic, the induced morphisms \( H^nL \to H^nM \) also extends naturally to a long exact sequence, by the following construction.

### Mapping cones

Let \( \mathcal{A} \) be an additive category.

**Definition 2.1.14.** Let \( f: X \to Y \) be a morphism of complexes in \( \mathcal{A} \). We define the mapping cone of \( f \) to be the complex \( \text{Cone}(f)^n = X[1]^n \oplus Y^n = X^{n+1} \oplus Y^n \) with differential

\[
d_{\text{Cone}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f[1]^n & d^n_Y \end{pmatrix} = \begin{pmatrix} -d^n_{X[n+1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}.
\]

Intuitively, for \( (x/y) \in X^{n+1} \oplus Y^n \), \( d_{\text{Cone}(f)}(x/y) = \begin{pmatrix} -d^{n+1}_X x \\ f^{n+1}x + d^n_Y y \end{pmatrix} \).

Note that the sign in the definition of the differential of \( X[1] \) makes \( \text{Cone}(f) \) a complex:

\[
d_{\text{Cone}(f)}d_{\text{Cone}(f)}^{-1} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \begin{pmatrix} -d^n_X & 0 \\ f^n & d^{n-1}_Y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_Xd^n_X & 0 \\ f^n f^{n+1}d^n_X & d^n_Yd^{n-1}_Y \end{pmatrix} = 0.
\]

**Example 2.1.15.** If \( X \) and \( Y \) are concentrated in degree 0, then \( \text{Cone}(f) \) can be identified with the complex \( X^0 \longrightarrow Y^0 \) concentrated on degrees \(-1\) and 0.

**Remark 2.1.16.** Let \( X \) and \( Y \) be CW complexes and let \( f: X \to Y \) be a cellular map. The (topological) mapping cone \( \text{Cone}(f) \) of \( f \) is obtained by gluing the base of the cone \( \text{Cone}(X) \) to \( Y \) via \( f \). If we let \( c \) denote the cone point, then \( \text{Cone}(\mathcal{C}_\bullet(f)) \) can be identified with \( \mathcal{C}_\bullet(\text{Cone}(f))/\mathcal{C}_\bullet(c) \). Here \( \mathcal{C}_\bullet(f) \) denotes the cellular chain complex. A chain complex \( (X_\bullet, d_\bullet) \) is regarded as a cochain complex by \( X^{-n} = X_n \), \( d^{-n} = d_n \).

Assume that \( \mathcal{A} \) is an abelian category. We have a short exact sequence of complexes

\[ 0 \to Y \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} X[1] \to 0, \]

where \( i: Y \to \text{Cone}(f) \) is the inclusion and \( p: \text{Cone}(f) \to X[1] \) is the projection. The short exact sequence induces a long exact sequence

\[
\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta} H^nY \xrightarrow{H^n i} H^n(\text{Cone}(f)) \xrightarrow{H^np} H^n(X[1]) \to \cdots.
\]

\[1\] We invite readers unfamiliar with CW complexes to replace “CW” and “cellular” by “simplicial.”
Proposition 2.1.17. Via the isomorphism $H^{n-1}(X[1]) \simeq H^n X$, the connecting morphism can be identified with $H^n f$.

The long exact sequence thus has the form

$$\cdots \to H^n X \xrightarrow{H^n f} H^n Y \xrightarrow{H^n i} H^n (\text{Cone}(f)) \xrightarrow{H^n p} H^{n+1} X \to \cdots.$$ 

Proof. The connecting morphism is constructed using the snake lemma applied to the commutative diagram

$$\begin{array}{ccc}
Y^{n-1}/B^{n-1}Y & \longrightarrow & C^{n-1}/B^{n-1}C \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z^nY \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z^nC \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z^{n+1}X,
\end{array}$$

where $C = \text{Cone}(f)$. We reduce by the Freyd-Mitchell Theorem to the case of modules. Let $x \in Z^n X$. Then $(x, 0) + B^{n-1} C$ is a lifting of $x + B^n X$. We conclude by $d_C^{n-1}(x, 0) = \left( \begin{array}{c} 0 \\ f^n(x) \end{array} \right)$.

Proposition 2.1.18. A morphism of complexes $f : X \to Y$ is a quasi-isomorphism if and only if its cone $\text{Cone}(f)$ is acyclic.

Proof. Indeed, by the long exact sequence, $H^n f$ is an isomorphism for all $n$ if and only if $H^n (\text{Cone}(f)) = 0$ for all $n$.

Proposition 2.1.19. Consider a short exact sequence of complexes $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. Then the map $\phi = (0, g) : \text{Cone}(f) \to Z$ is a quasi-isomorphism.

Proof. We have a short exact sequence

$$0 \to \text{Cone}(\text{id}_X) \xrightarrow{\psi} \text{Cone}(f) \xrightarrow{\phi} Z \to 0,$$

where $\psi$ is associated to the commutative square

$$\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & X \\
\downarrow \psi & & \downarrow \phi \\
X & \xrightarrow{f} & Y.
\end{array}$$

Since $\text{Cone}(\text{id}_X)$ is acyclic (in fact homotopy equivalent to zero), the long exact sequence implies that $\phi$ is a quasi-isomorphism.

Remark 2.1.20. We have a commutative diagram of long exact sequences

$$\begin{array}{ccc}
H^n X & \xrightarrow{H^n f} & H^n Y \\
\downarrow H^n f & & \downarrow H^n i \\
H^n X & \xrightarrow{H^n f} & H^n (\text{Cone}(f)) \\
\downarrow H^n i & & \downarrow \text{Hn} p \\
H^{n+1} X & \xrightarrow{H^{n+1} f} & H^{n+1} Y.
\end{array}$$

Indeed, for the commutativity of the square $(\ast)$ we reduce by the Freyd-Mitchell Theorem to the case of modules, and it suffices to note that for $(x, y) \in Z^n \text{Cone}(f)$, we have $f^n(x) + d^n y = 0$. By the five lemma, this gives another proof of Proposition 2.1.19.
2.2 Homotopy category of complexes, triangulated categories

Homotopy

Let \( \mathcal{A} \) be an additive category. Let \( X \) and \( Y \) be complexes in \( \mathcal{A} \). We let

\[
Ht(X, Y) = \prod_n \text{Hom}_\mathcal{A}(X^n, Y^{n-1})
\]

denote the abelian group of families of morphisms \( h = (h^n : X^n \to Y^{n-1})_{n \in \mathbb{Z}} \). Given \( h \), consider

\[
f^n = d^n_Y h^n + h^{n+1} d^n_X : X^n \to Y^n.
\]

We have

\[
d^n_Y f^n = d^n_Y d^n_Y h^n + d^n_Y h^{n+1} d^n_X = d^n_Y h^{n+1} d^n_X + h^{n+2} d^{n+1} X d^n_X = f^{n+1} d^n_X.
\]

Thus we get a morphism of complexes \( f : X \to Y \). We get a homomorphism of abelian groups

\[
(2.2.1) \quad Ht(X, Y) \to \text{Hom}_{C(\mathcal{A})}(X, Y).
\]

Definition 2.2.1. We say that a morphism of complexes \( f : X \to Y \) is null-homotopic if there exists \( h \in Ht(X, Y) \) such that \( f^n = d^n_Y h^n + h^{n+1} d^n_X \). We say that two morphisms of complexes \( f, g : X \to Y \) are homotopic if \( f - g \) is null-homotopic.

Lemma 2.2.2. Let \( f : X \to Y \), \( g : Y \to Z \) be morphisms of complexes in \( \mathcal{A} \). If \( f \) or \( g \) is null-homotopic, then \( gf \) is null-homotopic.

Proof. If \( f = dh + hd \) for \( h \in Ht(X, Y) \), then \( gf = gdh + ghd = d(gh) + (gh)d \), where \( gh \in Ht(X, Z) \). The other case is similar.

Definition 2.2.3. We define the homotopy category of complexes in \( \mathcal{A} \), \( \mathcal{K}(\mathcal{A}) \), as follows. The objects of \( \mathcal{K}(\mathcal{A}) \) are objects of \( C(\mathcal{A}) \), that is, complexes in \( \mathcal{A} \). For complexes \( X \) and \( Y \), we put

\[
\text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) = \text{Coker}(Ht(X, Y) \to \text{Hom}_{C(\mathcal{A})}(X, Y)).
\]

In other words, morphisms in \( \mathcal{K}(\mathcal{A}) \) are homotopy classes of morphisms of complexes.

An isomorphism in \( \mathcal{K}(\mathcal{A}) \) is called a homotopy equivalence. By definition, a morphism of complexes \( f : X \to Y \) is a homotopy equivalence if there exists a morphism of complexes \( g : Y \to X \) such that \( fg \) is homotopic to \( \text{id}_Y \) and \( gf \) is homotopic to \( \text{id}_X \). We say that two complexes are homotopy equivalent if they are isomorphic in \( \mathcal{K}(\mathcal{A}) \). A complex \( X \) is homotopy equivalent to \( 0 \) if and only if \( \text{id}_X \) is null-homotopic.

Remark 2.2.4. The category \( \mathcal{K}(\mathcal{A}) \) is an additive category and the functor \( C(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \) carrying a complex to itself and a morphism of complexes to its homotopy class is an additive functor.
Notation 2.2.5. Let $I \subseteq \mathbb{Z}$ be an interval. We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$, $K^f(\mathcal{A})$ denote the full subcategories of $K(\mathcal{A})$ consisting of complexes in $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^f(\mathcal{A})$, respectively. These are additive subcategories. The functor $C^{[0,0]}(\mathcal{A}) \to K^{[0,0]}(\mathcal{A})$ is an isomorphism of categories. We thus obtain an equivalence of categories between $\mathcal{A}$ and $K^{[0,0]}(\mathcal{A})$.

Remark 2.2.6. Let $\mathcal{A}$ be an abelian category. Note that $dh + hd$ induces zero morphisms on cohomology. Thus if $f, g: X \to Y$ are homotopic, then $H^n f = H^n g: H^n X \to H^n Y$. The additive functor $H^n: C(\mathcal{A}) \to \mathcal{A}$ factorizes through an additive functor

$$H^n: K(\mathcal{A}) \to \mathcal{A}.$$  

In particular, a homotopy equivalence is a quasi-isomorphism. The converse does not hold in general. Indeed, an acyclic complex is not homotopic to zero in general, as shown by the following lemma.

Similarly, the additive functors $\tau^{\leq n}, \tau^{\geq n}: C(\mathcal{A}) \to C(\mathcal{A})$ induce additive functors

$$\tau^{\leq n}, \tau^{\geq n}: K(\mathcal{A}) \to K(\mathcal{A}).$$

Lemma 2.2.7. Let $\mathcal{A}$ be an abelian category. Then a complex $X$ in $\mathcal{A}$ is homotopy equivalent to zero if and only if $X$ is acyclic and the short exact sequences

$$0 \to Z^n X \to X^n \to Z^{n+1} X \to 0$$

are split.

Thus a complex in $\mathcal{A}$ is homotopy equivalent to zero if and only if it is isomorphic to $(Z^n \oplus Z^{n+1})$, with $d^n: Z^n \oplus Z^{n+1} \to Z^{n+1} \rightarrow Z^{n+1} \oplus Z^{n+2}$. This holds in fact more generally for idempotent-complete\footnote{In a category $\mathcal{C}$, a morphism $e: X \to X$ such that $e^2 = e$ is called an idempotent. A typical example for $\mathcal{C}$ additive is the composition $e = gf: A \oplus B \xrightarrow{i} A \xrightarrow{=} A \oplus B$. We have $fg = id_A$. We say that an idempotent $e: X \to X$ splits if there exist morphisms $f: X \to Y$, $g: Y \to X$ such that $e = gf$ with $fg = id_Y$. We say that $\mathcal{C}$ is idempotent-complete if every idempotent in $\mathcal{C}$ splits. Every abelian category is idempotent-complete.} additive categories.

Proof. If the sequences are split short exact sequences, so that $X^n$ can be identified with $Z^n \oplus Z^{n+1}$, then $h^n: Z^n \oplus Z^{n+1} \to Z^n \to Z^{n-1} \oplus Z^n$ satisfies $hd + dh = id_X$. Conversely, if $hd + dh = id_X$, then $h^{n+1}$ restricted to $Z^{n+1} X$ provides a splitting of the short exact sequence.

Remark 2.2.8. The homotopy category brings us one step closer to the derived category of an abelian category $\mathcal{A}$. We will see that under the condition that $\mathcal{A}$ admits enough injectives, $D^+(\mathcal{A})$ is equivalent to $K^+(\mathcal{I})$, where $\mathcal{I}$ is the full subcategory of $\mathcal{A}$ spanned by injective objects.

Remark 2.2.9. Even for $\mathcal{A}$ abelian, $K(\mathcal{A})$ is not an abelian category in general. In fact, one can show that for $\mathcal{A}$ abelian, $K(\mathcal{A})$ admits kernels if and only if every short exact sequence in $\mathcal{A}$ splits (in this case $K(\mathcal{A})$ is equivalent to $\prod_{\mathbb{Z}} \mathcal{A}$ and is an abelian category)\footnote{Notation 2.2.5. Let $I \subseteq \mathbb{Z}$ be an interval. We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$, $K^f(\mathcal{A})$ denote the full subcategories of $K(\mathcal{A})$ consisting of complexes in $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^f(\mathcal{A})$, respectively. These are additive subcategories. The functor $C^{[0,0]}(\mathcal{A}) \to K^{[0,0]}(\mathcal{A})$ is an isomorphism of categories. We thus obtain an equivalence of categories between $\mathcal{A}$ and $K^{[0,0]}(\mathcal{A})$.} [V2, Propositions II.1.2.9, II.1.3.6].
Triangulated categories

Given a category $\mathcal{D}$ equipped with a functor $X \mapsto X[1]$, diagrams of the form $X \to Y \to Z \to X[1]$ are called triangles. It is sometimes useful to visualize such diagrams as

$$
\begin{array}{ccc}
Z & \rightarrow & X[1] \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
$$

A morphism of triangles is a commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & Y & \rightarrow & Z & \rightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \rightarrow & Y' & \rightarrow & Z' & \rightarrow & X'[1]
\end{array}
$$

Such a morphism is an isomorphism if and only if $f$, $g$, $h$ are isomorphisms.

**Definition 2.2.10** (Verdier). A triangulated category consists of the following data:

1. An additive category $\mathcal{D}$.
2. A translation functor $\mathcal{D} \to \mathcal{D}$ which is an equivalence of categories. We denote the functor by $X \mapsto X[1]$.
3. A collection of distinguished triangles $X \to Y \to Z \to X[1]$.

These data are subject to the following axioms:

- (TR1) The collection of distinguished triangles is stable under isomorphism.
- (TR2) Every morphism $f: X \to Y$ in $\mathcal{D}$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.
- (TR3) For every object $X$ of $\mathcal{D}$, $X \xrightarrow{id} X \to 0 \to X[1]$ is a distinguished triangle.
- (TR4) A diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y & \xrightarrow{f'} & U & \xrightarrow{f''} & X[1], \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z & \xrightarrow{g'} & W & \xrightarrow{g''} & Y[1], \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{h} & Z & \xrightarrow{h'} & V & \xrightarrow{h''} & X[1],
\end{array}
$$

with $h = gf$, there exists a distinguished triangle $U \xrightarrow{i} V \xrightarrow{i'} W \xrightarrow{i''} U[1]$ such that the following diagram commutes.
This notion was introduced by Verdier in his 1963 notes [V1] and 1967 thesis of doctorat d’État [V2]. Some authors call the translation functor the suspension functor and denote it by $\Sigma$. (TR4) is sometimes known as the octahedron axiom, as the four distinguished triangles and the four commutative triangles can be visualized as the faces of an octahedron.

**Remark 2.2.11.** The original definition included an axiom (TR3). May [M, Section 2] observed that this axiom can be deduced from (TR1) and (TR4), as we shall see in Proposition 2.2.18. He also observed that (TR2) follows from (TR1), (TR3), and the following weakened form of (TR2), as we shall see in Remark 2.2.26:

(T2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then the (clockwise) rotated diagram $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

**Remark 2.2.12.** We have an isomorphism of triangles

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{-f} & Y
\end{array}
\begin{array}{ccc}
Z & \xrightarrow{g} & Z[1] \\
\downarrow & & \downarrow \\
\text{id}_Y & & \text{id}_Y
\end{array}
\begin{array}{ccc}
X[1] & \xrightarrow{h} & X[1]
\end{array}
$$

Similarly, modifying exactly two signs in a triangle $T$ produces a triangle isomorphic to $T$.

In a category equipped with a translation functor, we write $[n]$ for $[1]$ composed $n$ times for $n \geq 0$. We often fix a quasi-inverse $[-1]$ of $[1]$ and we write $[-n]$ for $[-1]$ composed $n$ times.

**Remark 2.2.13.** (TR2) is equivalent to (T2) and the following dual of (T2):

(T2') If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then the counterclockwise rotated triangle $Z[1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$ is distinguished.

Rotating a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ thrice, we get $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$. Thus (TR2) is equivalent to (T2) and the following condition: if $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$ is a distinguished triangle, then the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished.

**Theorem 2.2.14.** Let $\mathcal{A}$ be an additive category. We equip $K(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.4. We say that a triangle in $K(\mathcal{A})$ is distinguished if it is isomorphic to a standard triangle, namely a triangle of the form $X \xrightarrow{f} Y \xrightarrow{g} \text{Cone}(f) \xrightarrow{h} X[1]$, where $g$ and $h$ are the canonical morphisms. Then $K(\mathcal{A})$ is a triangulated category.

**Proof.** The first two points of (TR1) are clear from the definition of distinguished triangles. For any complex $X$, $0 \rightarrow X \xrightarrow{\text{id}_X} X \rightarrow 0[1]$ is a distinguished triangle as $X$ can be identified with the cone of $0 \rightarrow X$. The third point of (TR1) follows thus from (TR2).
For (T2), it suffices to show, for every morphism $f: X \to Y$ of complexes, that there exists a homotopy equivalence $g: X[1] \to \text{Cone}(i)$ such that the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{i} & \text{Cone}(f) \\
\downarrow & & \downarrow p \\
Y & \xrightarrow{i} & \text{Cone}(f) \\
\end{array}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
\downarrow g & & \downarrow g \\
\end{array}
$$

commutes in $K(A)$. Here $i, p, i', p'$ denote the canonical morphisms. We have $\text{Cone}(i)^n = Y^{n+1} \oplus \text{Cone}(f)^n \simeq Y^{n+1} \oplus X^{n+1} \oplus Y^n$ and

$$
d^n_{\text{Cone}(i)} = \begin{pmatrix}
-d_{Y^{n+1}} & 0 & 0 \\
0 & -d_{X^{n+1}} & 0 \\
id_{Y^n} & f^{n+1} & d_Y
\end{pmatrix},
$$

We define $g^n: X[1]^n \to \text{Cone}(i)^n$ and $g'^n: \text{Cone}(i)^n \to X[1]^n$ by

$$
g^n = \begin{pmatrix}
id_{X^{n+1}} \\
0
\end{pmatrix}, \quad g'^n = (0, id_{X^{n+1}}, 0).
$$

It is clear that $g, g'$ are morphism of complexes, and $g'g = id_{X[1]}, g'i' = p, p'g = -f[1]$. Moreover,

$$
\text{id}_{\text{Cone}(i)^n} - g^n g'^n = \begin{pmatrix}
id_{Y^{n+1}} & f^{n+1} & 0 \\
0 & 0 & 0 \\
0 & 0 & id_{Y^n}
\end{pmatrix} = h^n + d^n_{\text{Cone}(i)} + d^n_{\text{Cone}(i)} h^n,
$$

where $h^n = \begin{pmatrix}
0 & 0 & id_{Y} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$.

For (T2'), it suffices to show, for every morphism $f: X \to Y$ of complexes, that there exists a homotopy equivalence $g: Y \to \text{Cyl}(f)$ such that the diagram

$$
\begin{array}{ccc}
\text{Cone}(f)[-1] & \xrightarrow{-p} & X \\
\downarrow & & \downarrow f \\
\text{Cone}(f)[-1] & \xrightarrow{-p} & X \\
\end{array}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
Y & \xrightarrow{i} & \text{Cone}(f) \\
\downarrow g & & \downarrow g \\
\text{Cyl}(f) & \xrightarrow{i'} & \text{Cone}(f) \\
\end{array}
$$

commutes in $K(A)$. Here $\text{Cyl}(f)$ denotes the mapping cone of $-p$, and is called the mapping cylinder of $f$. We have $\text{Cyl}(f)^n = \text{Cone}(f)[-1]^{n+1} \oplus X^n \simeq X^{n+1} \oplus Y^n \oplus X^n$ and

$$
d^n_{\text{Cyl}(f)} = \begin{pmatrix}
-d_{X^{n+1}} & 0 & 0 \\
0 & d_{Y} & 0 \\
id_{X^{n+1}} & d_{X} & d_{X}
\end{pmatrix}.$$
CHAPTER 2. DERIVED CATEGORIES AND DERIVED FUNCTORS

We define \( g : Y \to \text{Cyl}(Y) \) and \( g' : \text{Cyl}(f) \to Y \) by

\[
g^n = \begin{pmatrix} 0 \\ \text{id}_{Y^n} \\ 0 \end{pmatrix}, \quad g'^n = (0, \text{id}_{Y^n}, f^n).
\]

It is clear that \( g, g' \) are morphism of complexes, and \( g'g = \text{id}_Y, \; p'g = i, \; g'i' = f \). Moreover,

\[
\text{id}_{\text{Cyl}(Y)} - g^n g'^n = \begin{pmatrix} \text{id}_{X^{n+1}} & 0 & 0 \\ 0 & 0 & -f^n \\ 0 & 0 & \text{id}_X^n \end{pmatrix} = h^{n+1} d^n_{\text{Cyl}(Y)} + d^{n-1}_{\text{Cyl}(Y)} h^n,
\]

where \( h^n = \begin{pmatrix} 0 & 0 & -\text{id}_{X^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

For (TR4), we may assume that the three distinguished triangles are standard. Let

\[
i^n = \begin{pmatrix} \text{id}_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix}, \quad i'^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \text{id}_Z^n \end{pmatrix}, \quad i''^n = \begin{pmatrix} 0 & 0 \\ 0 & \text{id}_{Y^{n+1}} \\ 0 & 0 \end{pmatrix}.
\]

Then the diagram in (TR4) commutes in \( C(A) \). It remains to find a homotopy equivalence \( k : W \to \text{Cone}(i) \) such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i} & V \\
\downarrow & & \downarrow \\
U & \xrightarrow{i} & V \\
\end{array}
\begin{array}{ccc}
& \xrightarrow{i'} & W \\
& \downarrow & \downarrow k \\
& \xrightarrow{i''} & \text{Cone}(i) \\
\end{array}
\begin{array}{c}
\rightarrow U[1] \\
\end{array}
\]

commutes in \( K(A) \). Here \( j' \) and \( j'' \) are the canonical morphisms. We define \( k : W \to \text{Cone}(i) \) and \( k' : \text{Cone}(i) \to W \) by

\[
k^n = \begin{pmatrix} 0 \\ \text{id}_{Y^{n+1}} \\ 0 \\ 0 \\ \text{id}_Z^n \end{pmatrix}, \quad k'^n = \begin{pmatrix} 0 & \text{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & \text{id}_Z^n \end{pmatrix}.
\]

It is clear that \( k \) and \( k' \) are morphism of complexes and \( k'k = \text{id}_W, \; j''k = i'', \; k'j' = i' \). Moreover, \( \text{id}_{\text{Cone}(i)} - kk' = dl + ld \), where \( l^n = \begin{pmatrix} 0 & 0 & -\text{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

\[\square\]

Remark 2.2.15. By Remark 2.2.13 and using the isomorphism \( \text{Cone}(f) \cong \text{Cone}(-f[1])[-1] \), in the above proof of (TR2) it suffices to prove either (T2) or (T2').

Remark 2.2.16. Let \( f : X \to Y \) be a cellular map of CW complexes. The (topological) mapping cylinder \( \text{Cyl}(f) \) of \( f \) is obtained by gluing the base of the cylinder \( \text{Cyl}(X) \) of \( X \) to \( Y \) via \( f \). The mapping cylinder \( \text{Cyl}(C_\bullet(f)) \) can be identified with \( C_\bullet(\text{Cyl}(f)) \). The homotopy equivalence in the proof of (T2') mirrors the fact that \( Y \) is a deformation retract of \( \text{Cyl}(f) \).
Example 2.2.17. The stable homotopy category of spectra equipped with the suspension functor and the collection of (triangles isomorphic to) mapping cone triangles is a triangulated category.

Proposition 2.2.18. (TR1) and (TR4) imply the following property.

(TR3) Given a commutative diagram

\[
\begin{array}{cccccc}
X & \to^i & Y & \to^j & Z & \to^k & X[1] \\
\downarrow^f & & \downarrow^g & & \downarrow^h & & \downarrow^f[1] \\
X' & \to^{i'} & Y' & \to^{j'} & Z' & \to^{k'} & X'[1]
\end{array}
\]

in which both rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

Remark 2.2.19. The dotted arrow is not unique in general. This is the source of many troubles.

It seems that there is no known example of a category (equipped with a translation functor and a collection of distinguished triangles) satisfying (TR1), (TR2), and (TR3), but not (TR4) [N, Remark 1.3.15].

Proof of Proposition 2.2.18. By (TR1), we may extend \( gi = i'f \) to a distinguished triangle

\[
X \xrightarrow{gi} Y' \xrightarrow{j''} Z'' \xrightarrow{k''} X[1].
\]

Applying (TR1) to \( g \) and (TR4) to the distinguished triangles with bases \( g, i \), and \( gi \), we get a morphism \( Z \xrightarrow{h'} Z'' \) such that \( h'j = j''g \) and \( k = k''h' \). Similarly, applying (TR1) to \( f \) and (TR4) to the distinguished triangles with bases \( f, i' \), and \( gi \), we get \( Z'' \xrightarrow{h''} Z \) such that \( j' = h''j'' \) and \( f[1]k'' = k'h'' \). It suffices to take \( h = h''h' \).

Corollary 2.2.20. Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1] \) be a distinguished triangle. Then \( gf = 0 \).

Proof. By (TR1), \( X \xrightarrow{id_X} X \to 0 \to X[1] \) is a distinguished triangle. By (TR3), there exists a morphism \( 0 \to Z \) such that the diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{id_X} & X & \xrightarrow{0} & X[1] \\
\downarrow^f & & \downarrow^g & & \downarrow^{id_X[1]} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{X[1]} X[1]
\end{array}
\]

commutes. The commutativity of the square in the middle implies \( gf = 0 \).

Remark 2.2.21. Let \( D \) be a triangulated category. We endow \( D^{op} \) with the translation functor \([-1]\). We say that a triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1] \) in \( D^{op} \) is distinguished if \( Z \xrightarrow{h} Y \xrightarrow{f} X \xrightarrow{h[1]} Z[1] \) is a distinguished triangle in \( D \). Then \( D^{op} \) is a triangulated category.

The isomorphisms in Remark 2.1.5 induce an isomorphism of triangulated category \( K(A)^{op} \cong K(A^{op}) \).
Definition 2.2.22. Let $\mathcal{D}$ be a triangulated category and let $\mathcal{A}$ be an abelian category. An additive functor $H: \mathcal{D} \to \mathcal{A}$ is called a cohomological functor if for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$, the sequence $HX \xrightarrow{Hf} HY \xrightarrow{Hg} HZ$ is exact.

For a cohomological functor $H$, we write $H^n X$ for $H(X[n])$. Applying (TR2), we get a long exact sequence
$$\cdots \to H^n X \to H^n Y \to H^n Z \to H^{n+1} X \to \cdots.$$  

Example 2.2.23. For any abelian category $\mathcal{A}$, $H^n: K(\mathcal{A}) \to \mathcal{A}$ is a homological functor.

Proposition 2.2.24. Let $\mathcal{D}$ be a triangulated category. Let $W$ be an object of $\mathcal{D}$ and let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle. Then the sequences
$$\text{Hom}_\mathcal{D}(W, X) \to \text{Hom}_\mathcal{D}(W, Y) \to \text{Hom}_\mathcal{D}(W, Z),$$
$$\text{Hom}_\mathcal{D}(Z, W) \to \text{Hom}_\mathcal{D}(Y, W) \to \text{Hom}_\mathcal{D}(X, W)$$
are exact.

If $\mathcal{D}$ has small Hom sets, then the proposition means that the functors $\text{Hom}_\mathcal{D}(W, -): \mathcal{D} \to \text{Ab}$, $\text{Hom}_\mathcal{D}( -, W): \mathcal{D}^{\text{op}} \to \text{Ab}$ are cohomological functors.

Proof. Let us show that the first sequence is exact, the other case being similar. Since $gf = 0$, the composition is zero. Thus it suffices to show that for $j: W \to Y$ satisfying $gj = 0$, there exists $i: W \to X$ such that $j = fi$. Applying (TR1), (T2) (see Remark 2.2.11), (TR3), we get the following commutative diagram
$$\begin{array}{ccc}
W & \xrightarrow{0} & W[1] \\
\downarrow{j} & & \downarrow{\text{id}_{W[1]}} \\
Y & \xrightarrow{g} & Z \\
\downarrow{i[1]} & & \downarrow{j[1]} \\
X[1] & \xrightarrow{-f[1]} & Y[1].
\end{array}$$

Corollary 2.2.25. Let
$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
X' & \xrightarrow{h} & Z' \\
\downarrow{f[1]} & & \downarrow{f[1]} \\
X'[1]
\end{array}$$
be a morphism of distinguished triangles. If $f$ and $g$ are isomorphisms, so is the third one.

Thus triangles extending a morphism $X \to Y$ are unique up to non-unique isomorphisms.
2.2. HOMOTOPY CATEGORY, TRIANGULATED CATEGORIES

Proof. Let $W$ be any object of the triangulated category. Then we have a commutative diagram

\[
\begin{array}{cccccccc}
\text{Hom}(W, X) & \rightarrow & \text{Hom}(W, Y) & \rightarrow & \text{Hom}(W, Z) & \rightarrow & \text{Hom}(W, X[1]) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{Hom}(W, f) & \rightarrow & \text{Hom}(W, g) & \rightarrow & \text{Hom}(W, h) & \rightarrow & \text{Hom}(W, f[1]) & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{Hom}(W, X') & \rightarrow & \text{Hom}(W, Y') & \rightarrow & \text{Hom}(W, Z') & \rightarrow & \text{Hom}(W, X'[1]) & \rightarrow \\
\end{array}
\]

with exact rows. By the five lemma, $\text{Hom}(W, h)$ is an isomorphism. Therefore $h$ is an isomorphism by Yoneda’s lemma.

\[\square\]

Remark 2.2.26. We can now show that (TR1), (T2), and (TR3) imply (TR2) (see Remark 2.2.11). Indeed, Corollary 2.2.25 holds under these axioms. Let $Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y$ be a distinguished triangle. By (TR1), there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$. By (T2), $X \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]}$ and $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]}$ are distinguished triangles, hence are isomorphic by the corollary. It follows that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is isomorphic to the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$, hence is a distinguished triangle by (TR1).

Corollary 2.2.27. In a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$, $f$ is an isomorphism if and only if $Z$ is a zero object.

Proof. Applying Corollary 2.2.25 to the diagram (2.2.2), we see that $f$ is an isomorphism if and only if $h$ is an isomorphism.

\[\square\]

Triangulated functors

Definition 2.2.28. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories. A triangulated functor consists of the following data:

1. An additive functor $F: \mathcal{D} \rightarrow \mathcal{D}'$.
2. A natural isomorphism $\phi_X: F(X[1]) \simeq (FX)[1]$ of functors $\mathcal{D} \rightarrow \mathcal{D}'$.

These data are subject to the condition that $F$ carries distinguished triangles in $\mathcal{D}$ to distinguished triangles in $\mathcal{D}'$. That is, for any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, in $\mathcal{D}$, $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\phi_X \circ Fh} (FX)[1]$ is a distinguished triangle in $\mathcal{D}'$.

Let $(F, \phi), (F', \phi'): \mathcal{D} \rightarrow \mathcal{D}'$ be triangulated functors. A natural transformation of triangulated functors is a natural transformation $\alpha: F \rightarrow F'$ such that the following diagram commutes for all $X$:

\[
\begin{array}{ccc}
F(X[1]) & \xrightarrow{\phi_X} & (FX)[1] \\
\downarrow & & \downarrow \alpha(X[1]) \\
F'(X[1]) & \xrightarrow{\phi'_X} & (F'X)[1].
\end{array}
\]

Triangulated functors from $\mathcal{D}$ to $\mathcal{D}'$ form an additive category $\text{TrFun}(\mathcal{D}, \mathcal{D}')$. 
Example 2.2.29. Let $F: A \to B$ be an additive functor between additive categories. Then $F$ extends to an additive functor $C(F): C(A) \to C(B)$ with $(C(F)X)^n = F(X^n)$, which factorizes through a triangulated functor $K(F): K(A) \to K(B)$.

Remark 2.2.30. A composition of triangulated functors is a triangulated functor. If $F: D \to D'$ is a triangulated functor and $H: D' \to A$ is a cohomological functor, then $HF: D \to A$ is a cohomological functor.

Definition 2.2.31. Let $D$ be a triangulated category. A triangulated subcategory of $D$ consists of a subcategory $D'$ of $D$, stable under $[1]$, and a class of distinguished triangles such that $D'$ is a triangulated category and the inclusion functor $D' \to D$ is a triangulated functor.

Remark 2.2.32. Let $D'$ be a full triangulated subcategory of $D$. Then a triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $D'$ is a distinguished triangle in $D'$ if and only if it is a distinguished triangle in $D$. Indeed, the “only if” part is trivial and for the “if” part, note that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z'[1] \to X[1]$ in $D'$, and it then suffices to apply (TR3) and Corollary 2.2.25.

It follows that a full triangulated subcategory is determined by its set of objects. A nonempty set $D'$ of objects of $D$, stable under $[1]$, spans a full triangulated subcategory if and only if for every morphism $g: Y \to Z$ with $Y$ and $Z$ in $D'$, there exists a distinguished triangle $X \to Y \to Z \to X[1]$ with $X$ in $D'$. The only non-trivial point here is that in the “if” part, $D'$ is stable under $[-1]$ and direct sums up to isomorphisms in $D$. For this we use the fact that $Y \to 0 \to Y[1] \xrightarrow{\text{id}} Y[1]$ and $X \oplus Y \to X \to Y[1] \xrightarrow{\text{id}} Y[1]$ are distinguished triangles for $X$ and $Y$ in $D$ (Exercise).

Example 2.2.33. Let $A$ be an additive category. Then $K^+(A)$, $K^-(A)$, $K^b(A)$ are full triangulated subcategories of $K(A)$. On the other hand, for a nonempty interval $I \subset \mathbb{Z}$, $K^I(A)$ is not stable under translation unless $A$ has no nonzero objects.

2.3 Localization of categories

Proposition 2.3.1. Let $C$ be a category and let $S$ be a collection of morphisms. Then there exists a category $C[S^{-1}]$ and a functor $Q: C \to C[S^{-1}]$ such that

1. For any $s \in S$, $Q(s)$ is an isomorphism.
2. For any functor $F: C \to D$ such that $F(s)$ is invertible for all $s \in S$, there exists a unique functor $G: C[S^{-1}] \to D$ such that $F = GQ$.

Note that in (2) we require an equality of functors, not just natural isomorphism. The pair $(C[S^{-1}], Q)$ is clearly unique up to unique isomorphism (not just equivalence). We call $C[S^{-1}]$ the localization of $C$ with respect to $S$.

Proof. Let $\text{Ob}(C[S^{-1}]) = \text{Ob}(C)$. Consider diagrams in $C$ of the form

$$\cdots \to \cdots \xrightarrow{f} \cdots \to \cdots \xrightarrow{g} \cdots$$

such that $Q(f)$ and $Q(g)$ are isomorphisms.
where each $\leftarrow$ represents an element of $S$. More formally, such a diagram is a finite sequence $(f_i) = f_n \cdots f_0$, $f_i \in T = \text{Mor}(C) \bigsqcup S$, with source($f_{i+1}$) = target($f_i$). Here

$$\begin{align*}
\text{source}(\alpha(f)) &= \text{source}(f), & \text{target}(\alpha(f)) &= \text{target}(f), \\
\text{source}(\beta(s)) &= \text{target}(s), & \text{target}(\beta(s)) &= \text{source}(s),
\end{align*}$$

where $\alpha: \text{Mor}(C) \to T$, $\beta: S \to T$ are the inclusions. We adopt the convention that a sequence of length zero is uniquely determined by an object $X$ and we write $i_X$ for the sequence. Consider the equivalence relation stable under concatenation generated by the following relations:

- $\alpha(fg) \sim \alpha(f)\alpha(g)$, $\alpha(\text{id}_X) \sim i_X$.

- For $s: X \to Y$ in $S$, $\beta(s)\alpha(s) \sim \text{id}_X$ and $\alpha(s)\beta(s) \sim \text{id}_Y$.

We define morphisms of $C$ in $S$ as we shall see, in most of our applications, the sets of morphisms belong to $S$. That is, if it satisfies (M1), (M2), and (M3), then $S$ is a left multiplicative system in $C_{op}$. That is, if it satisfies (M1), (M2'), and (M3'), where (M2') and (M3') are (M2) and (M3) with all arrows reverted. We say that $S$ is a multiplicative system if it is both a left multiplicative system and a right multiplicative system.

For any collection $S$ of morphisms in a category $C$, and for any object $Y$ of $C$, we let $S_Y$ denote the full subcategory of $C_Y$ consisting of $(Y, s: Y \to Y')$ with $s$ in $S$. An morphism from $(Y', s)$ to $(Y'', t)$ is a morphism $f: Y' \to Y''$ in $C$ such that $t = fs$. Dually, for any object $X$ of $C$, we let $S_X$ denote the full subcategory of $C_{/X}$ consisting of $(X', s: X' \to X)$ with $s$ in $S$.

**Proposition 2.3.4.** Let $S$ be a right multiplicative system in a category $C$. 

---

**Remark 2.3.2.** If $C$ is a small category, then $C[S^{-1}]$ is a small category. If the Hom sets of $C$ are small, then the Hom sets of $C[S^{-1}]$ is not small in general. However, as we shall see, in most of our applications, the Hom sets of $C[S^{-1}]$ are small.
(1) The category $S_{Y/}$ is filtered for any object $Y$ of $C$.

(2) Moreover, for objects $X$ and $Y$ of $C$, the map

$$\text{colim}_{(Y', s) \in S_{Y/}} \hom_C(X, Y') \to \hom_C[S^{-1}](X, Y)$$

carrying a map $f : X \to Y'$ indexed by $s : Y \to Y'$ to $s^{-1}f$ is a bijection.

This property is sometimes summarized as follows: $C[S^{-1}]$ admits a calculus of left fractions [GZ, Section I.2]. A morphism from $X$ to $Y$ in $C[S^{-1}]$ is an equivalence class of diagrams of the form

$$X \overset{f}{\longrightarrow} Z \overset{g}{\longleftarrow} Y$$

in $C$ with $s \in S$, sometimes called “right roofs” [GM, Remark III.2.9] (“left roofs” for certain authors). Two such diagrams $(f, Z, s)$, $(f', Z', s')$ are equivalent if there exists a third diagram $(f'', Z'', s'')$ and a commutative diagram

\[
\begin{array}{ccc}
X & & Z \\
\downarrow f & & \downarrow s \\
\downarrow f'' & & \downarrow s'' \\
Y & & Z'.
\end{array}
\]

Proof. (1) Let $s : Y \to Y'$, $s' : Y \to Y''$ be in $S$, defining objects $(Y', s)$, $(Y'', s')$ of $S_{Y/}$. Applying (M2), we get $t : Y' \to Z$, $g : Y'' \to Z$ in $C$ with $t$ in $S$ such that $ts = gs'$. By (M1), $ts$ is in $S$. Then $t$ and $g$ define morphisms $(Y', s) \to (Z, ts)$ and $(Y'', s') \to (Z, ts)$ in $S_{Y/}$, respectively. Now let $u, v : (Y', s) \to (Y'', s')$ be morphisms in $S_{Y/}$. Then $s' = us = vs$. Applying (M3), we get $w : Y'' \to W$ in $S$ with $wu = wv$. By (M1), $ws'$ is in $S$, so that $w$ defines a morphism $(Y'', s') \to (W, ws')$ in $S_{Y/}$. Therefore, $S_{Y/}$ is filtered.

(2) We define a category $D$ by $\text{Ob}(D) = \text{Ob}(C)$ and

$$\hom_D(X, Y) = \text{colim}_{Y' \in S_{Y/}} \hom_C(X, Y').$$

Given $(f, Y', s) : X \to Y$, we let $[f, Y', s]$ denote its equivalence class in the colimit. Composition is defined as follows. Given $(f, Y', s) : X \to Y$ and $(g, Z', t) : Y \to Z$, we apply (M2) to $g$ and $s$ to get the commutative diagram

$$\begin{array}{ccc}
X & & Z'' \\
\downarrow f & & \downarrow s' \\
Y' & & Z' \\
\downarrow s & & \downarrow t \\
Y & & Z.
\end{array}$$

with $s'$ in $S$ and we set $[g, Z', t][f, Y', s] = [g'f, Z'', s't]$, where $s't$ is in $S$ by (M1). It is easy to check that this does not depend on the choices of $(g', Z'', s')$. Indeed, if
(g'', Z'', s'') is another choice, then by (M2) applied to s' and s'' we get

\[ \text{is}' = \text{is}' \]

with \( i: Z'' \to V, i': Z'' \to V, i \in S \). Since \( \text{id}' g = \text{id}' g = \text{id}' g = \text{id}' g \), applying (M3) to \( (i'g, i'g) \), we get \( v: V \to V' \) in \( S \) such that \( v g = v g \). Thus we get

\[ (g', Z'', s') \sim (v i' g, V', v i' s') \sim (g'', Z'', s''), \]

so that \( (g' f, Z''', s't) \sim (g'' f, Z'', s't) \). The identity \( X \to X \) is given by \( (\text{id}_X, X, \text{id}_X) \). To check the associativity of the composition, we apply (M2) to get the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{s} & & \downarrow{g} \\
Z & \xrightarrow{i} & W
\end{array}
\]

Consider the functor \( F: C \to D \) carrying \( X \to X \) and \( f: X \to Y \) to \([f, Y, \text{id}_Y] \).

It remains to show that the pair \((D, F)\) solves the same universal problem as \((C[S^{-1}], Q)\). For \( s: X \to Y \) in \( S \), \( F(s) = [s, Y, \text{id}_Y] \) has an inverse given by \([\text{id}_Y, Y, s] \).

For any functor \( F': C \to D' \) such that \( F'(s) \) is invertible for all \( s \) in \( S \), we define \( G: D \to D' \) by \( G([f, Y', s]) = F'(s)^{-1} F'(f) \). Note that for a morphism \((f, Y', s) \to (f', Y'', s') \), we have \( F'(s)^{-1} F'(f) = F'(s')^{-1} F'(f') \). Thus the definition of \( G \) does not depend on the choice of \((f, Y', s) \). Moreover, \( F' = GF \). The uniqueness of \( G \) is clear.

**Remark 2.3.5.** Dually, if \( S \) is a left multiplicative system, then \( S_X^\text{op} \) is a filtered category for any object \( X \) and the map

\[
\text{colim}_{(X', Y) \in S_X^\text{op}} \text{Hom}_C(X', Y) \to \text{Hom}_{C[S^{-1}]}(X, Y)
\]

is a bijection for objects \( X \) and \( Y \) in \( C \). If \( S \) is a multiplicative system, then the map

\[
\text{colim}_{(X', Y') \in S_X^\text{op} \times S_Y^\text{op}} \text{Hom}_C(X', Y') \to \text{Hom}_{C[S^{-1}]}(X, Y)
\]

is a bijection for objects \( X \) and \( Y \) in \( C \).

**Remark 2.3.6.** Coproducts seldom commute with products in \( \text{Set} \). For nonempty sets \( X, X', Y, Y' \), the canonical map \( (X \times X') \coprod (Y \times Y') \to (X \coprod Y) \times (X' \coprod Y') \) is not a bijection.

**Remark 2.3.7.** If \( S \) is a right multiplicative system and \( C \) admits finite coproducts, then \( C[S^{-1}] \) admits finite coproducts and the localization functor \( Q: C \to C[S^{-1}] \) preserves finite coproducts by the proposition.

If \( S \) is a multiplicative system of an additive category \( A \), then \( A[S^{-1}] \) is an additive category and the localization functor \( Q: A \to A[S^{-1}] \) is an additive functor.
Localization of triangulated categories

Let \( \mathcal{D} \) be a triangulated category and let \( \mathcal{N} \) be a full triangulated subcategory. We let \( S_{\mathcal{N}} \) denote the collection of morphisms \( f: X \to Y \) in \( \mathcal{D} \) such that there exists a distinguished triangle \( X \xrightarrow{f} Y \to Z \to X[1] \) in \( \mathcal{D} \) with \( Z \) in \( \mathcal{N} \).

**Proposition 2.3.8.** \( S_{\mathcal{N}} \) is a multiplicative system.

**Proof.** We check the axioms of a right multiplicative system.

(M1) Since there exists a zero object in \( \mathcal{N} \), \( \text{id}_X \) is in \( S_{\mathcal{N}} \) by (TR1). Let \( f: X \to Y \), \( g: Y \to Z \) be in \( S_{\mathcal{N}} \). There exist distinguished triangles \( X \xrightarrow{f} Y \to U \to X[1] \), \( Y \xrightarrow{g} Z \to W \to Y[1] \) with \( U \) and \( W \) in \( \mathcal{N} \). By (TR1), there exists a distinguished triangle \( X \xrightarrow{gf} Z \to V \to X[1] \). By (TR4), there exists a distinguished triangle \( U \to V \to W \to U[1] \). Thus \( V \) is isomorphic to an object of \( \mathcal{N} \). By (TR2), it follows that \( gf \) is in \( S_{\mathcal{N}} \).

(M2) Let \( f: X \to Y \), \( s: X \to X' \) with \( s \) in \( S_{\mathcal{N}} \). By (TR2), there exists a distinguished triangle \( Z \xrightarrow{g} X \xrightarrow{g} X' \to Z[1] \) with \( Z \) in \( \mathcal{N} \). Applying (TR1) and (TR3), we obtain a morphism of distinguished triangles

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow{id}_Z & & \downarrow{f} \\
Z & \xrightarrow{fg} & Y \\
\end{array}
\begin{array}{ccc}
X & \xrightarrow{s} & X' \\
\downarrow{id}[1] & & \downarrow{id} \\
X' & \xrightarrow{tf'} & Z'[1].
\end{array}
\]

By (TR2), \( t \) is in \( S_{\mathcal{N}} \).

(M3) It suffices to show that for \( f: X \to Y \) such that there exists \( s: W \to X \) in \( S_{\mathcal{N}} \) with \( fs = 0 \), there exists \( t: Y \to Z \) in \( S_{\mathcal{N}} \) such that \( tf = 0 \). We have a distinguished triangle \( W \xrightarrow{g} X \xrightarrow{g} X' \to W[1] \) with \( X' \) in \( \mathcal{N} \). By the long exact sequence of \( \text{Hom}_\mathcal{D} \), there exists \( f': X' \to Y \) such that \( f = f'g \). By (TR1), we get a distinguished triangle \( X' \xrightarrow{f'} Y \xrightarrow{f} Z \to X'[1] \). By (TR2), \( t \) is in \( S_{\mathcal{N}} \). Moreover, \( tf' = 0 \) so that \( tf = 0 \).

Similarly one shows that \( S_{\mathcal{N}} \) is a left multiplicative system. \( \square \)

We write \( \mathcal{D}/\mathcal{N} = \mathcal{D}[S_{\mathcal{N}}^{-1}] \). Let \( Q: \mathcal{D} \to \mathcal{D}/\mathcal{N} \) be the localization functor. The composition \( \mathcal{D} \xrightarrow{[1]} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{N} \) carries \( S_{\mathcal{N}} \) to isomorphisms, thus factors uniquely through a functor \([1]: \mathcal{D}/\mathcal{N} \to \mathcal{D}/\mathcal{N} \). We say that a triangle in \( \mathcal{D}/\mathcal{N} \) is distinguished if it is isomorphic to the image of a distinguished triangle in \( \mathcal{D} \) under \( Q \).

**Proposition 2.3.9.** \( \mathcal{D}/\mathcal{N} \) is a triangulated category. For any object \( X \) of \( \mathcal{N} \), \( QX \) is a zero object. Moreover, for any triangulated functor \( F: \mathcal{D} \to \mathcal{D}' \) such that \( FX \simeq 0 \) for every object \( X \) of \( \mathcal{N} \), there exists a unique functor \( G: \mathcal{D}/\mathcal{N} \to \mathcal{D}' \) such that \( F = GQ \).

We call \( \mathcal{D}/\mathcal{N} \) the quotient category of \( \mathcal{D} \) by \( \mathcal{N} \).

**Proof.** The axioms for \( \mathcal{D}/\mathcal{N} \) follow from the axioms for \( \mathcal{D} \). (The implication for (TR4) is not completely trivial, because (TR4) is applicable to the liftings of the three distinguished triangles only up to isomorphisms.) The other assertions follow from the fact that, for any triangulated functor \( F: \mathcal{D} \to \mathcal{D}' \), \( FX \simeq 0 \) for every object \( X \) of \( \mathcal{N} \) if and only if \( Ff \) is an isomorphism for every morphism \( f \) in \( S_{\mathcal{N}}^{-1} \). \( \square \)
2.4. DERIVED CATEGORIES

**Definition 2.3.10.** A full triangulated subcategory \( N \) of a triangulated category \( D \) is said to be thick ("saturated" in the terminology of Verdier’s thesis [V2, II.2.1.6]) if it is stable under direct summand in \( D \).

**Remark 2.3.11.** Let \( F: D \to D' \) be a triangulated functor. We let \( \text{Ker}(F) \) denote the kernel of \( F \), namely the full subcategory of \( D \) spanned by objects \( X \) such that \( FX \cong 0 \). Then \( \text{Ker}(F) \) is a thick subcategory of \( D \). Moreover \( F \) can be decomposed as \( D \xrightarrow{Q} D/\text{Ker}(F) \xrightarrow{G} D' \), where \( \text{Ker}(G) \) is spanned by zero objects.

**Remark 2.3.12.** Let \( N \) be a full triangulated subcategory of \( D \). It was shown in Verdier’s thesis [V2, Corollaire II.2.2.11] (and independently by Rickard) that the following conditions are equivalent:

1. \( N \) is thick.
2. \( N = \text{Ker}(Q: D \to D/N) \) (namely, an object \( X \) of \( D/N \) is zero if and only if \( X \) is in \( N \)).

It follows that in general \( \text{Ker}(Q: D \to D/N) \) is the smallest thick subcategory of \( D \).

2.4 Derived categories

Let \( \mathcal{A} \) be an abelian category. We let \( N(\mathcal{A}) \) denote the full subcategory of \( K(\mathcal{A}) \) consisting of acyclic complexes. Then \( N(\mathcal{A}) \) is a triangulated subcategory of \( K(\mathcal{A}) \).

**Definition 2.4.1.** We call \( D(\mathcal{A}) = K(\mathcal{A})/N(\mathcal{A}) \) the derived category of \( \mathcal{A} \).

Thus \( D(\mathcal{A}) \) is a triangulated category. By definition, \( D(\mathcal{A}) = K(\mathcal{A})[S^{-1}] \), where \( S \) is the collection of quasi-isomorphisms in \( K(\mathcal{A}) \). Objects of \( D(\mathcal{A}) \) are complexes in \( \mathcal{A} \) and we have

\[
\text{Hom}_{D(\mathcal{A})}(X, Y) \cong \text{colim}_{(Y', s) \in S_Y} \text{Hom}_{\mathcal{C}}(X, Y') \cong \text{colim}_{(X', s) \in S_X} \text{Hom}_{\mathcal{C}}(X', Y).
\]

In general, \( D(\mathcal{A}) \) does not have small Hom sets, even if \( \mathcal{A} \) has small Hom sets. See however Remark 2.4.23 below. A triangle in \( D(\mathcal{A}) \) is distinguished if and only if it is isomorphic to a standard triangle \( \xrightarrow{f} Y \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} X[1] \).

The functors \( H^n: K(\mathcal{A}) \to \mathcal{A} \) carry quasi-isomorphisms to isomorphisms, hence induce cohomological functors

\[
H^n: D(\mathcal{A}) \to \mathcal{A}.
\]

For any distinguished triangle \( X \to Y \to Z \to X[1] \) in \( D(\mathcal{A}) \), we have a long exact sequence

\[
\cdots \to H^{n-1}Z \to H^nX \to H^nY \to H^nZ \to H^{n+1}X \to \cdots.
\]

The functors \( \tau^{\leq n}, \tau^{\geq n}: K(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A}) \) induce additive functors

\[
\tau^{\leq n}, \tau^{\geq n}: D(\mathcal{A}) \to D(\mathcal{A}).
\]

These are not triangulated functors unless all objects of \( \mathcal{A} \) are zero objects.
**Proposition 2.4.2.**  
(1) A morphism \( f : X \to Y \) in \( D(\mathcal{A}) \) is an isomorphism if and only if \( H^n f \) is an isomorphism for every \( n \).

(2) An object \( Z \) of \( D(\mathcal{A}) \) is a zero object if and only if \( H^n Z \) is a zero object for every \( n \).

**Proof.** By the long exact sequences of cohomology objects associated to distinguished triangles, the two assertions are equivalent. The “only if” parts are trivial. The “if” part of (2) is also trivial. \( \square \)

For any short exact sequence of complexes \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \), we get a distinguished triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1] \) in \( D(\mathcal{A}) \), where \( \delta = (0, g) : \text{Cone}(f) \to Z \) is the quasi-isomorphism (Proposition 2.1.19). By Remark 2.1.20 the long exact sequence associated to the distinguished triangle is

\[
H^n X \xrightarrow{H^n f} H^n Y \xrightarrow{H^n g} H^n Z \xrightarrow{\delta} H^{n+1} X,
\]

where \( \delta \) is the connecting morphism in the long exact sequence associated to the short exact sequence.

**Remark 2.4.3.** For a short exact sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{A} \), we get a morphism \( h : Z \to X[1] \) in \( D(\mathcal{A}) \). We have \( H^n h = 0 \) for every \( n \), but \( h \) is not the zero morphism unless the short exact sequence splits.

**Example 2.4.4.** Let \( X \) be a complex and let \( b \) be an integer. We have a short exact sequence of complexes

\[
0 \to \tau^{\leq b} X \to X \to \tau^{\leq b} X \to 0.
\]

Although \( X/\tau^{\leq b} X \) is in \( C^{\geq b}(\mathcal{A}) \) but not in \( C^{\geq b+1}(\mathcal{A}) \) in general, the morphism of complexes \( X/\tau^{\leq b} X \to \tau^{\geq b+1} X \) is a quasi-isomorphism. Thus we get a distinguished triangle in \( D(\mathcal{A}) \)

\[
\tau^{\leq b} X \to X \to \tau^{\geq b+1} X \to (\tau^{\leq b} X)[1].
\]

If \( a \leq b < c \) are integers (\( a \) could be \( -\infty \) and \( c \) could be \( +\infty \)), applying the above to \( \tau^{[a,c]} X \), we get a distinguished triangle in \( D(\mathcal{A}) \)

\[
\tau^{[a,b]} X \to \tau^{[a,c]} X \to \tau^{[b+1,c]} X \to (\tau^{[a,b]} X)[1].
\]

**Example 2.4.5.** More trivially the short exact sequence of complexes

\[
0 \to \sigma^{[b+1,c]} X \to \sigma^{[a,b]} X \to \sigma^{[a,c]} X \to 0
\]

induces a distinguished triangle

\[
\sigma^{[b+1,c]} X \to \sigma^{[a,b]} X \to \sigma^{[a,c]} X \to (\sigma^{[b+1,c]} X)[1]
\]

**Notation 2.4.6.** Let \( I \subseteq \mathbb{Z} \) be an interval. We let \( D^I(\mathcal{A}) \) denote the full subcategory of \( D(\mathcal{A}) \) consisting of complexes \( X \) such that \( H^n X = 0 \) for \( n \notin I \). We let \( D^+(\mathcal{A}) \) (resp. \( D^-(\mathcal{A}) \), resp. \( D^b(\mathcal{A}) \)) denote the full subcategory of \( D(\mathcal{A}) \) consisting of complexes \( X \) such that \( H^n X = 0 \) for \( n \leq 0 \) (resp. \( n \geq 0 \), resp. \( |n| \gg 0 \)).
The full subcategories $D^+(A), D^-(A), D^b(A)$ are triangulated subcategories of $D(A)$.

**Proposition 2.4.7.** The functor $H^0: D^{[0,0]}(A) \to A$ is an equivalence of categories.

**Proof.** Consider the functor $F: A \to D^{[0,0]}(A)$ carrying $A$ to a complex $X$ concentrated in degree 0 with $X^0 = A$. We have $H^0FA \simeq A$. For any complex $X$ concentrated in degree 0, $X \simeq \tau_{[0,0]}X \simeq FH^0X$. □

To give descriptions of $D^*(A)$ in terms of $K^*(A)$, we need a general result on colimits.

**Definition 2.4.8.** We say that a category $C$ is connected if for every pair of objects $X$ and $Y$, there exists a sequence of objects $X = X_0, \ldots, X_n = Y$ such that for each $0 \leq i \leq n - 1$, there exists either a morphism $X_i \to X_{i+1}$ or a morphism $X_{i+1} \to X_i$. We say that a functor $\phi: J \to I$ is cofinal if $(i \downarrow \phi)$ is nonempty and connected for every object $i$ of $I$. We say that a subcategory is cofinal if the inclusion functor is cofinal.

**Proposition 2.4.9.** Let $\phi: J \to I$ be a functor of categories. The following conditions are equivalent:

1. $\phi$ is cofinal.
2. For every functor $F: I \to C$, the functor $C_{F/} \to C_{F\phi/}$ carrying $(X, (f_i: Fi \to X))$ to $(X, (f_{i\phi(j)}: F\phi(j) \to X))$ is an isomorphism of categories.
3. For every functor $F: I \to C$ and every colimit diagram $(X, (f_i: Fi \to X))$ of $F$, $(X, (f_{i\phi(j)}: F\phi(j) \to X))$ is a colimit diagram of $F\phi$.

**Proof.**

1. $\Rightarrow$ 2. We construct an inverse $C_{F\phi/} \to C_{F/}$ as follows. Let $(X, (g_j: F\phi(j) \to X))$ be an object of $C_{F\phi/}$. For each object $i$ of $I$, and each object $x = (j, \alpha: i \to \phi(j))$ of $(i \downarrow \phi)$, we put $f_{i,x} = g_j F(\alpha): F(i) \to X$. For a morphism $\beta: x \to x' = (j', \alpha')$, $f_{i,x} = g_j F(\alpha) = g_{j'} F\phi(\beta) F(\alpha) = g_{j'} F(\alpha') = f_{i,x'}$.

Since $(i \downarrow \phi)$ is nonempty and connected, $f_{i,x}$ is independent of $x$ and we let $f_i$ denotes the common value. For a morphism $a: i \to i'$ in $I$, and for $x = (j, \alpha')$ in $(i' \downarrow \phi)$,$f_{i',F(a)} = g_j F(\alpha') F(a) = g_j F(\alpha'a) = f_i$ since $(j, \alpha'a)$ is in $(i \downarrow \phi)$. We have thus constructed an object $(X, (f_i: Fi \to X))$ of $C_{F/}$. This construction is clearly functorial and provides the inverse as claimed.

2. $\Rightarrow$ 3. This follows from the definition of colimit: a colimit diagram of $F$ is by definition an initial object of $C_{F/}$ and similarly for $F\phi$.

3. $\Rightarrow$ 1. (1) We take $C$ to be the category of sets in a universe for which the Hom sets of $I$ are small. Let $i$ be an object of $I$ and let $F = \operatorname{Hom}_I(i, -): I \to C$. The set colim $F = \operatorname{colim}_{i' \in I} \operatorname{Hom}_I(i, i')$ can be identified with the set of connected components of the category $C_{i/}$, which is a singleton because $C_{i/}$ admits the initial object $(i, \text{id}_i)$. By (3), colim $F\phi = \operatorname{colim}_{i' \in I} \operatorname{Hom}_I(i, \phi(j))$, which can be identified with the set of connected components of $(i \downarrow \phi)$, is a singleton. In other words, $(i \downarrow \phi)$ is connected. □
Example 2.4.10. If \( I \) admits a final object \( i \), then the functor \( \{ \ast \} \to I \) carrying \( \ast \) to \( i \) is cofinal. In this case, for any functor \( F : I \to C \), \( F(i) \) is a colimit of \( F \).

Proposition 2.4.11. Let \( I \) be a filtered category and let \( J \) be a full subcategory. Let \( \iota : J \to I \) be the inclusion functor. Then \( J \) is a cofinal subcategory of \( I \) if and only if for every object \( i \) of \( I \), there exist an object \( j \) in \( J \) and a morphism \( i \to j \). In this case, \( J \) is a filtered category.

The first assertion means the inclusion functor \( \iota \) is cofinal if and only if \( (i \downarrow \iota) \) is nonempty for every object \( i \) of \( I \).

Proof. The “only if” part of the first assertion is trivial. To show the “if” part of the first assertion, let \( (i, a : i \to j) \) and \( (i, a' : i \to j') \) be objects of \( (i \downarrow \iota) \). Since \( I \) is filtered, there exist an object \( k \) of \( I \) and morphisms \( b : j \to k \), \( b' : j' \to k \). Furthermore, since \( I \) is filtered, we may assume \( ba = b'a' \). By assumption, we may assume that \( k \) is in \( J \). Then we get morphisms \( (j, a) \xrightarrow{b} (k, ba) \xrightarrow{b'} (j', a') \) in \( (i \downarrow \iota) \).

Now let \( J \) be a cofinal full subcategory of \( I \). For \( j \) and \( j' \) in \( J \), there exist an object \( k \) in \( I \) and morphisms \( j \to k \), \( j' \to k \). By the cofinality of \( J \), we may assume that \( k \) is in \( J \). For \( j \Rightarrow j' \) in \( J \), there exists \( j' \to k \) in \( I \) equalizing the arrows. By the cofinality of \( J \), we may assume that \( k \) is in \( J \). \( \square \)

For \( Y \in D^+(A) \), where \( * \) is \( + \) or \( \geq n \), we let \( S^n_{Y/} \) denote the full subcategory of \( S^*_Y \) consisting of pairs \( (Y', s : Y \to Y') \) such that \( Y' \in K^+(A) \). For \( X \in D^+(A) \), where \( * \) is \( - \) or \( \leq n \), we let \( S^n_{/X} \) denote the full subcategory of \( S^*_Y \) consisting of pairs \( (X', s : X' \to X) \) such that \( X' \in K^+(A) \).

Proposition 2.4.12. (1) Let \( * \) be \( + \) or \( \geq n \). Let \( Y \) be a complex in \( D^+(A) \). Then \( S^n_{Y/} \) is a cofinal full subcategory of \( S^*_Y \).

(2) Let \( * \) be \( - \) or \( \leq n \). Let \( X \) be a complex in \( D^+(A) \). Then \( (S^n_{/X})^{op} \) is a cofinal full subcategory of \( (S^*_X)^{op} \).

Proof. We treat the case where \( * \) is \( \geq n \), the other cases being similar. Let \( Y \) be in \( K_{\geq n}^+(A) \) and let \( (Y', a : Y \to Y') \) be a quasi-isomorphism. Then \( f : Y' \to \tau^{-n}Y'' = Y'' \) provides a morphism \( (Y', a) \to (Y'', fa) \) in \( S^*_Y \) with \( (Y'', fa) \) in \( S^*_{Y/} \). \( \square \)

Corollary 2.4.13. Let \( * \) be \( + \), \( - \), or \( b \). The functor \( K^+(A) \to D^+(A) \) induces an equivalence of triangulated categories \( K^+(A)/N^+(A) \to D^+(A) \), where \( N^+(A) = N(A) \cap K^+(A) \).

Proof. The functor is clearly essentially surjective, and it suffices to show that the functor is fully faithful. We first treat the case \( * = + \). For \( X, Y \) in \( K^+(A) \), by the cofinality of \( S^*_{Y/} \) in \( S^*_Y \), we have

\[
\text{Hom}_{K^+(A)/N^+(A)}(X, Y) \simeq \text{colim} \text{Hom}_{K(A)}(X, Y')
\]

\[
\simeq \text{colim} \text{Hom}_{K(A)}(X, Y) \simeq \text{Hom}_{D^+(A)}(X, Y).
\]

The case \( * = - \) is similar.
Finally let \( * = b \). Let \( X, Y \in K^{[m,n]}(A) \). We construct an inverse of the map

\[
\text{Hom}_{K^b(A)/N^b(A)}(X, Y) \simeq \colim_{(X', Y') \in (S^b_{/A})^p \times S^b_{/A}} \text{Hom}_K(A)(X', Y') \\
\rightarrow \colim_{(X', Y') \in (S_{/A})^p \times S_{/A}} \text{Hom}_K(A)(X', Y') \simeq \colim_{(X', Y') \in (S_{/A}^n)^p \times S_{/A}^m} \text{Hom}_K(A)(X', Y') \\
\simeq \text{Hom}_{D^b(A)}(X, Y)
\]

For \( X' \in K^{\leq n}(A) \) and \( Y' \in K^{\geq m}(A) \), any morphism of complexes \( f: X' \rightarrow Y' \) factorizes as

\[ X' \rightarrow \tau^{\geq m} X' \overset{g}{\rightarrow} \tau^{\leq n} Y' \rightarrow Y'. \]

It is easy to check that \( f \mapsto g \) provides the inverse as claimed. \( \square \)

**Corollary 2.4.14.** Let \( X \) be an object of \( D^{\leq n}(A) \) and let \( Y \) be an object of \( D^{\geq n+1}(A) \). Then \( \text{Hom}_{D(A)}(X, Y) = 0 \).

**Proof.** Indeed, \( \text{Hom}_{D(A)}(X, Y) \simeq \colim_{(X', Y') \in (S_{/A}^n)^p \times S_{/A}^{n+1}} \text{Hom}_K(A)(X', Y') = 0. \) \( \square \)

**Corollary 2.4.15.** For \( X \) in \( D^{\leq n}(A) \), \( Y \) in \( D(A) \), \( Z \) in \( D^{\geq n}(A) \), the maps

\[ \text{Hom}_{D(A)}(X, \tau^{\leq n} Y) \rightarrow \text{Hom}_{D(A)}(X, Y), \quad \text{Hom}_{D(A)}(\tau^{\geq n} Y, Z) \rightarrow \text{Hom}_{D(A)}(Y, Z) \]

are isomorphisms.

Thus the functor \( \tau^{\leq n}: D(A) \rightarrow D^{\leq n}(A) \) is a right adjoint of the inclusion functor \( D^{\leq n}(A) \rightarrow D(A) \); the functor \( \tau^{\geq n}: D(A) \rightarrow D^{\geq n}(A) \) is a left adjoint of the inclusion functor \( D^{\geq n}(A) \rightarrow D(A) \).

**Proof.** The distinguished triangle \( \tau^{\leq n} Y \rightarrow Y \rightarrow \tau^{\geq n+1} Y \) induces a long exact sequence

\[ \text{Hom}(X, (\tau^{\geq n+1} Y)[-1]) \rightarrow \text{Hom}(X, \tau^{\leq n} Y) \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(X, \tau^{\geq n+1} Y). \]

Since \( \tau^{\geq n+1} Y \) and \((\tau^{\geq n+1} Y)[-1]\) are in \( D^{\geq n+1}(A) \), the first and fourth terms are zero. The first assertion follows. The proof of the second assertion is similar. \( \square \)

**Resolutions**

Let \( M \) be an \( R \)-module. Choosing a set of generator, we get an exact sequence \( F^0 \rightarrow M \rightarrow 0 \), where \( F^0 \) is a free \( R \)-module. Choosing a set of relations among the generators, we get an exact sequence \( F^{-1} \rightarrow F^0 \rightarrow M \rightarrow 0 \), where \( F^{-1} \) is a free \( R \)-module. Further choosing relations among the relations, we get, by induction, an exact sequence

\[ \cdots \rightarrow F^{-n} \rightarrow \cdots \rightarrow F^0 \rightarrow M \rightarrow 0, \]

where each \( F^i \) is a free \( R \)-module. Such an exact sequence induces a quasi-isomorphism \( F^\bullet \rightarrow M \), and is called a free resolution of \( M \).
Theorem 2.4.16. Let \( J \subseteq A \) be a full additive subcategory. Assume that for every object \( X \) of \( A \), there exists a monomorphism \( X \to Y \) with \( Y \in J \).

(1) For every \( K \in C_{\geq m}(A) \), there exist \( L \in C_{\geq n}(J) \) and a quasi-isomorphism \( f: K \to L \) such that \( \tau_{\geq m}f \) is a monomorphism of complexes for each \( m \).

(2) The functor \( K^+ (J) \to D^+ (A) \) induces an equivalence of triangulated categories

\[
K^+ (J) / N^+ (J) \to D^+ (A),
\]

where \( N^+ (J) = N(A) \cap K^+ (J) \).

In (1) the condition on \( \tau_{\geq m}f \) means that \( f \) is a monomorphism of complexes and the morphism \( K^i / B^i K \to L^i / B^i L \) induced by \( f \) is a monomorphism for each \( i \).

In (2) \( N^+ (J) \) is a thick triangulated subcategory of \( K^+ (J) \).

Proof. (1) It suffices to construct \( L_m = (\cdots \to L^m \to 0 \to \cdots) \in C_{[n,m]} (J) \) and a morphism \( f_m: K \to L_m \) of complexes for each \( m \) such that \( f_m^i \) and \( K^i / B^i K \to L^i / B^i L \) are monomorphisms for each \( i \leq m \). \( H^i f_m \) is an isomorphism for each \( i < m \), \( L_m = \sigma_{\leq m} L_{m+1} \) and \( f_m \) equals the composite \( K \to L_{m+1} \to L_m \). We proceed by induction on \( m \). For \( m < n \), we take \( L_m = 0 \). Given \( L_m \), we construct \( L_{m+1} \) as follows. Form the pushout square

\[
\begin{CD}
K^m / B^m K @>>> L^m / B^m L \\
\downarrow @. \downarrow \\
K^{m+1} @>>> X.
\end{CD}
\]

By induction hypothesis, the upper horizontal arrow is a monomorphism. It follows that we have a commutative diagram

\[
\begin{CD}
0 @>>> K^m / B^m K @>>> L^m / B^m L @>>> Z @>>> 0 \\
@. \downarrow @. \downarrow @. \downarrow \\
0 @>>> K^{m+1} @>>> X @>>> Z @>>> 0
\end{CD}
\]

with exact rows. By assumption, there exists a monomorphism \( X \to L^{m+1} \) with \( L^{m+1} \) in \( J \). We define \( f^{m+1}: K^{m+1} \to L^{m+1} \) and \( d^m_L : L^m \to L^{m+1} \) by the obvious compositions. Then \( f_{m+1} \) is a morphism of complexes. It is clear that \( f^{m+1} \) is a monomorphism. Applying the snake lemma to the above diagram, we see that \( K^{m+1} / B^{m+1} K \to L^{m+1} / B^{m+1} L \) is a monomorphism and \( H^m f \) is an isomorphism.

(2) This follows from (1) and the following lemma.

Lemma 2.4.17. Let \( K \) be a triangulated category and let \( N, J \) be full triangulated subcategories of \( K \), with \( \text{Ob}(N) \) stable under isomorphisms. Assume that for each \( X \in K \), there exists a morphism \( X \to Y \) in \( S_X \) such that with \( Y \in J \). Then the triangulated functor \( F: J / J \cap N \to K / N \) is an equivalence of triangulated categories.

Proof. By assumption, \( F \) is essentially surjective. Let us show that \( F \) is fully faithful. For \( K \in K \), the full subcategory \( (S^*_N)_K / \subseteq (S^*_N)_K / \) consisting of pairs \( (L, f: K \to L) \)
with \( L \in \mathcal{J} \) is cofinal. Moreover, since \( \text{Ob}(\mathcal{N}) \) is stable under isomorphisms, we have \( (S^2_{\mathcal{J}})_{K/} = (S^2_{\mathcal{J} \cap \mathcal{N}})_{K/} \). For \( X \in K \), \( F \) induces

\[
\text{Hom}_{\mathcal{J}/\mathcal{J} \cap \mathcal{N}}(X, K) \simeq \colim_{L \in (S^2_{\mathcal{J} \cap \mathcal{N}})_{K/}} \text{Hom}_K(X, L) \\
\simeq \colim_{L \in (S_{\mathcal{N}})_{K/}} \text{Hom}_K(X, L) \simeq \text{Hom}_{K/\mathcal{N}}(X, K).
\]

Dually we have the following.

**Theorem 2.4.18.** Let \( \mathcal{J} \subseteq \mathcal{A} \) be a full additive subcategory. Assume that for every object \( X \) of \( \mathcal{A} \), there exists an epimorphism \( Y \to X \) with \( Y \) in \( \mathcal{J} \).

1. For every \( K \in C^{\leq n}(\mathcal{A}) \), there exist \( L \in C^{\leq n}(\mathcal{J}) \) and a quasi-isomorphism \( f: L \to K \) such that \( \tau^{\leq m} f \) is an epimorphism of complexes for each \( m \).
2. The functor \( K^- (\mathcal{J}) \to D^- (\mathcal{A}) \) induces an equivalence of triangulated categories

\[
K^- (\mathcal{J})/N^- (\mathcal{J}) \to D^- (\mathcal{A}),
\]

where \( N^- (\mathcal{J}) = N(\mathcal{A}) \cap K^- (\mathcal{J}) \).

**Definition 2.4.19.** We say that \( \mathcal{A} \) admits enough injectives if for every object \( X \) of \( \mathcal{A} \), there exists a monomorphism \( X \to I \) with \( I \) injective. We say that \( \mathcal{A} \) admits enough projectives if for every object \( X \) of \( \mathcal{A} \), there exists an epimorphism \( P \to X \) with \( P \) projective.

**Corollary 2.4.20.** Let \( \mathcal{A} \) be an abelian category with enough injectives. We let \( I \) denote the full subcategory of \( \mathcal{A} \) consisting of injective objects. Then the triangulated functor \( K^+ (\mathcal{I}) \to D^+ (\mathcal{A}) \) is an equivalence of triangulated categories.

It follows that for \( X, I \in K^+ (\mathcal{I}) \), \( \text{Hom}_{K(\mathcal{A})}(X, I) \xrightarrow{\sim} \text{Hom}_{D(\mathcal{A})}(X, I) \). This extends to \( X \in K(\mathcal{A}) \) and for this the assumption on \( \mathcal{A} \) can be dropped by Propositions 2.4.24 and 2.4.26 below.

**Proof.** This follows from Theorem 2.4.16 and the lemma below. \( \square \)

**Lemma 2.4.21.** Let \( \mathcal{A} \) be an abelian category. We let \( \mathcal{I} \) denote the full subcategory of \( \mathcal{A} \) consisting of injective objects. Then \( N^+ (\mathcal{I}) \) is equivalent to zero.

**Proof.** Let \( L \in K^+ (\mathcal{I}) \) be an acyclic complex. Then \( L \) breaks into short exact sequences

\[
0 \to Z^n L \to L^n \to Z^{n+1} L \to 0.
\]

One shows by induction on \( i \) that \( Z^n L \) is injective and the sequence splits. Thus \( L \) is homotopy equivalent to 0. \( \square \)

Dually we have the following.

**Corollary 2.4.22.** Let \( \mathcal{A} \) be an abelian category with enough projectives. We let \( \mathcal{P} \) denote the full subcategory of \( \mathcal{A} \) consisting of projective objects. Then \( N^- (\mathcal{P}) \) is equivalent to zero and the triangulated functor \( K^- (\mathcal{P}) \to D^- (\mathcal{A}) \) is an equivalence of triangulated categories. Moreover, for \( L \in K(\mathcal{A}) \), \( P \in K^- (\mathcal{P}) \), \( \text{Hom}_{K(\mathcal{A})}(P, L) \xrightarrow{\sim} \text{Hom}_{D(\mathcal{A})}(P, L) \).
Remark 2.4.23. By the corollaries, if $\mathcal{A}$ has small Hom sets and admits enough injectives (resp. projectives), then $D^+(\mathcal{A})$ (resp. $D^-\mathcal{A}$) has small Hom sets.

Proposition 2.4.24. Let $\mathcal{A}$ be an abelian category. For any complex $I$, the following conditions are equivalent:

1. $\text{Hom}_{K(\mathcal{A})}(X, I) = 0$ for all $X \in N(\mathcal{A})$.
2. $\text{Hom}_{K(\mathcal{A})}(X, I) \rightarrow \text{Hom}_{D(\mathcal{A})}(X, I)$ is an isomorphism for all $X \in K(\mathcal{A})$.

Proof. (2) $\Rightarrow$ (1). Clear.

(1) $\Rightarrow$ (2). We have $\text{Hom}_{D(\mathcal{A})}(X, I) \cong \colim_{(X', s) \in (\mathcal{S}/x)^{op}} \text{Hom}_{K(\mathcal{A})}(X', I)$. Applying (1) to the cone of $s$, we see that $\text{Hom}_{K(\mathcal{A})}(s, I) : \text{Hom}_{K(\mathcal{A})}(X, I) \rightarrow \text{Hom}_{K(\mathcal{A})}(X', I)$ is an isomorphism.

Definition 2.4.25. A complex $I$ is said to be \textit{homotopically injective} if it satisfies the conditions of the above proposition.

We let $K_{hi}(\mathcal{A}) \subseteq K(\mathcal{A})$ denote the full subcategory spanned by homotopically injective complexes. The functor $K_{hi}(\mathcal{A}) \rightarrow D(\mathcal{A})$ is fully faithful.

Proposition 2.4.26. Let $\mathcal{I} \subseteq \mathcal{A}$ denote the full subcategory consisting of injective objects. Let $X \in N(\mathcal{A})$, $I \in K(\mathcal{I})$. Assume $X \in K^+$ or $I \in K^+$. Then $\text{Hom}_{K(\mathcal{A})}(X, I) = 0$. In particular, we have $K^+(\mathcal{I}) \subseteq K_{hi}(\mathcal{A})$.

Proof. Let $f : X \rightarrow I$ be a morphism of complexes. We construct a homotopy $h$ such that $f^n = h^{n+1}d^n_X + d^{n-1}h^n$ as follows. Assume $h^m$, $m \leq n$ constructed. We have

$$ (f^n - d^n h^n) d^n_X = d^{n-1} f^{n-1} - d^n h^{n-1} = 0. $$

Thus $f^n - d^n h^n$ factorizes via $d^n : X^n / B^n X \cong Z^{n+1} X$ through $g^n : Z^{n+1} X \rightarrow I^n$. We take $h^{n+1} : X^{n+1} \rightarrow I^n$ to be an extension of $g^n$.

Dually one defines homotopically projective complexes.

Grothendieck categories

Example 2.4.27. Let $R$ be a small ring. The abelian category $R\text{-Mod}$ admits enough injectives and enough projectives.

Example 2.4.28. Let $X$ be a small topological space. The category $\text{Shv}(X)$ is an abelian category with enough injectives, but not enough projectives in general.

To show that $\text{Shv}(X)$ admits enough injectives, consider, for every point $x \in X$, the stalk functor $i_x^* : \text{Shv}(X) \rightarrow \text{Ab}$ defined by $i_x^* F = \text{colim}_{U \in \text{Nbhd}(x)^{op}} F(U)$, where $\text{Nbhd}(x)$ is the partially ordered set of open neighborhoods of $x$. This functor admits a right adjoint $i_x : \text{Ab} \rightarrow \text{Shv}(X)$ defined by $(i_x A)(U) = A$ if $x \in U$ and $(i_x A)(U) = 0$ if $x \notin U$. Since $\text{Nbhd}(x)^{op}$ is filtered, the functor $i_x^*$ is exact. Moreover, $i_x$ is clearly exact. Let $F$ be a sheaf of abelian groups on $X$. We choose $i_x^* F \hookrightarrow I_x$ with $I_x$ injective for each $x \in X$. We have

$$ F \hookrightarrow \prod_x i_x^* F \hookrightarrow \prod_x i_x I_x = I. $$
Since $i_x^*$ is exact, the right adjoint $i_x$ preserves injectives, so that $I$ is injective.

One can show that if $X$ is a locally connected topological space, then $\text{Shv}(X)$ admits enough projectives if and only if $X$ is an Alexandrov space (namely, if any (infinite) intersection of open subsets is open).

**Definition 2.4.29.** Let $\mathcal{C}$ be a category and let $G$ be an object of $\mathcal{C}$. We say that $G$ is a generator of $\mathcal{C}$ if every morphism $f : X \to Y$ such that $\text{Hom}_\mathcal{C}(G,X) \to \text{Hom}_\mathcal{C}(G,Y)$ is a bijection is an isomorphism.

**Definition 2.4.30.** A Grothendieck category is an abelian category $\mathcal{A}$ with small Hom-sets admitting a generator and satisfying the following axiom:

(AB5) $\mathcal{A}$ admits small colimits and small filtered colimits are exact.

**Example 2.4.31.** $R$-$\text{Mod}$ and $\text{Shv}(X)$ are Grothendieck categories. The $R$-module $R$ is a generator of $R$-$\text{Mod}$. A generator of $\text{Shv}(X)$ is $\bigoplus U \mathbb{Z}$, where $U$ runs through open subsets of $X$, and $\mathbb{Z}_U$ is the presheaf carrying $V \subseteq U$ to $\mathbb{Z}$ and other $V$ to $0$. Indeed, $\text{Hom}_{\text{Shv}(X)}(\mathbb{Z}_U,F)$ $\simeq F(U)$.

For $R$ nonzero, $R$-$\text{Mod}^{\text{op}}$ is not a Grothendieck category. For $X$ nonempty, $\text{Shv}(X)^{\text{op}}$ is not a Grothendieck category. Indeed, sequential limit $\lim_{\text{op}}$ is not exact in $R$-$\text{Mod}$ or $\text{Shv}(X)$.

**Theorem 2.4.32** (Grothendieck [G, Théorème 1.10.1]). Grothendieck categories admit enough injectives.

**Theorem 2.4.33.** Let $\mathcal{A}$ be a Grothendieck category. There are enough homotopically injective complexes: for every complex $I$ in $\mathcal{A}$, there exists a quasi-isomorphism $I \to I'$ with $I'$ homotopically injective. In particular, the functor $K_{\text{hi}}(\mathcal{A}) \to D(\mathcal{A})$ is an equivalence of categories.

It follows from the theorem that $D(\mathcal{A})$ has small hom-sets when $\mathcal{A}$ does. We refer the reader to [KS2, Corollary 14.1.8] for a proof (of a generalization) of the theorem.

**Proposition 2.4.34.** An abelian category $\mathcal{A}$ admitting a generator and small colimits (for example, a Grothendieck category) satisfies

(AB3*) $\mathcal{A}$ admits small limits.

We refer the reader to [KS2, Proposition 5.2.8, Corollary 5.2.10] for a generalization of the proposition to categories that are not necessarily abelian categories.

**Lemma 2.4.35.** Let $\mathcal{C}$ be a category with small Hom-sets and admitting a generator $G$ and fiber products. Then for any object $X$, the set of subobjects of $X$ is small.

By a subobject of $X$, we mean an isomorphism class of objects $(Y,i)$ of $\mathcal{C}/X$ with $i$ a monomorphism.

**Proof.** Consider the map $\phi$ from the set of subobjects of $X$ to the set of subsets of $\text{Hom}(G,X)$ carrying a subobject $Y$ of $X$ to $\text{Im}(\text{Hom}(G,Y) \to \text{Hom}(G,X))$. Since $\text{Hom}(G,X)$ is a small set, it suffices to show that $\phi$ is an injection. If $\phi(Y) = \phi(Y')$, then $\text{Hom}(G,p_Y)$ and $\text{Hom}(G,p_{Y'})$ are bijections. Here $p_Y : Y \times_X Y' \to Y$ and $p_{Y'} : Y \times_X Y' \to Y'$ are the projections. Thus $p_Y$ and $p_{Y'}$ are isomorphisms. It follows that $Y$ equals $Y'$ as subobjects of $X$. 

\[\square\]
Lemma 2.4.36. Let $C$ be a category with small Hom-sets and admitting a generator $G$ and equalizers. Let $X$ be an object such that $G^{\text{IIHom}(G,X)}$ exists. Then the morphism $G^{\text{IIHom}(G,X)} \to X$ is an epimorphism.

Proof. Indeed, $\text{Hom}(G,-)$ is faithful and the map $\text{Hom}(G, G^{\text{IIHom}(G,X)}) \to \text{Hom}(G, X)$ is surjective.

Proof of Proposition 2.4.34. Let $F : I \to A$ be a small diagram. The category $A_{/F}$ admits small colimits and the forgetful functor $A_{/F} \to A$ preserves such colimits. We need to show that it admits a final object. By Theorem 1.4.16 applied to $(A_{/F})^{\text{op}}$, it suffices to find a small set of objects of $A_{/F}$ that is weakly final. For any object $X$ of $A$, $\text{Nat}(\Delta X, F)$ is a small set. Let $Z = G^{\text{IINat}(\Delta G,F)}$. There is an obvious morphism $\Delta Z \to F$. For any cone $u : \Delta X \to F$, form the pushout

$$
\begin{array}{ccc}
G^{\text{IIHom}(G,X)} & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

in $A$. The morphism $X \to Y$ lifts to a morphism with source $(X,u)$ in $A_{/F}$. By Lemma 2.4.36, the vertical arrow on the left is an epimorphism. It follows that the same holds for the vertical arrow on the right. Thus the set of cones $v : \Delta Y \to F$ with $Y$ a quotient of $Z$ is weakly final and it suffices to apply Lemma 2.4.35.

2.5 Extensions

Let $\mathcal{A}$ be an abelian category.

Notation 2.5.1. For $K, L \in D(\mathcal{A})$, the hyper Ext groups are defined to be

$$
\text{Ext}^n(K, L) = \text{Hom}_{D(\mathcal{A})}(K, L[n]).
$$

For $n = 0$, $\text{Ext}^0(K, L) = \text{Hom}_\mathcal{A}(K, L)$.

Let $X$ and $Y$ be objects of $\mathcal{A}$. For $n < 0$, $\text{Ext}^n(X, Y) = 0$.

For any short exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{A}$, we have short exact sequences

$$
0 \to \text{Hom}(X, Y') \to \text{Hom}(X, Y) \to \text{Hom}(X, Y'') \\
\to \text{Ext}^1(X, Y') \to \text{Ext}^1(X, Y) \to \text{Ext}^1(X, Y'') \to \ldots,
$$

$$
0 \to \text{Hom}(Y'', X) \to \text{Hom}(Y, X) \to \text{Hom}(Y', X) \\
\to \text{Ext}^1(Y'', X) \to \text{Ext}^1(Y, X) \to \text{Ext}^1(Y', X) \to \ldots,
$$

Remark 2.5.2. We have $\text{Ext}^n_{\mathcal{A}^{\text{op}}}(X, Y) \simeq \text{Ext}^n_{\mathcal{A}}(Y, X)$. 

Yoneda extensions

For \( n \geq 1 \), we will now give an interpretation of \( \text{Ext}^n(X, Y) \), which is due to Yoneda for \( n \geq 2 \).

**Definition 2.5.3.** Let \( n \geq 1 \). An \( n \)-extension of \( X \) by \( Y \) is an exact sequence

\[
0 \to Y \to K^{-n+1} \to \cdots \to K^0 \to X \to 0.
\]

An extension of \( Y \) by \( X \) is a 1-extension of \( Y \) by \( X \), namely, a short exact sequence

\[
0 \to Y \to K^0 \to X \to 0.
\]

A morphism of \( n \)-extensions of \( X \) by \( Y \) is a morphism of exact sequences inducing identities on \( X \) and on \( Y \).

By the snake lemma, a morphism of 1-extensions of \( X \) by \( Y \) is necessarily an isomorphism. This fails for \( n \)-extensions, \( n \geq 2 \) in general. We let \( E^n(X, Y) \) denote the set of equivalence classes of \( n \)-extensions of \( X \) by \( Y \), the equivalence relation being generated by morphisms.

We have seen how to produce a morphism \( X \to Y[1] \) in \( D(A) \) from an extension of \( X \) by \( Y \). More generally, given an \( n \)-extension of \( X \) by \( Y \) as above, we have a commutative diagram

\[
\begin{array}{c}
\cdots \to 0 \to 0 \to \cdots \to X \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \to Y \to K^{-n+1} \to \cdots \to K^0 \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \to Y \to 0 \to \cdots \to 0 \to \cdots
\end{array}
\]


giving morphisms of complexes \( X \xleftarrow{s} K \to Y[n] \). Note that \( s \) is a quasi-isomorphism, so that the morphisms induce a morphism \( X \to Y[n] \) in \( D(A) \). If \( E \to E' \) is a morphism of \( n \)-extensions, then \( E \) and \( E' \) induce the same morphism \( X \to Y[n] \).

We thus obtain a map \( \phi : E^n(X, Y) \to \text{Ext}^n(X, Y) \).

**Proposition 2.5.4.** The map \( \phi : E^n(X, Y) \to \text{Ext}^n(X, Y) \) is a bijection.

**Proof.** We construct an inverse \( \psi : \text{Ext}^n(X, Y) \to E^n(X, Y) \) as follows. Let \( X \xleftarrow{s} K \to Y[n] \) represent a morphism \( X \to Y[n] \) in \( D(A) \). For any quasi-isomorphism \( t : K' \to K \), we may replace \( (s, K, f) \) by \( (st, K', ft) \), without changing the class of \( (s, K, f) \). Conversely, if for a quasi-isomorphism \( t : K \to K' \), we have \( s = st \) and \( f = ft \), then we may replace \( (s, K, f) \) by \( (s', K', f') \). Thus, by truncation, we may assume that \( K \in K^{-n,0}(A) \). Then we have an exact sequence

\[
0 \to K^{-n} \to \cdots \to K^0 \xrightarrow{e} X \to 0.
\]

We will further modify \( K \) to make \( K^{-n} \) equal to \( Y \). Let \( L \) be \( Y \xrightarrow{\text{id}} Y \) put in degrees \(-n - 1\) and \(-n\). Consider the morphism of complexes \( g : Y[n] \to L \) given
by \( g^{-n} = \text{id}_Y \). Let \( K' = \text{Cone}(gf)[-1] \) and let \( p: K' \to K \) be the projection, which is a quasi-isomorphism since \( L \) is acyclic. We define a section \( t: K \to K' \) of \( p \) as follows. For \( i \neq -n \), let \( t' \) be the inclusion. Let \( t^{-n} = (\text{id}, -f^{-n}): K^{-n} \to K^{-n} \oplus Y \). Then \( t \) is a quasi-isomorphism. Let \( s' = sp: K' \to X \) and define \( f': K' \to Y[n] \) by minus the projection. Then we may replace \((s,K,f)\) by \((s',K',f')\). Let \( M \) be \( K^{-n} \xrightarrow{\text{id}} K^{-n} \) put in degrees \(-n\) and \(-n + 1\). Consider the morphism of complexes \( i: M \to K' \) given by the commutative diagram

\[
\begin{array}{cccccccccc}
\vdots & \rightarrow & 0 & \rightarrow & K^{-n} & \xrightarrow{\text{id}} & K^{-n} & \rightarrow & 0 & \rightarrow & \vdots \\
\downarrow & & \downarrow & & (\text{id},0) & & (d_{K}^{-n},-f^{-n}) & & \downarrow & & \\
\vdots & \rightarrow & 0 & \rightarrow & K^{-n} \oplus Y & \rightarrow & K^{-n+1} \oplus Y & \rightarrow & K^{-n+2} & \rightarrow & \vdots.
\end{array}
\]

We have \( s'i = 0 \) (the case \( n = 1 \) needs special attention) and \( f'i = 0 \). Let \( t': K' \to K'' = \text{Coker}(i) \). Then \( t' \) is a quasi-isomorphism since \( M \) is acyclic and \( i \) is a monomorphism of complexes. There exist \( s'': K'' \to X, f'': K'' \to Y[n] \) such that \( s' = s''t', f' = f''t' \). We may assume that \( K''_{-n} = Y \) with \( t^{-n} \) given by the projection. Then we have \( f'' = \text{id}_Y \). We define \( \psi([s,K,f]) \in E^n(X,Y) \) to be the class of

\[
0 \to Y = K''_{-n} \to \cdots \to K^0 \xrightarrow{s^0} X \to 0.
\]

Then \( \phi\psi([s,K,f]) = [s',K'',f'] = [s,K,f] \). Moreover, it is clear that \( \psi\phi(E) = E \).

The zero element of \( \text{Ext}^n(X,Y) \) corresponds to the class of the \( n \)-extension given by the direct sum of \( X \xrightarrow{\text{id}} X \) (put in degrees 0 and 1) and \( Y \xrightarrow{\text{id}} Y \) (put in degrees \(-n\) and \(-n + 1\)). In particular, \( 0 \in \text{Ext}^1(X,Y) \) corresponds to the class of split short exact sequences.

The group structure \( \text{Ext}^n(X,Y) \) can be described in terms of \( n \)-extensions.

**Remark 2.5.5.** Let \( X, Y, Z \) be objects of \( A \). For \( E \in E^n(X,Y), E' \in E^m(Y,Z) \), represented by

\[
0 \to Y \xrightarrow{d_{K}^{-n}} K^{-n+1} \to \cdots \to K^0 \to X \to 0,
\]

\[
0 \to Z \to K'^{-m+1} \to \cdots \to K^0 \xrightarrow{s^0} Y \to 0.
\]

We define \( E'' = E' \circ E \in E^{n+m}(X,Y) \) to be the class of the spliced exact sequence

\[
0 \to Z \xrightarrow{(-1)^nd_{K}^{-m}} K'^{-m+1} \to \cdots \xrightarrow{(-1)^n(d_{K}^{-1})} K^0 \xrightarrow{d_{K}^{-n}s^0} K^{-n+1} \to \cdots \to K^0 \to X \to 0.
\]

Then \( \phi(E' \circ E) \) is the composite \( X \xrightarrow{\phi(E)} Y[n] \xrightarrow{\phi(E')[n]} Z[n + m] \). Indeed, we have a commutative diagram

\[
\begin{array}{ccc}
K'' & \xrightarrow{g} & K' \noarrow{f} \noarrow{f[n]} \noarrow{Z[n + m]} \\
\downarrow t & & \downarrow s' \noarrow{Y[n]} \\
K & \xrightarrow{s} \noarrow{X} & K'[n] \noarrow{f'[n]} \noarrow{Z[n + m]}
\end{array}
\]
with \( s'' = st, f'' = f'[n]g \), where \( g^i = \text{id} \) for \(-m - n \leq i \leq -n\), \( t^i = \text{id} \) for \(-n + 1 \leq i \leq 0\) and \( t^{-n} = s'0 \).

We could define the composition \( E' \circ E \) without adding signs. To make it compatible with \( \phi \), we need to modify \( \phi \) by a factor of \((-1)^{n(n+1)}\) (or \((-1)^{n(n-1)}\)).

**Corollary 2.5.6.** Let \( X \) and \( Y \) be objects of \( \mathcal{A} \) and let \( m,n \geq 0 \) be integers. Every element \( e'' \in \text{Ext}^{n+m}(X,Y) \) has the form \( e'' = e'[n] \circ e \) for some object \( Z \) of \( \mathcal{A} \) and some \( e \in \text{Ext}^n(X,Z), e' \in \text{Ext}^m(Z,Y) \).

**Proof.** We may assume \( m,n \geq 1 \). An \((n + m)\)-extension of \( Y \) by \( X \)

\[
0 \to Y \to K^{-n-m+1} \to \cdots \to K^{-n} \xrightarrow{d^{-n}} K^{-n+1} \to \cdots \to X \to 0
\]

can be decomposed into exact sequences

\[
0 \to Y \to K^{-n-m+1} \to \cdots \to K^{-n} \to \text{Im}(d^{-n}) \to 0,
\]

\[
0 \to \text{Im}(d^{-n}) \to K^{-n+1} \to \cdots \to X \to 0.
\]

\[ \square \]

**Homological dimension**

**Proposition 2.5.7.** Let \( X \) be an object of \( \mathcal{A} \) and let \( m \geq 0 \) be an integer. The following conditions are equivalent:

1. \( \text{Ext}^m(X,Y) = 0 \) for every object \( Y \) of \( \mathcal{A} \).
2. \( \text{Ext}^n(X,Y) = 0 \) for every object \( Y \) of \( \mathcal{A} \) and every \( n \geq m \).

Dually, for an object \( Y \) of \( \mathcal{A} \), the following conditions are equivalent:

1. \( \text{Ext}^m(X,Y) = 0 \) for every object \( X \) of \( \mathcal{A} \).
2. \( \text{Ext}^n(X,Y) = 0 \) for every object \( X \) of \( \mathcal{A} \) and every \( n \geq m \).

**Proof.** That (2) implies (1) is trivial. That (1) implies (2) follows from Corollary 2.5.6.

**Definition 2.5.8.** Let \( X \) and \( Y \) be objects of \( \mathcal{A} \). The **projective dimension of \( X \)** and **injective dimension of \( Y \)** are defined to be

\[
\text{proj}.\dim(X) = \sup\{n \in \mathbb{Z} \mid \text{Ext}^n(X,Y) \neq 0 \text{ for some } Y\},
\]

\[
\text{inj}.\dim(Y) = \sup\{n \in \mathbb{Z} \mid \text{Ext}^n(X,Y) \neq 0 \text{ for some } X\}.
\]

The **homological dimension** of \( \mathcal{A} \) is defined to be

\[
\text{hom}.\dim(\mathcal{A}) = \sup\{n \in \mathbb{Z} \mid \text{Ext}^n(X,Y) \neq 0 \text{ for some } X,Y\}.
\]

The **left global dimension** of a ring \( R \) is defined to be

\[
1.\text{gl}.\dim(R) = \text{hom}.\dim(\text{Mod}-R).
\]

Dually, the **right global dimension** of a ring \( R \) is defined to be

\[
\text{r}.\text{gl}.\dim(R) = \text{hom}.\dim(\text{Mod}-R).
\]
We adopt the convention \( \sup \emptyset = -\infty \). The above dimensions take values in \( \mathbb{N} \cup \{ \pm \infty \} \). Proposition 2.5.7 gives equivalent conditions for \( \text{proj.dim}(X) < m \). By definition,

\[
\text{hom.dim}(\mathcal{A}) = \sup_{X \in \mathcal{A}} \text{proj.dim}(X) = \sup_{Y \in \mathcal{A}} \text{inj.dim}(Y).
\]

**Remark 2.5.9.** The following conditions are equivalent:

- \( X = 0 \);
- \( \text{proj.dim}(X) = -\infty \);
- \( \text{inj.dim}(X) = -\infty \).

**Remark 2.5.10.** The following conditions are equivalent:

- \( \text{proj.dim}(X) \leq 0 \);
- \( \text{Ext}^1(X, Y) = 0 \) for all \( Y \);
- \( X \) is projective.

Dually, the following conditions are equivalent:

- \( \text{inj.dim}(Y) \leq 0 \);
- \( \text{Ext}^1(X, Y) = 0 \) for all \( X \);
- \( Y \) is injective.

Moreover, \( \text{hom.dim}(\mathcal{A}) \leq 0 \) if and only if every short exact sequence in \( \mathcal{A} \) splits. Thus \( \text{l.gl.dim}(R) \leq 0 \) if and only if \( R \) is a semisimple ring.

**Proposition 2.5.11** (Dimension shifting). Let \( 0 \to X' \to P^{-k+1} \to \cdots \to P^0 \to X \to 0 \) be an exact sequence with \( P^n \) projective. Then \( \text{Ext}^n(X', Y) \simeq \text{Ext}^{n+k}(X, Y) \) for all \( Y \) and all \( n \geq 1 \). In particular,

\[
\max\{\text{proj.dim}(X'), 0\} = \max\{\text{proj.dim}(X) - k, 0\}.
\]

Moreover, if \( \text{proj.dim}(X) \geq k \), then we have

\[
\text{proj.dim}(X') = \text{proj.dim}(X) - k.
\]

**Proof.** For the first assertion, decomposing the exact sequence into short exact sequence, we reduce by induction to the case \( k = 1 \). In this case, the assertion follows from the long exact sequence

\[
0 = \text{Ext}^n(P^0, Y) \to \text{Ext}^n(X', Y) \to \text{Ext}^{n+1}(X, Y) \to \text{Ext}^{n+1}(P^0, Y) = 0.
\]

The first assertion implies that \( \text{proj.dim}(X') < n \) if and only if \( \text{proj.dim}(X) < n + k \) for all \( n \geq 1 \), hence (2.5.1). If \( \text{proj.dim}(X) \geq k \), then \( X' \) is nonzero (by the “if” part of Corollary 2.5.12 below), and (2.5.2) follows.

**Corollary 2.5.12.** If \( \mathcal{A} \) admits enough projectives, then \( \text{proj.dim}(X) \leq n \) if and only if there exists a projective resolution of \( X \) concentrated in \([-n, 0] \) (namely, an exact sequence \( 0 \to P^{-n} \to \cdots \to P^0 \to X \to 0 \) with \( P^i \) projective).

Note that the “if” part holds without the assumption on \( \mathcal{A} \).

**Proof.** The “if” part follows from (2.5.1):

\[
\text{proj.dim}(X) \leq \max\{\text{proj.dim}(P^{-n}), 0\} + n = n.
\]
For the “only if” part, by the assumption there exists an exact sequence $0 \to P^{-n} \to \cdots \to P^0 \to X \to 0$ with $P^i$ projective for $i \geq -n + 1$. It follows from \([2.5.1]\) that
\[
\text{proj.dim}(P^{-n}) \leq \max\{\text{proj.dim}(X) - n, 0\} = 0,
\]
namely $P^{-n}$ is projective.

\[\begin{proof}
\end{proof}\]

Corollary 2.5.13. If $\mathcal{A}$ admits enough projectives, then $\text{hom.dim}(\mathcal{A}) \leq 1$ if and only if all subobjects of projectives are projective. Dually if $\mathcal{A}$ admits enough injectives, then $\text{hom.dim}(\mathcal{A}) \leq 1$ if and only if all quotients of injectives are injective.

Example 2.5.14. For $\mathcal{A} = R\text{-Mod}$, we get that $\text{l.gl.dim}(R) \leq 1$ if and only if $R$ is left hereditary. Moreover, we obtain Proposition \[\text{1.8.31}\]

Dually $\text{r.gl.dim}(R) \leq 1$ if and only if $R$ is right hereditary. Since there are left hereditary rings that are not right hereditary, $\text{l.gl.dim}(R) \neq \text{r.gl.dim}(R)$ in general.

Proposition 2.5.15. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor preserving projectives. Then $\text{proj.dim}(FX) \leq \text{proj.dim}(X)$.

\[\begin{proof}
\end{proof}\]

Example 2.5.16. Let $R$ and $S$ be rings and let $R M_S$ be a bimodule such that $R M$ is projective. The functor $R M_S \otimes_S -: S\text{-Mod} \to R\text{-Mod}$ carries projectives to projectives (this follows either from the characterization of projective modules are direct summands of free modules or from the existence of an exact right adjoint $\text{Hom}_R(R M_S, -): R\text{-Mod} \to S\text{-Mod}$). If $M_S \otimes_S -$ is an exact functor\[3\] (for example, if $S$ is a projective right $R$-module), then $\text{proj.dim}_R(M \otimes_S N) \leq \text{proj.dim}_S(N)$ for every $S$-module $N$.

This applies in particular to the case where $R \to S$ is a homomorphism of rings and $M = R S_S$. In this case $R S_S \otimes_S -: S\text{-Mod} \to R\text{-Mod}$ can be identified with the functor defined by restriction of scalars $S N \mapsto R N$. Thus if $S$ is a projective (left) $R$-module, we have $\text{proj.dim}_R(N) \leq \text{proj.dim}_S(N)$ for every $S$-module $N$.

Theorem 2.5.17. Let $R$ be a ring. For any $R[t]$-module $N$, we have
\[
\text{proj.dim}_{R[t]}(N) \leq \text{proj.dim}_R(N) + 1.
\]
In particular, $\text{l.gl.dim}(R[t]) \leq \text{l.gl.dim}(R) + 1$.

Note that $R[t]$ is a free left $R$-module, so that $\text{proj.dim}_R(N) \leq \text{proj.dim}_{R[t]}(N)$.

\[\begin{proof}
\end{proof}\]

We may assume that $d = \text{proj.dim}_R(N) < \infty$. For $d \geq 1$, choose a short exact sequence of $R[t]$-modules $0 \to N' \to P \to N \to 0$ such that $R[t] P$ is projective. Then $N' \neq 0$ and $R[t] P$ is projective, so that we have identities $\text{proj.dim}_{R[t]}(N') = \text{proj.dim}_{R[t]}(N) - 1$ and $\text{proj.dim}_{R[t]}(N') = d - 1$. We are thus reduced to the case of $d - 1$. Thus, by induction, we may assume $d = 0$, namely that $R N$ is projective. Then we have the following projective resolution of $R[t] N$
\[
0 \to R[t] \otimes_R N \xrightarrow{f} R[t] \otimes_R N \xrightarrow{g} N \to 0,
\]
\[\text{In this case we say that } M \text{ is a flat right } R\text{-module.}\]
where \( g(1 \otimes n) = n \) and \( f(1 \otimes n) = 1 \otimes tn - t \otimes n \). It follows that \( \text{proj.dim}_{R[t]}(N) \leq 1 \).

**Remark 2.5.18.** In fact, we have \( \text{l.gl.dim}(R[t]) = \text{l.gl.dim}(R) + 1 \). We refer the reader to [GM, Theorem III.5.16] for a categorical statement.

**Corollary 2.5.19.** For any semisimple ring \( R \), we have \( \text{l.gl.dim}(R[x_1, \ldots, x_n]) \leq n \).

Combining this with the theorem of Quillen and Suslin, we get the following.

**Corollary 2.5.20** (Hilbert’s syzygy theorem). Let \( k \) be a field and let \( R = k[x_1, \ldots, x_n] \). Then any \( R \)-module \( N \) admits a free resolution concentrated in \([-n,0]\). Moreover, for any exact sequence

\[
F^{-n+1} \xrightarrow{d^{-n+1}} F^{-n+2} \to \cdots \to F^0 \to N \to 0
\]

of \( R \)-modules with \( F^i \) free, \( F^{-n} = \ker(d^{-n+1}) \) is free.

Here \( F^{-n} \) is sometimes called the \( n \)-th \(((n-1)\text{-th according to some convention}) syzygy of the (partial) free resolution of \( N \). The word “syzygy” comes from astronomy, in which it describes the alignment of three celestial bodies. The formulation of the theorem in Hilbert’s 1890 paper is for finitely generated graded \( R \)-modules. It follows from Nakayama’s lemma that finitely generated graded \( R \)-modules whose underlying \( R \)-modules are projective are graded free.

**Proposition 2.5.21.** Let \( R \) be a ring, let \( M \) be an \( R \)-module, and let \( n \geq 0 \) be an integer. Then \( \text{inj.dim}(M) \leq n \) if and only if \( \text{Ext}_{R}^{n+1}(R/I, M) = 0 \) for every left ideal \( I \) of \( R \).

There is no obvious analogue of the proposition for projective dimensions. Whitehead asks whether for every abelian group \( A \) with \( \text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0 \) (which implies \( \text{Ext}^1_{\mathbb{Z}}(A, B) = 0 \) for every finitely generated abelian group \( B \)) implies that \( A \) is free (or, equivalently, projective). Shelah proved that Whitehead’s problem is undecidable within \( \text{ZFC} \) even assuming the Continuum Hypothesis.

**Proof.** The “only if” part is clear. For the “if” part, take an exact sequence \( 0 \to M \to I^0 \to \cdots \to I^{n-1} \to N \to 0 \) with \( I^i \) injective. Then \( \text{Ext}_{R}^{1}(R/I, N) \simeq \text{Ext}_{R}^{n+1}(R/I, M) = 0 \). Thus the restriction map \( \text{Hom}_{R}(R, N) \to \text{Hom}_{R}(I, N) \) is surjective. It follows from Baer’s test that \( N \) is injective.

**Corollary 2.5.22.** For any ring \( R \), we have \( \text{l.gl.dim}(R) = \sup_{I} \text{proj.dim}(R/I) \), where \( I \) runs through left ideals of \( R \).

### Hyper \( \text{Ext} \) groups

For \( K, L \in D(A) \), we have long exact sequences

\[
\cdots \to \text{Ext}^{n}(K, \tau \leq m L) \to \text{Ext}^{n}(K, L) \to \text{Ext}^{n}(K, \tau \geq m+1 L) \to \text{Ext}^{n+1}(K, \tau \leq m L) \to \cdots ,
\]

\[
\cdots \to \text{Ext}^{n}(\tau \geq m+1 K, L) \to \text{Ext}^{n}(K, L) \to \text{Ext}^{n}(\tau \leq m K, L) \to \text{Ext}^{n+1}(\tau \geq m+1 K, L) \to \cdots .
\]

Thus for \( K \in D^{-}(A) \) and \( L \in D^{+}(A) \), \( \text{Ext}^{n}(K, L) \) is obtained from the groups \( \text{Ext}^{n}(H^{k}[-k], H^{l}[-l]) \simeq \text{Ext}^{n+k-l}(H^{k}K, H^{l}L) \) (zero for all but finitely many pairs \((k,l)\)) by successively taking subquotients and extensions. Here we used the fact that \( \text{Ext}^{n}(\tau \leq k K, \tau \geq l L) = 0 \) for \( n + k - l > 0 \).
Lemma 2.5.23. Assume $d = \text{hom.dim}(\mathcal{A}) < \infty$. Let $K, L \in D^b(\mathcal{A})$ with $K \in D^{\geq k}$ and $L \in D^{\leq l}$. For $k - l + n > d$, we have $\text{Ext}^n(K, L) = 0$. For $k - l + n = d$, the map $\text{Ext}^n(K, L) \to \text{Ext}^n((H^k K)[-k], (H^l L)[-l]) \simeq \text{Ext}^d(H^k K, H^l L)$ is a bijection.

Proof. The first assertion follows from the above description of $\text{Ext}^n(K, L)$. The map in the second assertion is the composite of the maps

$$\text{Ext}^n(K, L) \to \text{Ext}^n((H^k K)[-k], L) \to \text{Ext}^n((H^k K)[-k], (H^l L)[-l]),$$

which are bijections since $\text{Ext}^i(\tau_{\geq k+1} K, L) = 0$ and $\text{Ext}^i((H^k K)[-k], \tau_{\leq l-1} L) = 0$ for $i = n, n+1$ by the first assertion.

Proposition 2.5.24. If $\text{hom.dim}(\mathcal{A}) \leq 1$, then for every $K \in D^b(\mathcal{A})$, and every integer $n$, we have we have $K \simeq \bigoplus_n (H^n K)[-n]$.

The decomposition is not canonical. We note that $D^b(\mathcal{A})$ is not the direct sum of $\mathcal{A}[-n]$ unless $\text{hom.dim}(\mathcal{A}) \leq 0$.

Proof. Indeed, for any integer $n$, consider the distinguished triangle

$$\tau_{\leq n} K \to K \to \tau_{\geq n+1} K \xrightarrow{h} (\tau_{\leq n} K)[1]$$

By the lemma, $\text{Ext}^i(\tau_{\geq n+1} K, \tau_{\leq n} K) = 0$, so that $h = 0$. It follows that $K \simeq \tau_{\leq n} K \oplus \tau_{\geq n+1} K$ (Exercise). We conclude by induction.

Corollary 2.5.25 (Künneth formula for hyper Ext). Assume $\text{hom.dim}(\mathcal{A}) \leq 1$. For every $K \in D^-(\mathcal{A})$ and every $L \in D^+(\mathcal{A})$, we have a split short exact sequence

$$0 \to \bigoplus_{l-k=n} \text{Ext}^1(H^k K, H^l L) \xrightarrow{f} \text{Ext}^n(K, L) \xrightarrow{g} \bigoplus_{l-k=n} \text{Hom}(H^k K, H^l L) \to 0.$$  

In particular, for $K \in D^-(\mathcal{A})$ and $Y \in \mathcal{A}$, we have a split short exact sequence

$$0 \to \text{Ext}^1(H^{1-n} K, Y) \to \text{Ext}^n(K, Y) \to \text{Hom}(H^{-n} K, Y) \to 0.$$

The exact sequences are canonical, with $g$ carrying $a: K \to L[n]$ to the family of $H^k a: H^k K \to H^{k+n} L$ (zero for all but finitely many $k$), and $f$ given by the maps

$$\text{Ext}^1(H^k K, H^l L) \simeq \text{Ext}^n(\tau_{\geq k} K, \tau_{\leq l} L) \to \text{Ext}^n(K, L).$$

Here we used Lemma 2.5.23. The splittings are not canonical.

Proof. We have $L \in D^{\geq k}$ for some $k$. Then $\text{Ext}^n(K, L) \to \text{Ext}^n(\tau_{\geq k-n} K, L)$ is an isomorphism. Thus we may assume $K \in D^b(\mathcal{A})$. Similarly, we may assume $L \in D^b(\mathcal{A})$. Then, by the proposition,

$$\text{Ext}^n(K, L) \simeq \bigoplus_{k,l} \text{Ext}^{n+k-l}(H^k K, H^l L).$$

\[]
2.6 Derived functors

Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and let \( F: \mathcal{A} \to \mathcal{B} \) be an additive functor. We have remarked that \( F \) extends to an additive functor \( C(F): C(\mathcal{A}) \to C(\mathcal{B}) \), which induces a triangulated \( K(F): K(\mathcal{A}) \to K(\mathcal{B}) \). The composite

\[
K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B}) \to D(\mathcal{B})
\]

factorizes through a functor \( D(F): D(\mathcal{A}) \to D(\mathcal{B}) \) if and only if \( F \) is exact. Indeed, \( F \) is exact if and only if it carries \( N(\mathcal{A}) \) into \( N(\mathcal{B}) \), as acyclic complexes break into short exact sequences. Note that \( D(F) \) is equipped with the structure of a triangulated functor. Even when \( F \) is not exact, it is possible to define a localization of \( K(F) \) in many cases.

**Definition 2.6.1.** Let \( F: \mathcal{K} \to \mathcal{K}' \) be a triangulated functor and let \( \mathcal{N} \subseteq \mathcal{K}, \mathcal{N}' \subseteq \mathcal{K}' \) be full triangulated subcategories. Let \( Q: \mathcal{K} \to \mathcal{K}/\mathcal{N} \) and \( Q': \mathcal{K}' \to \mathcal{K}'/\mathcal{N}' \) be the localization functors. A right derived functor of \( F \) with respect to \( \mathcal{N} \) and \( \mathcal{N}' \) is an initial object of \( (Q'F \downarrow -Q) \), where \( -Q: \text{TrFun}(\mathcal{K}/\mathcal{N}, \mathcal{K}'/\mathcal{N}') \to \text{TrFun}(\mathcal{K}, \mathcal{K}'/\mathcal{N}') \) is the functor induced by composition with \( Q \). In other words, a right derived functor is a pair \( (RF, \epsilon) \), where \( RF: \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}' \) is a triangulated functor, and \( \epsilon: Q'F \to (RF)Q \) is a natural transformation of triangulated functors, such that for every such pair \( (G, \eta) \), there exists a unique natural transformation of triangulated functors \( \alpha: RF \to G \) such that \( \eta = (\alpha Q)\epsilon \). A left derived functor of \( F \) with respect to \( \mathcal{N} \) and \( \mathcal{N}' \) is a final object of \( (-Q \downarrow Q'F) \).

Right (resp. left) derived functors of \( F \) with respect to \( \mathcal{N} \) and \( \mathcal{N}' \) are unique up to unique natural isomorphism.

**Remark 2.6.2.**
- The above definition only depends on the category \( \mathcal{K}' \) via its localization \( \mathcal{K}'/\mathcal{N}' \).
- A similar notion of derived functors can be defined for localizations of (non triangulated) categories \( \mathcal{C} \to \mathcal{C}[S^{-1}] \).

**Proposition 2.6.3.** Let \( F: \mathcal{K} \to \mathcal{K}' \) be a triangulated functor and let \( \mathcal{N} \subseteq \mathcal{K}, \mathcal{N}' \subseteq \mathcal{K}' \) be full triangulated subcategories. Let \( \mathcal{J} \subseteq \mathcal{K} \) be a full triangulated subcategory satisfying the following conditions:

1. For every \( X \in \mathcal{K} \), there exists \( X \to Y \in \mathcal{S}_\mathcal{J} \) with \( Y \in \mathcal{J} \).
2. For every \( Y \in \mathcal{J} \cap \mathcal{N}' \), \( FY \in \mathcal{N}' \).

Then the right derived functor \( (RF: \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}', \epsilon) \) exists and the restriction of \( \epsilon \) to \( \mathcal{J} \) is a natural isomorphism.

By (2), the restriction of \( F \) to \( \mathcal{J} \) induces a functor \( \bar{F}: \mathcal{J}/\mathcal{J} \cap \mathcal{N} \to \mathcal{K}'/\mathcal{N}' \). The restriction of \( \epsilon \) to \( \mathcal{J} \) induces a natural isomorphism from \( \bar{F} \) to the composite

\[
\mathcal{J}/\mathcal{J} \cap \mathcal{N} \xrightarrow{\phi} \mathcal{K}/\mathcal{N} \xrightarrow{RF} \mathcal{K}'/\mathcal{N}'
\]

**Proof.** By Lemma [2.4.17], \( \phi \) is an equivalence. For each \( X \), we choose \( f: X \to Y \) as in (1). Then there exists a quasi-inverse \( \psi \) of \( \phi \) such that \( \psi X = Y \). We define \( RF = \bar{F}\psi \). In other words, \( (RF)X = FY \). Then \( Ff \) defines \( \epsilon: Q'F \to (RF)Q \). For a pair \( (G, \eta) \), we define \( \alpha: RF \to G \) by taking \( \alpha_X \) to be the composite \( FY \xrightarrow{\eta_Y} GY \xrightarrow{(Gf)^{-1}} GX \). \( \square \)
The proof shows that $\epsilon_X: FX \to RFX$ can be computed by choosing $f: X \to Y$ in $S_N$ with $Y \in J$ and taking $\epsilon_X = Ff: FX \to FY$.

**Remark 2.6.4.** One important case is $K = K^+(A), N = N^+(A)$, and $J = K^+(I)$, where $I \subseteq A$ denotes the full subcategory spanned by injectives. Since in this case $J \cap N$ is equivalent to zero, the conditions of Proposition 2.6.3 are satisfied whenever $A$ admits enough injectives.

Another important case is $K = K(A), N = N(A)$, and $J = K_{\text{id}}(A)$. Since in this case $J \cap N$ is equivalent to zero, the conditions of Proposition 2.6.3 are satisfied whenever there are enough homotopically injective complexes.

**Definition 2.6.5.** Let $F: A \to B$ be an additive functor between abelian categories. By a right derived functor of $F$, we mean a right derived functor $RF: D^+(A) \to D^+(B)$ of $K^+(F): K^+(A) \to K^+(B)$ with respect to $N^+(A)$ and $N^+(B)$. If $RF$ exists, we put $R^nFK = H^nRFK \in B$ for $K \in D^+(A)$ (sometimes called the hypercohomology of $K$ with respect to $RF$). The functor $R^nF: A \to B$ is called the $n$-th right derived functor of $F$.

By a left derived functor of $F$, we mean a left derived functor of $K^-(F): K^-(A) \to K^-(B)$ with respect to $N^-(A)$ and $N^-(B)$. If $LF$ exists, we put $L_nFX = H^{-n}LFX \in B$ for $X \in A$. The functor $L_nF: A \to B$ is called the $n$-th left derived functor of $F$.

**Remark 2.6.6.** Consider a short exact sequence $0 \to X' \to X \to X'' \to 0$ in $A$. If $RF$ exists, then we have a distinguished triangle

$$RFX' \to RFX \to RFX'' \to RFX'[1],$$

which induces a long exact sequence

$$\cdots \to R^nFX' \to R^nFX \to R^nFX'' \to R^{n+1}FX' \to \cdots.$$  \hspace{1cm} (2.6.1)

Similarly, if $LF$ exists, then we have a long exact sequence

$$\cdots \to L_nFX' \to L_nFX \to L_nFX'' \to L_{n-1}FX' \to \cdots.$$

**Definition 2.6.7.** Let $F: A \to B$ is an additive functor between abelian categories. A full additive subcategory $J \subseteq A$ is said to be $F$-injective if it satisfies the following conditions:

(a) For every $X \in A$, there exists a monomorphism $X \to Y$ with $Y \in J$.

(b) For every $L \in N^+(J)$, $FL$ is acyclic.

A full additive subcategory $J \subseteq A$ is said to be $F$-projective if $J^{\text{op}} \subseteq A^{\text{op}}$ is $F$-injective.

Note that these conditions are equivalent to the conditions of Proposition 2.6.3 applied to the functor $K^+(F): K^+(A) \to K^+(B)$, the subcategories $N^+(A), N^+(B)$, and the subcategory $K^+(J)$. Indeed, (b) is the same as (2). By Theorem 2.4.16, (a) implies (1). Conversely, for $X \in A$, by (1) there exists a quasi-isomorphism $X \to L$ with $L \in N^+(J)$. Since $X \to Z^0L \to H^0L$ is an isomorphism, $X \to Z^0L$ is a monomorphism. It follows that $X \to Z^0L \to L^0$ is a monomorphism.
By Proposition 2.6.3 if there exists an $F$-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$, then the right derived functor $(RF: D^+(\mathcal{A}) \to D^+(\mathcal{B}), \epsilon)$ exists and the restriction of $\epsilon$ to $K^+(\mathcal{J})$ is a natural isomorphism.

The terminology is not completely standard. Our definition here follows [KS2, Definitions 10.3.2, 13.3.4]. The same authors gave a more restrictive definition of $F$-injective categories in their previous book [KS1, Definition 1.8.2] ((a), (b′) below, and $\mathcal{J}$ stable under isomorphisms in $\mathcal{A}$).

**Proposition 2.6.8.** Condition (b′) below implies (b).

(b′) Every monomorphism $X' \to X$ in $\mathcal{A}$ with $X', X \in \mathcal{J}$ can be completed into a short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$ with $X'' \in \mathcal{J}$ such that the sequence

$$0 \to FX' \to FX \to FX'' \to 0$$

is exact.

**Proof.** Let $L \in K^+(\mathcal{J})$ be an acyclic complex. Then $L$ breaks into short exact sequences

$$0 \to Z^n L \to L^n \to Z^{n+1} L \to 0.$$

By (b), one shows by induction on $n$ that $Z^n L$ is isomorphic to an object in $\mathcal{J}$ and we have short exact sequences

$$0 \to F(Z^n L) \to F(L^n) \to F(Z^{n+1} L) \to 0,$$

so that $K^+(F)(L)$ is acyclic. \hfill \Box

**Remark 2.6.9.** If $\mathcal{A}$ admits enough injectives, then the full subcategory $\mathcal{I} \subseteq \mathcal{A}$ consisting of injective objects satisfies conditions (a) and (b′) for every $F$.

**Proposition 2.6.10.** Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and let $\mathcal{J} \subseteq \mathcal{A}$ be an $F$-injective subcategory. The functor $RF$ carries $D^{\geq n}(\mathcal{A})$ into $D^{\geq n}(\mathcal{B})$. In particular, $R^0 F$ is left exact and $R^n FX = 0$ for $X \in \mathcal{A}$ and $n < 0$. Moreover, $F$ is left exact if and only if the morphism $FX \to R^0 FX$ is an isomorphism for all $X \in \mathcal{A}$.

**Proof.** The first assertion follows from Proposition 2.6.8 and Theorem 2.4.16 (1). The second assertion follows from the first one and the long exact sequence (2.6.1). For the third assertion, the “if” part is then trivial. Finally, assume that $F$ is left exact and let $X \in \mathcal{A}$. Choose a quasi-isomorphism $X \to L$ with $L \in K^{\geq 0}(\mathcal{J})$, corresponding to an exact sequence

$$0 \to X \to L^0 \to L^1 \to \cdots .$$

Applying $F$, we obtain an exact sequence

$$0 \to FX \to FL^0 \to FL^1 .$$

Thus $R^0 FX \simeq H^0 FL \simeq FX$. \hfill \Box
Remark 2.6.11. If \( \mathcal{J} \) satisfies (a) and (b') for \( F \), then the same holds for \( R^0F \). Indeed, the second part of condition (b') for \( R^0F \) follows from the long exact sequence (2.6.1) and \( RFX = FX \) for \( X \in \mathcal{J} \). Moreover, the natural transformation \( F \to R^0F \) induces a natural isomorphism \( RF \sim R(R^0F) \).

Proposition 2.6.12. Let \( F : A \to B \) be a left exact functor between abelian categories admitting an \( F \)-injective subcategory \( \mathcal{J} \subseteq A \). Then the full subcategory \( \mathcal{I} \) of \( \mathcal{A} \) spanned by objects \( X \) such that \( R^nFX = 0 \) for all \( n \geq 1 \) satisfies (a) and (b') for \( F \). In particular, \( \mathcal{I} \) is \( F \)-injective.

Such objects are sometimes said to be \( F \)-acyclic.

Proof. It is clear that \( \mathcal{J} \subseteq \mathcal{I} \). Condition (a) for \( \mathcal{I} \) then follows from condition (a) for \( \mathcal{J} \). For any short exact sequence \( 0 \to X' \to X \to X'' \to 0 \) with \( X' \) and \( X \) in \( \mathcal{I} \), it follows from the long exact sequence that \( X'' \) is in \( \mathcal{I} \) and the sequence obtained by applying \( F \) is exact.

Proposition 2.6.13. Let \( F : A \to B, G : B \to C \) be additive functors between abelian categories. Let \( \mathcal{I} \subseteq A \) be an \( F \)-injective subcategory and let \( \mathcal{J} \subseteq B \) be a \( G \)-injective subcategory. Assume that \( F \) carries \( \mathcal{I} \) into \( \mathcal{J} \). Then \( \mathcal{I} \) is a \( GF \)-injective subcategory and the natural transformation \( \eta_L : R(GF) \to (RG)(RF) \) given by the universal property of right derived functors is a natural isomorphism.

This applies in particular to the case where \( \mathcal{I} \subseteq A \) and \( \mathcal{J} \subseteq B \) are the full subcategories spanned by injectives if \( A \) and \( B \) admit enough injectives and \( F \) preserves injectives.

Proof. It is clear that \( \mathcal{I} \) is \( GF \)-injective. For the second assertion, note that for \( L \in K^+(\mathcal{I}) \), the composite \( (GF)L \to GF(L) \to RG(RF)L \) and \( \epsilon_L \) are both isomorphisms in \( D^+(\mathcal{C}) \), and hence so is \( \eta_L \).

We leave it to the reader to give dual statements of the above.

Example 2.6.14. Let \( G \) be a group. By a (left) \( G \)-module, we mean an abelian group equipped with a (left) \( G \)-action, or equivalently, a (left) \( \mathbb{Z}G \)-module, where \( \mathbb{Z}G = \mathbb{Z}[G] \) is the group ring. The functor \( \text{Ab} \to \text{ZG-Mod} \) carrying an abelian group \( A \) to \( A \) equipped with trivial \( G \)-action admits a right adjoint \( (-)^G : \text{ZG-Mod} \to \text{Ab} \) and a left adjoint \( (-)_G : \text{ZG-Mod} \to \text{Ab} \), which can be described as follows.

For a \( G \)-module \( M \), \( M^G \) is the maximal \( G \)-invariant subgroup of \( M \), which is the group of \( G \)-invariants of \( M \). Since \( \text{ZG-Mod} \) admits enough injectives and enough projectives, these functors admit left and right derived functors. Moreover, \( M_G \) is the maximal \( G \)-invariant quotient group of \( M \), called the group of \( G \)-coinvariants of \( M \). We define \( H^n(G, -) \) to be the \( n \)-th right derived functor of \( (-)^G \), and \( H^n(G, -) \) to be the \( n \)-th left derived functor of \( (-)_G \). For a \( G \)-module \( M \), we call \( H^n(G, M) \) the \( n \)-th cohomology group of \( G \) with coefficients in \( M \), and \( H^n(G, M) \) the \( n \)-th homology group of \( G \) with coefficients in \( M \). Thus \( H^0(G, M) = M^G \) and \( H_0(G, M) = M_G \). For a short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) of \( G \)-modules, we have long exact sequences

\[
0 \to M^G \to M^G \to M''^G \to H^1(G, M') \to \cdots,
\]

\[
\cdots \to H_1(G, M'') \to M'_G \to M_G \to M''_G \to 0.
\]
$H^n(G, M)$ can be computed as the $n$-th cohomology of $I^n$, where $M \to I$ is an injective resolution. Dually, $H_n(G, M)$ can be computed as the $-n$-th cohomology of $P_G$, where $P \to M$ is a projective resolution. We will give better recipes for the computation later.

**Example 2.6.15.** Let $X$ be a topological space. The global section functor

$$\Gamma(X, -): \text{Shv}(X) \to \text{Ab}$$

is left exact. Since $\text{Shv}(X)$ admits enough injectives, $\Gamma(X, -)$ admits a right derived functor $R\Gamma(X, -): D^+(\text{Shv}(X)) \to D^+(\text{Ab})$. We write $H^n(X, -)$ for the $n$-th right derived functor of $R\Gamma(X, -)$. For $\mathcal{F} \in \text{Shv}(X)$, $H^n(X, \mathcal{F})$ is called the $n$-th cohomology group of $X$ with coefficients in $\mathcal{F}$. We have $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. By extension, for $K \in D^+(\text{Shv}(X))$, we write $H^n(X, K)$ for $H^n(\mathcal{F}, K)$, which is sometimes called the $n$-th hypercohomology of $X$ with coefficients in $K$. A sheaf $\mathcal{F}$ on $X$ is called flabby ($\text{flasque}$ in French) if the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective for every inclusion $U \subseteq V$ of open subsets of $X$. Any sheaf can be canonically embedded into a flabby sheaf: $\mathcal{F} \hookrightarrow \prod_{x \in X} i_{x*}i_{x*}^\ast \mathcal{F}$. Using Zorn’s lemma, one can show that the full subcategory of $\text{Shv}(X)$ spanned by flabby sheaves satisfies (a) and (b)’ for $\Gamma(X, -)$.

The functor $\Gamma(X, -)$ admits an exact left adjoint, carrying an abelian group $M$ to the constant sheaf $M_X$ on $X$ of value $M$. The sheaf $M_X$ is the sheafification of the constant presheaf $U \mapsto M$. For $X$ locally contractible, we have $H^n(X, M_X) \cong \check{H}^n_{\text{sing}}(X, M)$, where $H^n_{\text{sing}}(X, M)$ is the cohomology of the singular cochain complex $C^\ast(X, M)$. To see this, consider the sequence

$$0 \to M_X \to C^0(X, M) \to \cdots \to C^n(X, M) \to \cdots,$$

where $C^n(X, M)$ is the sheafification of $U \mapsto C^n(U, M)$. The sequence is exact. Indeed, the cohomology of the sequence at $C^n(X, M)$ is the sheafification of $U \mapsto \check{H}^n_{\text{sing}}(U, M)$, which is zero by local contractibility. Moreover, $C^n(X, M)$ is flabby. Thus, $H^n(X, M) \cong H^n(\Gamma(X, C^\ast(X, M)))$. A subdivision argument shows that the morphism of complexes $C^\ast(X, M) \to \Gamma(X, C^\ast(X, M))$, surjective in each degree, is a quasi-isomorphism.

**Example 2.6.16.** Let $f: X \to Y$ be a continuous map of topological spaces. The left exact functor $f_*: \text{Shv}(X) \to \text{Shv}(Y)$ defined by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ admits an exact left adjoint $f^*: \text{Shv}(Y) \to \text{Shv}(X)$ defined by $(f^*\mathcal{G})(U) = \text{colim}_{f(U) \subseteq V} \mathcal{G}(V)$, where the colimit runs through the filtered category $(U \downarrow f^{-1})^{\text{op}}$. Thus $f^*$ extends to a functor

$$f^*: D(\text{Shv}(Y)) \to D(\text{Shv}(X)).$$

Moreover $f_*$ admits a right derived functor $Rf_*: D^+(\text{Shv}(X)) \to D^+(\text{Shv}(Y))$. The full subcategory of $\text{Shv}(X)$ spanned by flabby sheaves satisfies (a) and (b)’ for $f_*$. In the special case where $Y$ is a point, $f_* = \Gamma(X, -)$.

For a sequence of continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have natural isomorphisms $f^*g^* \cong (gf)^*$ and $R(gf)_* \cong Rg_*Rf_*$.

**Remark 2.6.17.** We have already seen one important case (second part of Remark 2.6.4) where derived functors exist between unbounded derived categories. There are other such cases. We refer to [KS2, Chapters 14] for details.
2.7  Double complexes, derived \( Hom \)

The goal of this section and the following is to study derive functors of the \( Hom \) and tensor functors. Since these functors have two variables, we need some generalities on double complexes.

Double complexes

Let \( \mathcal{A} \) be an additive category.

**Definition 2.7.1.** We define the category of double complexes in \( \mathcal{A} \) to be \( C^2(\mathcal{A}) = C(C(\mathcal{A})) \). Thus a double complex consists of objects \( X^{i,j} \) for \( i, j \in \mathbb{Z} \) and differentials \( d_I: X^{i,j} \rightarrow X^{i+1,j}, \ d_H: X^{i,j} \rightarrow X^{i,j+1} \) such that \( d_I^2 = 0, \ d_H^2 = 0, \ d_Id_H = d_Hd_I \).

**Definition 2.7.2.** Let \( X \) be a double complex in \( \mathcal{A} \). We define two complexes in \( \mathcal{A} \) with \( (\text{tot}_\oplus X)^n = \bigoplus_{i+j=n} X^{i,j} \) (if the coproducts exist) and \( (\text{tot}_\Pi X)^n = \prod_{i+j=n} X^{i,j} \) (if the products exist). The differentials are defined as follows. Let \( i + j = n \). The composition \( X^{i,j} \rightarrow (\text{tot}_\oplus X)^n \xrightarrow{d^n} (\text{tot}_\oplus X)^{n+1} \) is given by

\[
(2.7.1) \quad d_I^{i,j} + (-1)^i d_H^{i,j}.
\]

The composition \( (\text{tot}_\Pi X)^{n-1} \xrightarrow{d^{n-1}} (\text{tot}_\Pi X)^n \rightarrow X^{i,j} \) is given by

\[
(2.7.2) \quad d_I^{i-1,j} + (-1)^i d_H^{i,j-1}.
\]

**Remark 2.7.3.** The sign in (2.7.1) and (2.7.2) ensures that \( d^2 = 0 \). If \( Y \) is the transpose of \( X \) defined by \( Y^{i,j} = X^{j,i} \) and by swapping the two differentials, then we have an isomorphism \( \text{tot}_\oplus X \simeq \text{tot}_\ominus Y \) given by \( (-1)^{ij} \text{id}_{X^{i,j}} \). The same holds for \( \text{tot}_\Pi \).

In the literature, a variant of Definition 2.7.1 with \( d_Id_H + d_Hd_I = 0 \) is sometimes used. If we adopt this variant, then (2.7.1) can be simplified to \( d = d_I + d_H \). The two definitions correspond to each other by multiplying \( d_H^{i,j} \) by the sign \( (-1)^i \).

**Definition 2.7.4.** We say that a double complex \( X \) is biregular if for every \( n \), \( X^{i,j} = 0 \) for all but finitely many pairs \( (i, j) \) with \( i + j = n \). We let \( C^2_{\text{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A}) \) denote full subcategory consisting of biregular double complexes. It is an additive subcategory.

If \( X^{i,j} = 0 \) for \( i, j \ll 0 \) (\( X \) concentrated in a translate of the first quadrant) or \( X^{i,j} = 0 \) for \( i, j \gg 0 \) (\( X \) concentrated in a translate of the third quadrant), then \( X \) is biregular. If \( X^{i,j} = 0 \) for \( |i| \gg 0 \) (concentrated in a vertical stripe) or \( X^{i,j} = 0 \) for \( |j| \gg 0 \) (concentrated in a horizontal stripe), then \( X \) is biregular.

**Remark 2.7.5.** If \( X \) is a biregular double complex, then \( \text{tot}_\oplus X \) and \( \text{tot}_\Pi X \) exist and we have \( \text{tot}_\oplus X \cong \text{tot}_\ominus Y \). We will simply write \( \text{tot}X \). We get an additive functor \( \text{tot}: C^2(\mathcal{A}) \rightarrow C(\mathcal{A}) \).

**Example 2.7.6.** Let \( f: L \rightarrow M \) be a morphism of complexes in \( \mathcal{A} \). We define a double complex \( X \) by \( X^{-1,j} = L^j, \ X^0,j = M^j, \ X^{i,j} = 0 \) for \( i \neq -1, 0, \ d_I^{-1,j} = f^j, \ d_H \) given by \( d_L \) and \( d_M \). Then \( \text{tot}X = \text{Cone}(f) \).
Let $A, A', A''$ be additive categories. Let $F: A \times A' \to A''$ be a functor that is additive in each variable. Then $F$ extends to a functor $C^2(F): C(A) \times C(A') \to C^2(A'')$ additive in each variable. For $X \in C(A), Y \in C(A')$, the double complex $C^2(F)(X,Y)$ is defined by $C^2(F)(X,Y)^{i,j} = F(X^i,Y^j)$, with $d^2_{ij} = F(d_X^i, d_Y^j), d^2_{ij} = F(id_X, d_Y^j)$.

**Example 2.7.7.** Let $R$ be a ring. The functor $- \otimes_R - : \text{Mod-}R \times R\text{-Mod} \to \text{Ab}$ is additive in each variable. Then it extends to

$$- \otimes_R - : C(\text{Mod-}R) \times C(R\text{-Mod}) \to C^2(\text{Ab}).$$

**Example 2.7.8.** Let $A$ be an additive category with small Hom sets. The functor $\text{Hom}_A: A^{op} \times A \to \text{Ab}$ is additive in each variable. We have an isomorphism $C(A)^{op} \simeq C(A^{op})$, carrying $(X,d)$ to $((X^{-n}),(-1)^n d^{-n-1})$. Thus $\text{Hom}_A$ extends to a functor

$$\text{Hom}_A^*: C(A)^{op} \times C(A) \to C^2(\text{Ab}),$$

additive in each variable. For $X,Y \in C(A)$, $\text{Hom}_A^*(X,Y)^{i,j} = \text{Hom}_A(X^{-j},Y^i)$, with

$$d^2_{ij} = \text{Hom}_A(X^{-j}, d_Y^i), \quad d^2_{ij} = \text{Hom}_A((-1)^j d_X^{j+1}, Y^i).$$

We define $\text{Hom}_A^*$ as the composite functor

$$C(A)^{op} \times C(A) \xrightarrow{\text{Hom}_A^*} C^2(\text{Ab}) \xrightarrow{\text{tot}} C(\text{Ab}).$$

We have

$$\text{Hom}_A^*(X,Y)^n = \prod_{j \in \mathbb{Z}} \text{Hom}_A(X^j, Y^{n+j}),$$

and for $f = (f^j) \in \text{Hom}_A^*(X,Y)^n$,

$$(d^nf)^j = d_Y^{j+n} f^j + (-1)^{n+1} f^{j+1} d_X^j.$$

**Proposition 2.7.9.** We have

$$Z^0 \text{Hom}_A^*(X,Y) \simeq \text{Hom}_{C(A)}(X,Y),$$

$$B^0 \text{Hom}_A^*(X,Y) \simeq \text{Im}(\text{Ht}(X,Y) \to \text{Hom}_{C(A)}(X,Y)),$$

$$H^0 \text{Hom}_A^*(X,Y) \simeq \text{Hom}_{K(A)}(X,Y).$$

**Proof.** We have $d^0(f) = df - fd$, so that $d^0(f) = 0$ if and only if $f: X \to Y$ is a morphism of complexes. We have $\text{Ht}(X,Y) = \text{Hom}_A^*(X,Y)^{-1}$, and for $h \in \text{Ht}(X,Y)$,

$$d^{-1}(h) = dh + hd. \quad \Box$$

**Definition 2.7.10.** Let $D$, $D'$, $D''$ be triangulated categories. A **triangulated bifunctor** is a functor $F: D \times D' \to D''$ equipped with natural isomorphisms $F(X[1],Y) \simeq F(X,Y)[1], F(X,Y[1]) \simeq F(X,Y)[1]$, such that the following diagram anticommutes

$$\begin{array}{ccc}
F(X[1],Y[1]) & \xrightarrow{F} & F(X,Y[1]) \\
\downarrow & & \downarrow \\
F(X[1],Y)[1] & \xrightarrow{F} & F(X,Y)[2]
\end{array}$$

and such that $F$ is triangulated in each variable.

Note that $\text{Hom}_A^*$ factorizes through a triangulated bifunctor $K(A)^{op} \times K(A) \to K(\text{Ab})$. 

2.7. DOUBLE COMPLEXES, DERIVED HOM

Derived Hom

Definition 2.7.11. Let $F: \mathcal{K} \times \mathcal{K}' \to \mathcal{K}''$ be a triangulated bifunctor and let $\mathcal{N} \subseteq \mathcal{K}$, $\mathcal{N}' \subseteq \mathcal{K}'$, $\mathcal{N}'' \subseteq \mathcal{K}''$ be full triangulated subcategories. Let $Q: \mathcal{K} \to \mathcal{K}/\mathcal{N}$, $Q': \mathcal{K}' \to \mathcal{K}'/\mathcal{N}'$, and $Q'': \mathcal{K}'' \to \mathcal{K}''/\mathcal{N}''$ be the localization functors. A right derived bifunctor of $F$ with respect to $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}''$ is an initial object of $(Q''F \downarrow -(Q \times Q'))$, where $-(Q \times Q'): \text{TrBiFun}(\mathcal{K}/\mathcal{N}, \mathcal{K}'/\mathcal{N}'\mathcal{K}''/\mathcal{N}'') \to \text{TrBiFun}(\mathcal{K}, \mathcal{K}'/\mathcal{N}'\mathcal{K}''/\mathcal{N}'')$ is the functor induced by composition with $Q \times Q'$. In other words, a right derived bifunctor is a pair $(RF, \epsilon)$, where $RF: \mathcal{K}/\mathcal{N} \times \mathcal{K}'/\mathcal{N}' \to \mathcal{K}''/\mathcal{N}''$ is a triangulated bifunctor, and $\epsilon: Q''F \to (RF)(Q \times Q')$ is a natural transformation of triangulated bifunctors, such that for every such pair $(G, \eta)$, there exists a unique transformation of triangulated bifunctors $\alpha: RF \to G$ such that $\eta = (\alpha(Q \times Q'))\epsilon$. A left derived functor of $F$ with respect to $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}''$ is a final object of $(-(Q \times Q') \downarrow Q''F)$.

Remark 2.7.12. The above definition only depends on the category $\mathcal{K}''$ via its localization $\mathcal{K}''/\mathcal{N}''$.

Proposition 2.7.13. Let $F: \mathcal{K} \times \mathcal{K}' \to \mathcal{K}''$ be a triangulated bifunctor and let $\mathcal{N} \subseteq \mathcal{K}$, $\mathcal{N}' \subseteq \mathcal{K}'$, $\mathcal{N}'' \subseteq \mathcal{K}''$ be full triangulated subcategories. Let $J \subseteq \mathcal{K}$ be a full triangulated subcategory satisfying the following conditions:

1. For every $X \in \mathcal{K}$, there exists $X \to Y$ in $\mathcal{S}_\mathcal{N}$ with $Y \in J$.
2. For $Y \in J \cap \mathcal{N}$ and $X' \in \mathcal{K}'$, we have $F(Y, X') \in \mathcal{N}''$.
3. For $Y \in J$ and $X' \in \mathcal{N}'$, we have $F(Y, X') \in \mathcal{N}''$.

Then the right derived bifunctor $(RF: \mathcal{K}/\mathcal{N} \times \mathcal{K}'/\mathcal{N}' \to \mathcal{K}''/\mathcal{N}'')$, $\epsilon$) exists and the restriction of $\epsilon$ to $J \times \mathcal{K}'$ is a natural isomorphism.

Under the assumptions of the proposition, for every $X' \in \mathcal{K}'$, $RF(-, X') \mathcal{K}/\mathcal{N} \to \mathcal{K}''/\mathcal{N}''$ is a right derived functor of $F(-, X'): \mathcal{K} \to \mathcal{K}''$. By (2) and (3), the restriction of $F$ to $J \times \mathcal{K}'$ induces a functor $\tilde{F}: (J/\mathcal{J} \cap \mathcal{N}) \times (\mathcal{K}'/\mathcal{N}') \to \mathcal{K}''/\mathcal{N}''$.

Proof. The proof is very similar to the proof of Proposition 2.6.3.

Let $\mathcal{A}$ be an abelian category with small Hom sets.

Proposition 2.7.14. Assume that $\mathcal{A}$ admits enough injectives. Then the triangulated bifunctor

$$\text{Hom}_\mathcal{A}^\bullet: K(\mathcal{A})^{\text{op}} \times K^+(\mathcal{A}) \to K(\text{Ab})$$

admits a right derived bifunctor

$$\text{RHom}_\mathcal{A}: D(\mathcal{A})^{\text{op}} \times D^+(\mathcal{A}) \to D(\text{Ab})$$

such that, for $M \in K^+(\mathcal{A})$ with injective components and $L \in K(\mathcal{A})$, we have

$$\text{Hom}_\mathcal{A}^\bullet(L, M) \tilde{\to} \text{RHom}_\mathcal{A}(L, M).$$
Proof. We need to show that for $L \in K(A)$, $M \in K^+(A)$, $M^n$ injective for all $n$, with $L$ or $M$ acyclic, then $\text{Hom}_A^\bullet(L, M)$ is acyclic. Indeed,

$$H^n\text{Hom}_A^\bullet(L, M) \simeq \text{Hom}_{K(A)}(L, M[n]) \simeq \text{Hom}_{D(A)}(L, M[n]) = 0.$$ 

Here we used Propositions 2.4.24 and 2.4.26.

**Remark 2.7.15.** Assume that $A$ has enough injectives. For $L \in D(A)$, $M \in D^+(A)$, we have

$$H^n\text{RHom}_A(L, M) \simeq H^n\text{Hom}_A^\bullet(L, M') \simeq \text{Hom}_{K(A)}(L, M'[n]) \simeq \text{Ext}^n(L, M),$$

where we have taken a quasi-isomorphism $M \to M' \in K^+(A)$ such that $M'$ has injective components. In particular, for $X \in A$, $\text{Ext}^n(X, -)$ is the $n$-th right derived functor of $\text{Hom}(X, -)$.

**Remark 2.7.16.** If there are enough homotopically injective complexes, then the triangulated bifunctor

$$\text{Hom}_A^\bullet: K(A)^{\text{op}} \times K(A) \to K(\text{Ab})$$

admits a right derived bifunctor

$$\text{RHom}_A: D(A)^{\text{op}} \times D(A) \to D(\text{Ab})$$

such that, for $M \in K_{\text{hi}}(A)$ and $L \in K(A)$, we have

$$\text{Hom}_A^\bullet(L, M) \sim \text{RHom}_A(L, M).$$

Dually, we have the following.

**Proposition 2.7.17.** Assume that $A$ admits enough projectives. Then the triangulated bifunctor

$$\text{Hom}_A^\bullet: K^-(A)^{\text{op}} \times K(A) \to K(\text{Ab})$$

admits a right derived bifunctor

$$\text{RHom}_A: D^-(A)^{\text{op}} \times D(A) \to D(\text{Ab})$$

such that for $L \in K^-(A)$ with projective components and $M \in K(A)$, we have

$$\text{Hom}_A^\bullet(L, M) \sim \text{RHom}_A(L, M).$$

**Remark 2.7.18.** In the case where $A$ admits enough injectives and enough projectives, the functors $\text{RHom}$ defined in Propositions 2.7.14 and 2.7.17 are isomorphic when restricted to $D^-(A) \times D^+(A)$. Indeed, for $L \in D^-(A)$ and $M \in D^+(A)$, $\text{RHom}(L, M)$ can be computed by finding quasi-isomorphisms $L' \to L$ and $M \to M'$ such that $L'$ has projective components and $M'$ has injective components and taking $\text{Hom}_A^\bullet(L, M)$.

**Remark 2.7.19.** If $A$ admits enough injectives or enough projectives, then $\text{RHom}_A(-, -)$ carries $(D^{\leq m}(A))^{\text{op}} \times D^{\geq n}(A)$ to $D^{\geq n-m}(\text{Ab})$. 
Remark 2.7.20. Assume that \( \mathcal{A} \) admits enough injectives. We have an isomorphism \( R\text{Hom}_\mathcal{A}(L, M) \simeq R\text{Hom}_\mathcal{A}(M, L) \), natural in \( L \in D(\mathcal{A}) \) and \( M \in D^+(\mathcal{A}) \). Here on the right hand side we regard \( L \in D(\mathcal{A}^{\text{op}}) \) and \( M \in D^-(\mathcal{A}^{\text{op}}) \).

Example 2.7.21. Let \( R \) be a ring and let \( \mathcal{A} = R\text{-Mod} \). Let \( r \in R \) be an element such that \( R \xrightarrow{r} R \) is a monomorphism. Then we have a projective resolution \( 0 \to R \xrightarrow{r} R \to R/\mathbb{Z} \to 0 \), so that \( R\text{Hom}_R(R/\mathbb{Z}, M) \) is computed by the complex \( M \xrightarrow{r} M \) put in degrees 0 and 1. Thus if \( R \) is a PID and \( s \in R \) is nonzero, then \( \text{Ext}_R^i(R/rR, R/sR) \simeq R/(r, s)R \) for \( i = 0, 1 \) and, by Proposition 2.5.24, we have \( R\text{Hom}_R(R/rR, R/sR) \simeq R/(r, s)R \oplus (R/(r, s)R)[-1] \).

Proposition 2.7.22. Let \( \mathcal{A} \) and \( \mathcal{B} \) be additive categories admitting enough injectives and let \( F : \mathcal{B} \to \mathcal{A} \) be an exact functor admitting a right adjoint \( G : \mathcal{A} \to \mathcal{B} \). Then for \( X \in D(\mathcal{B}) \) and \( Y \in D^+(\mathcal{A}) \), we have

\[
R\text{Hom}_\mathcal{A}(FX, Y) \simeq R\text{Hom}_\mathcal{B}(X, RG Y), \\
\text{Hom}_{D(\mathcal{A})}(FX, Y) \simeq \text{Hom}_{D(\mathcal{B})}(X, RG Y).
\]

In particular, \( RG : D^+(\mathcal{A}) \to D^+(\mathcal{B}) \) is a right adjoint of the functor \( F : D^+(\mathcal{B}) \to D^+(\mathcal{A}) \).

Proof. We may replace \( Y \) by a complex in \( K^+(\mathcal{A}) \) with injective components. Then \( RG Y \) is computed by \( GY \). Since \( F \) is exact, \( G \) preserves injectives, so that \( GY \) is in \( K^+(\mathcal{B}) \) with injective components. The first isomorphism is given by

\[
\text{Hom}_\mathcal{A}^*(FX, Y) \simeq \text{Hom}_\mathcal{B}^*(X, GY), \\
\text{Hom}_\mathcal{A}^*(FX, Y) \simeq \text{Hom}_\mathcal{B}^*(X, GY).
\]

Applying \( H^0 \), we get the second isomorphism. \( \square \)

Example 2.7.23. Let \( f : X \to Y \) be a continuous map. The functor \( Rf_* : D^+(\text{Shv}(X)) \to D^+(\text{Shv}(Y)) \) is a right adjoint of the functor \( f^* : D^+(\text{Shv}(Y)) \to D^+(\text{Shv}(X)) \).

Double complexes and acyclicity

Let \( \mathcal{A} \) be an abelian category. For a double complex \( X \) in \( \mathcal{A} \), we put

\[
H_i(X)^{ij} = \text{Ker}(d_{ij}^i)/\text{Im}(d_{i-1,j}^{ij}), \\
H_{ii}(X)^{ij} = \text{Ker}(d_{ii}^i)/\text{Im}(d_{ii-1}^{ij}).
\]

The full additive subcategory \( C^2_{\text{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A}) \) is stable under subobjects and quotients. Thus \( C^2_{\text{reg}}(\mathcal{A}) \) is an abelian category and the inclusion functor is exact.

Proposition 2.7.24. Let \( X \) be a biregular double complex such that \( H^j_i(X) \) is acyclic for every \( i \). Then \( \text{tot} X \) is acyclic.

A similar statement holds for \( H^j_{II} \), which generalizes the fact that the cone of a quasi-isomorphism is acyclic.
Proof. For each \( m \), there exists \( N \) such that \( H^m(\text{tot}X) = H^m(\text{tot}(\tau^{\leq n}I)) \) for all \( n \geq N \). It suffices to show that \( H^m(\text{tot}(\tau^{\leq n}I)) = 0 \) for all \( n \geq N \). We proceed by induction on \( n \) (for a fixed \( m \)). For \( n \ll 0 \), \((\text{tot}(\tau^{\leq n}I)) m = 0 \). Assume that \( H^m(\tau^{\leq n}I X) = 0 \) and consider the short exact sequence of double complexes

\[
0 \to \tau^{\leq n-1}I X \to \tau^{\leq n}I X \to Y \to 0,
\]

where \( Y = (B^n_l X \xrightarrow{f} Z^n_l X) \) is concentrated on the columns \( n - 1 \) and \( n \). Applying \( \text{tot} \), we get the exact sequence of complexes

\[
0 \to \text{tot}\tau^{\leq n-1}I X \to \text{tot}\tau^{\leq n}I X \to \text{tot}Y \to 0.
\]

We have a quasi-isomorphism \( \text{tot}(Y)[n] \simeq \text{Cone}((-1)^n f) \to H^n_l(X) \). It follows \( \text{tot}Y \) is acyclic. Taking long exact sequence, we get

\[
H^m(\tau^{\leq n}I X) \simeq H^m(\text{tot}\tau^{\leq n-1}I X) = 0.
\]

\[\square\]

**Corollary 2.7.25.** Let \( X \) be a biregular double complex such that \( X^{\bullet,j} \) is acyclic for every \( j \) (namely, every row of \( X \) is acyclic). Then \( \text{tot}X \) is acyclic.

A similar statement holds for columns of \( X \): if \( X^{i,\bullet} \) is acyclic for every \( i \), then \( \text{tot}X \) is acyclic.

**Corollary 2.7.26.** Let \( f : X \to Y \) be a morphism of biregular double complexes such that \( H^i_l(f) : H^i_l(X) \to H^i_l(Y) \) is a quasi-isomorphism for each \( i \). Then \( \text{tot}(f) : \text{tot}(X) \to \text{tot}(Y) \) is a quasi-isomorphism.

**Proof.** We let \( W = \text{Cone}_H(f) \) with \( W^{i,j} = X^{i,j+1} \oplus Y^{i,j} \). Then \( H^i_l(W) \simeq \text{Cone}(H^i_l(f)) \) is acyclic. By the proposition applied to \( W \), \( \text{tot}(W) \simeq \text{Cone}(\text{tot}(f)) \) is acyclic. \[\square\]

**Corollary 2.7.27.** Let \( f : X \to Y \) be a morphism of biregular double complexes such that \( f^{\bullet,j} : X^{\bullet,j} \to Y^{\bullet,j} \) is a quasi-isomorphism for each \( j \). Then \( \text{tot}(f) : \text{tot}(X) \to \text{tot}(Y) \) is a quasi-isomorphism.

### 2.8 Flat modules, derived tensor product

Let \( R \) be a ring.

**Flat modules**

For any left \( R \)-module \( L \) and right \( R \)-module \( M \), the functors \( L \otimes_R - \) and \( - \otimes_R M \) are left adjoint functors, hence commute with small colimits. In particular, they are right exact.

**Definition 2.8.1.** A left \( R \)-module \( M \) is said to be flat if the functor

\[
- \otimes_R M : \text{Mod-}R \to \text{Ab}
\]

is exact. A right \( R \)-module \( L \) is said to be flat if the functor \( L \otimes_R - \) is exact.
**Remark 2.8.2.** A right $R$-module is flat if and only if the corresponding left $R^{op}$-module is flat.

**Remark 2.8.3.** Flat $R$-modules are stable under direct sum, direct summand, and filtered colimits.

**Lemma 2.8.4.** Every projective $R$-module is flat.

*Proof.* The $R$-module $R$ is flat as $- \otimes_R R$ is the identity functor. Free $R$-modules are direct sums of copies of the $R$-module $R$, hence are flat. Every projective $R$-module is a direct summand of a free $R$-module, hence is flat. \qed

The converse does not hold. Indeed, since filtered colimits of abelian groups are exact, filtered colimits of flat $R$-modules is flat. By contrast, projective $R$-modules are not stable under filtered colimits.

**Example 2.8.5.** $\mathbb{Q} = \text{colim} \frac{1}{n} \mathbb{Z}$, $n$ ordered by division, is a flat $\mathbb{Z}$-module but is not free, hence not projective.

We will later show that flatness is equivalent to projectivity under a finiteness condition.

**Theorem 2.8.6** (Lazard, Govorov). Every flat $R$-module is a filtered colimit of free $R$-modules.

We refer the reader to [L1, Theorem 4.34] for a proof.

**Derived tensor product**

The composite functor

$$C(\text{Mod}-R) \times C(\text{R-Mod}) \xrightarrow{- \otimes_R -} C^2(\text{Ab}) \xrightarrow{\text{tot} \otimes} C(\text{Ab})$$

induces a triangulated bifunctor

$$\text{tot} \otimes (- \otimes_R -) : K(\text{Mod}-R) \times K(\text{R-Mod}) \to K(\text{Ab}).$$

**Proposition 2.8.7.** The triangulated bifunctor

$$\text{tot} \otimes (- \otimes_R -) : K(\text{Mod}-R) \times K^-(\text{R-Mod}) \to K(\text{Ab}).$$

admits a left derived bifunctor

$$- \otimes^L_R - : D(\text{Mod}-R) \times D^-(\text{R-Mod}) \to D(\text{Ab})$$

such that for all $M \in K^-(\text{R-Mod})$ with projective components and $L \in K(\text{Mod}-R)$, we have

$$L \otimes^L_R M \cong \text{tot} \otimes (L \otimes_R M).$$
Proof. Note that $R$-$\text{Mod}$ admits enough projectives. By Proposition 2.7.13, it suffices to show that for $L \in C(\text{Mod}-R)$, $M \in C^-(\text{Mod}-R)$, $M^n$ projective for all $n$, with $L$ or $M$ acyclic, $\text{tot}_\oplus(L \otimes_R M)$ is acyclic. If $M$ is acyclic, then the image of $M$ in $K^-(\text{Mod}-R)$ is zero and the assertion is trivial. Assume now that $L$ is acyclic. We have $L \simeq \text{colim}_{n \in (\mathbb{Z}, \leq)} \tau^\leq L$. It follows that $\text{tot}_\oplus\text{colim}_n (\tau^\leq_n L \otimes_R M)$ is acyclic. If $M$ is acyclic, then the image of $M$ in $K^-(\text{Mod}-R)$ is zero and the assertion is trivial. Assume now that $L$ is acyclic. We have $L \simeq \text{colim}_{n \in (\mathbb{Z}, \leq)} \tau^\leq_n L$. It follows that $\text{total}_\oplus\text{colim}_n (\tau^\leq_n L \otimes_R M)$ is acyclic. By duality, we get the following.

Proposition 2.8.8. The triangulated bifunctor
\[ \text{tot}_\oplus(- \otimes_R -): K^-(\text{Mod}-R) \times K(\text{Mod}-R) \to K(\text{Ab}) \]

admits a left derived bifunctor
\[ - \otimes_R^L: D^-(\text{Mod}-R) \times D(\text{Mod}-R) \to D(\text{Ab}) \]
such that for all $L \in K^-(\text{Mod}-R)$ with projective components and $M \in K(\text{Mod}-R)$, we have
\[ L \otimes_R^L M \Rightarrow \text{tot}_\oplus(L \otimes_R M). \]

The functors defined in Propositions 2.8.7 and 2.8.8 are isomorphic when restricted to $D^-(\text{Mod}-R) \times D^-(\text{Mod}-R)$. Moreover, for $L \in D^{\leq a}(\text{Mod}-R)$, $M \in D^{\leq b}(\text{Mod}-R)$, $L \otimes^L M \in D^{\leq a+b}(\text{Ab})$.

Definition 2.8.9. For $L \in D(\text{Mod}-R)$, $M \in D(\text{Mod}-R)$, with $L \in D^-$ or $M \in D^-$, we define the hyper $\text{Tor}$ by
\[ \text{Tor}^R_n(L, M) = H^{-n}(L \otimes_R^L M). \]

Thus for $X \in \text{Mod}-R$ and $Y \in R$-$\text{Mod}$, $\text{Tor}_{n}^{R}(X, -)$ is the $n$-th left derived functor of $X \otimes_R -$ and $\text{Tor}_{n}^{R}(-, Y)$ is the $n$-th left derived functor of $- \otimes_R Y$. We have $\text{Tor}_{n}^{R}(X, Y) = 0$ for $n < 0$ and $\text{Tor}_{0}^{R}(X, Y) = X \otimes_R Y$.

Proposition 2.8.10. Let $Y$ be a left $R$-module. Then the following conditions are equivalent:

1. $Y$ is flat;
2. $\text{Tor}_{n}^{R}(X, Y) = 0$ for all right $R$-module $X$;
3. $\text{Tor}_{n}^{R}(X, Y) = 0$ for all right $R$-module $X$ and all $n \geq 1$.

It follows then from the long exact sequence for $\text{Tor}$ that flat $R$-modules are stable under extension.
Proof. (3) ⇒ (2). Obvious.

(2) ⇒ (1). Since Tor\(_1^n(\_Y) = 0\), the long exact sequence implies that \( Y \) is flat.

(1) ⇒ (3). Let \( X' \rightarrow X \) be a projective resolution of \( X \), giving rise to the exact sequence
\[
\cdots \rightarrow X'_{n-1} \rightarrow X'_n \rightarrow X \rightarrow 0.
\]
Since \( Y \) is flat, the induced sequence
\[
\cdots \rightarrow X'_{n-1} \otimes_R Y \rightarrow X'_n \otimes_R Y \rightarrow X \otimes_R Y \rightarrow 0.
\]
is exact. Thus \( X \otimes_R^L Y \simeq X' \otimes_R Y \simeq X \otimes_R Y \) in \( D(\text{Ab}) \). It follows that Tor\(_n^R(X, Y) = 0\).

\[\Box\]

**Corollary 2.8.11.** Let
\[
0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0
\]
be a short exact sequence of \( R \)-modules.

(1) If \( F'' \) is flat, then for any right \( R \)-module \( X \), the induced sequence
\[
0 \rightarrow X \otimes_R F' \rightarrow X \otimes_R F \rightarrow X \otimes_R F'' \rightarrow 0
\]
is exact.

(2) If \( F \) and \( F'' \) are flat, then \( F' \) is flat.

\[\text{Proof.}\] The proof is the same as the proof of Proposition 2.6.12. The long exact sequence has the form
\[
\text{Tor}_n^R(X, F'') \rightarrow \text{Tor}_n^R(X, F') \rightarrow \text{Tor}_n^R(X, F).
\]

(1) For \( n = 0 \), we get the injectivity of \( X \otimes_R F' \rightarrow X \otimes_R F \).

(2) For \( n \geq 1 \), we have Tor\(_{n+1}^R(X, F'') = \text{Tor}_n^R(X, F) = 0\), which implies Tor\(_n^R(X, F') = 0\). Hence \( F' \) is flat.

\[\Box\]

**Corollary 2.8.12.** Let \( L \in C(\text{Mod}-R) \), \( M \in C(\text{R-Mod}) \). If \( L \in C^-(\text{Mod}-R) \) with flat components, or \( M \in C^-(\text{R-Mod}) \) with flat components, then we have
\[
L \otimes_R^LM \simeq \text{tot}_\oplus(L \otimes M).
\]

\[\text{Proof.}\] We treat the case \( M \in C^-(\text{R-Mod}) \), the other case being similar. Choose a quasi-isomorphism \( M' \rightarrow M \), where \( M' \in C^-(\text{R-Mod}) \) has projective components. Then \( L \otimes_R^LM \simeq \text{tot}_\oplus(L \otimes_R M') \simeq \text{tot}_\oplus(L \otimes_R M) \). Here in the last isomorphism we used the lemma below.

\[\Box\]

**Lemma 2.8.13.** Let \( L \in C(\text{Mod}-R) \), \( M \in C^-(\text{R-Mod}) \). Assume \( M \) acyclic and \( M^n \) flat for all \( n \). Then \( \text{tot}_\oplus(L \otimes_R M) \) is acyclic.

\[\text{Proof.}\] We have \( L \simeq \text{colim}_{n \in (\mathbb{N}, \leq)} \tau^{\leq n}L \). It follows that
\[
\text{tot}_\oplus(L \otimes_R M) \simeq \text{colim}_n \text{tot}_\oplus(\tau^{\leq n}L \otimes_R M).
\]
Since filtered colimits are exact in \( \text{Ab} \), we may assume that \( L \in C^-(\text{Mod}-R) \). Then \( L \otimes_R M \) is biregular. Since \( M \) is acyclic, \( M \) splits into short sequences
\[
0 \rightarrow Z^n M \rightarrow M^n \rightarrow Z^{n+1} M \rightarrow 0.
\]
We prove that $Z^n M$ is flat for all $n$ by descending induction. For $n > 0$, $M^n = 0$. If $Z^{n+1} M$ is flat, then, by Corollary 2.8.1, $Z^n M$ is flat. It follows that for each $i$, $L^i \otimes_R M$ splits into short exact sequences, hence is acyclic. Thus $\text{tot}_{\otimes}(L \otimes_R M)$ is acyclic.

**Proposition 2.8.14.** We have an isomorphism $L \otimes_{L^1}^R M \simeq M \otimes_{L^0}^R L$, natural in $L \in D(\text{Mod}-R)$ and $M \in D^{-}(\text{R-Mod})$. Here on the right hand side we regard $L \in D(\text{R}^{op}-\text{Mod})$ and $M \in D(\text{Mod}-R)$.

**Proof.** We may assume $M \in C^-(\text{R-Mod})$ with flat components. Then the isomorphism is given by the isomorphism of double complexes $L \otimes_R M \simeq M \otimes_{R^{op}}^L L$. □

**Derived tensor product and derived Hom**

Let $R$ and $S$ be rings. We have the following derived functors

- $\otimes_{L^1}^S : D((R,S)-\text{Mod}) \times D^{-}(S-\text{Mod}) \to D(R-\text{Mod})$,
- $\otimes_{L^1}^R : D^{-}(\text{Mod}-R) \times D((R,S)-\text{Mod}) \to D(\text{Mod}-S)$,
- $\text{RHom}_{R-\text{Mod}} : D((R,S)-\text{Mod}) \times D^{+}(R-\text{Mod}) \to D(S-\text{Mod})$,
- $\text{RHom}_{S-\text{Mod}} : D^{-}(R-\text{Mod}) \times D((R,S)-\text{Mod}) \to D(\text{Mod}-S)$,
- $\text{RHom}_{S-\text{Mod}} : D((R,S)-\text{Mod}) \times D^{+}(\text{Mod}-S) \to D(\text{Mod}-R)$,
- $\text{RHom}_{S-\text{Mod}} : D^{-}(\text{Mod}-S) \times D((R,S)-\text{Mod}) \to D(R-\text{Mod})$.

**Theorem 2.8.15.** We have isomorphisms

$$(L \otimes_{L^1}^R M) \otimes_{L^1}^S N \simeq L \otimes_{L^1}^R (M \otimes_{L^1}^S N),$$

$\text{RHom}_{R-\text{Mod}}(M \otimes_{L^1}^R N, K) \simeq \text{RHom}_{S-\text{Mod}}(N, \text{RHom}_{R-\text{Mod}}(M, K))$,

$\text{RHom}_{S-\text{Mod}}(L \otimes_{L^1}^R M, P) \simeq \text{RHom}_{S-\text{Mod}}(L, \text{RHom}_{S-\text{Mod}}(M, P))$,

natural in $L \in D^{-}(\text{Mod}-R)$, $M \in D((R,S)-\text{Mod})$, $N \in D^{-}(S-\text{Mod})$, $K \in D^{+}(R-\text{Mod})$, $P \in D^{+}(\text{Mod}-S)$.

**Proof.** We may assume $K, P \in C^+$ with injective components and $L, N \in C^-$ with projective components (for the first isomorphism it suffices to take $L, N \in C^-$ with flat components). Then the isomorphisms are given by isomorphisms of triple complexes. □

**Proposition 2.8.16.** Let $R \to S$ be a ring homomorphism. We have isomorphisms

$\text{RHom}_{R-\text{Mod}}(N, K) \simeq \text{RHom}_{S-\text{Mod}}(N, \text{RHom}_{R-\text{Mod}}(S, K))$,

$\text{RHom}_{S-\text{Mod}}(S \otimes_R^L K', N) \simeq \text{RHom}_{R-\text{Mod}}(K', N)$,

natural in $N \in D(R-\text{Mod})$, $K \in D^{+}(S-\text{Mod})$, $K' \in D^{-}(S-\text{Mod})$.

For $N \in D^{-}$ (resp. $D^{+}$), the first (resp. second) isomorphism is a special case of the theorem.

**Proof.** We may assume $K \in C^+$ with injective components and $K' \in C^-$ with projective components. Then $\text{Hom}_{R-\text{Mod}}^*(S, K)^n$ is an injective $S$-module and $S \otimes_R K'^m$ is a projective $S$-module. □

**Remark 2.8.17.** The derived functors and natural isomorphisms above all extend to unbounded derived categories.
Flat dimension

The following generalizes Proposition 2.8.10.

**Proposition 2.8.18.** Let $Y$ be a left $R$-module and let $m \geq 0$ be an integer. The following conditions are equivalent:

1. There exists a flat resolution $Y'$ of $Y$ concentrated in $[-m+1,0]$.
2. $\text{Tor}_n^R(X,Y) = 0$ for every right $R$-module $X$.
3. $\text{Tor}_n^R(X,Y) = 0$ for every right $R$-module $X$ and $n \geq m$.
4. $- \otimes_R Y$ carries $D^{\geq 0}(\text{Mod-}R)$ to $D^{\geq -m+1}(\text{Ab})$.

**Proof.** (1) $\Rightarrow$ (4). Indeed for $L \in C^{\geq 0}(\text{Mod-}R)$, $L \otimes_R Y \simeq \text{tot}(L \otimes_R Y)$, and the latter is concentrated in $[-m+1, +\infty)$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (1). For $m = 0$, taking $X = R$, we get $Y = 0$. For $m \geq 1$, we apply the lemma below (with $k = m - 1$) to get the flat resolution. \hfill \Box

**Lemma 2.8.19 (Dimension shifting).** Let $0 \to Y' \to F^{-k+1} \to \cdots \to F^0 \to Y$ be an exact sequence of left $R$-modules with $F^i$ flat. Then $\text{Tor}_n^R(X,Y') \simeq \text{Tor}_{n+k}^R(X,Y)$ for $n \geq 1$.

**Proof.** Decomposing the exact sequence into short exact sequences, we reduce by induction to the case $k = 1$. In this case, the assertion follows from the long exact sequence. \hfill \Box

**Definition 2.8.20.** Let $X$ be a right $R$-module and let $Y$ be a left $R$-module. The **flat dimensions** (or Tor-dimensions) of $X$ and $Y$ are defined to be

$$\text{fl.dim}(X) = \sup \{ n \in \mathbb{Z} | \text{Tor}_n^R(X,Y) \neq 0 \text{ for some } Y \},$$

$$\text{fl.dim}(Y) = \sup \{ n \in \mathbb{Z} | \text{Tor}_n^R(X,Y) \neq 0 \text{ for some } X \}.$$  

The **weak dimension** of $R$ is defined to be

$$\text{w.dim}(R) = \sup \{ n \in \mathbb{Z} | \text{Tor}_n^R(X,Y) \neq 0 \text{ for some } X,Y \}.$$  

The above dimensions take values in $\mathbb{N} \cup \{+\infty\}$. Proposition 2.8.18 gives equivalent conditions for $\text{fl.dim}(Y) < m$. We have $\text{fl.dim}(Y) = -\infty$ if and only if $Y = 0$. By definition,

$$\text{w.dim}(R) = \sup_{X \in \text{Mod-}R} \text{fl.dim}(X) = \sup_{Y \in \text{Mod-}R} \text{fl.dim}(Y),$$

so that $\text{w.dim}(R^{op}) = \text{w.dim}(R)$.

**Remark 2.8.21.** Since projective modules are flat, we have

$$\text{fl.dim}(X) \leq \text{proj.dim}(X), \quad \text{fl.dim}(Y) \leq \text{proj.dim}(Y),$$

$$\text{w.dim}(R) \leq \min \{ \text{l.gl.dim}(R), \text{r.gl.dim}(R) \}.$$
Künneth formula

**Proposition 2.8.22** (Künneth formula for hyper Tor). Assume that $R$ is left and right hereditary. Let $L \in D(\text{Mod}-R)$, $M \in D(R\text{-Mod})$ such that either (a) $L, M \in D^-$, or (b) $L \in D^b$, or (c) $M \in D^b$. Then we have a split short exact sequence

$$0 \to \bigoplus_{l+m=-n} (H^lL \otimes_R H^mM) \xrightarrow{f} \text{Tor}_n^R(L, M) \xrightarrow{g} \bigoplus_{l+m=1-n} \text{Tor}_1^R(H^lL, H^mM) \to 0.$$  

Here $f$ and $g$ are induced by

$$H^lL \otimes_R H^mM \simeq \text{Tor}_n^R(\tau_{\le l}L, \tau_{\le m}M) \to \text{Tor}_n^R(L, M),$$

$$\text{Tor}_n^R(L, M) \to \text{Tor}_n^R(\tau_{\ge l}L, \tau_{\ge m}M) \simeq \text{Tor}_1^R(H^lL, H^mM).$$

The splitting is not canonical.

**Proof.** We may assume $L \in D^b$. By Proposition 2.5.24 we have

$$\text{Tor}_n^R(L, M) \simeq \bigoplus_{l,m} \text{Tor}_{l+m+n}(H^lK, H^mL).$$

\[ \square \]

**Remark 2.8.23.** Recall that the singular (resp. cellular) (co)homology of a topological space (resp. CW complex) $X$ with coefficients in an abelian group $M$ is defined by

$$H_n(X, M) = H^{-n}(C_\bullet(X) \otimes M), \quad H^n(X, M) = H^n\text{Hom}^\bullet(C_\bullet(X), M).$$

Here $C_\bullet(X) \in C(\mathbb{C}^\mathbb{R})$ denotes the singular (resp. cellular) chain complex, which is a complex of free abelian groups. In other words,

$$H_n(X, M) = \text{Tor}_n(C_\bullet(X), M), \quad H^n(X, M) = \text{Ext}^n(C_\bullet(X), M).$$

Since $\mathbb{Z}$ is hereditary, we get split short exact sequences

$$0 \to H_n(X) \otimes M \to H_n(X, M) \to \text{Tor}_1(H_{n-1}(X), M) \to 0,$n

$$0 \to \text{Ext}^1(H_{n-1}(X), M) \to H^n(X, M) \to \text{Hom}(H_n(X), M) \to 0,$n

where $H_n(X) = H_n(X, \mathbb{Z})$. These sequences are known as universal coefficient theorems.

For topological spaces $X$ and $Y$, the Eilenberg-Zilber theorem provides an isomorphism $C_\bullet(X \times Y) \simeq \text{tot}(C_\bullet(X) \otimes C_\bullet(Y))$ in $K(\text{Ab})$. (For CW complexes, we have an isomorphism in $C(\text{Ab})$.) Thus we have $H_n(X \times Y) \simeq \text{Tor}_n(C_\bullet(X), C_\bullet(Y))$. Applying the Künneth formula, we get a split short exact sequence

$$0 \to \bigoplus_{l+m=n} H_l(X) \otimes H_m(Y) \to H_n(X \times Y) \to \bigoplus_{l+m=n-1} \text{Tor}_1(H_l(X), H_m(X)) \to 0.$$n

This is also called the Künneth formula.
More flatness tests

**Definition 2.8.24.** Let $R$ be a domain and let $M$ be an $R$-module. The set of $m \in M$ such that $rm = 0$ for some nonzero $r \in R$ is a submodule called the *torsion submodule* of $M$. We say that $M$ is *torsion-free* if its torsion submodule is zero.

Note that any submodule of a torsion-free $R$-module is torsion-free.

**Proposition 2.8.25.** Let $R$ be a domain. Any flat $R$-module $M$ is torsion-free.

*Proof.* It suffices to show that for any nonzero $r \in R$, the map $g : M \to M$ carrying $m$ to $rm$ is injection. Consider the homomorphism $f : R \to R$ carrying $x$ to $rx$, which is an injection. By the flatness of $M$, the map $f \otimes_R M : R \otimes_R M \to R \otimes_R M$, which can be identified with $g$, is an injection.

**Proposition 2.8.26.** Let $R$ be a commutative domain. If (a) finitely generated submodules of free $R$-modules are projective, then (b) every torsion-free $R$-module $M$ is flat.

Note that the assumption is satisfied if $R$ is a Dedekind domain.

*Proof.* Since $M$ is a filtered colimit of its finitely generated submodules, we may assume $M$ finitely generated. In this case, by the assumption and the following lemma, $M$ is projective.

**Lemma 2.8.27.** Let $R$ be a commutative domain. Every finitely generated torsion-free $R$-module $M$ can be embedded into $R^n$ for some $n$.

*Proof.* Let $K$ be the fraction field of $R$. Then $M \simeq M \otimes_R R \subseteq M \otimes_R K \simeq K^n$ for some $n$. Since $M$ is finitely generated, there exists $r \in R$ such that the image of $M$ is contained in $r^{-1}R^n$.

**Remark 2.8.28.** (1) Similarly to Kaplansky’s theorem, one can show that for any ring $R$, Condition (a) of Proposition 2.8.26 is equivalent to the condition that every finitely generated left ideal of $R$ is projective. A ring satisfying this condition is said to be *left semi-hereditary*. See [L1, Theorem 2.29] for details.

(2) One can show that for a commutative domain, Conditions (a) and (b) of Proposition 2.8.26 are equivalent. A commutative domain satisfying these equivalent conditions is called a *Prüfer domain* (or *Prüfer ring*). See [L1, Theorem 4.69] for details.

Next we present Lambek’s theorem. Let $R$ be a ring. Given a left $R$-module $M$, the *character module* of $M$ is $M^* = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ which is a right $R$-module. Recall that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. It has the following “cogenerator” property.

**Lemma 2.8.29.** For any abelian group $A$ and nonzero element $x \in A$, there exists a homomorphism of abelian groups $f : A \to \mathbb{Q}/\mathbb{Z}$ with such that $f(x) \neq 0$.

---

*4In the literature the character module is often denoted by $M'$ rather than $M^*$. (*)
Proof. There exists a nonzero homomorphism $\mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, this extends to a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$.

Remark 2.8.30. It follows that every abelian group can be embedded into a product of $\mathbb{Q}/\mathbb{Z}$. Such a product is sometimes said to be “cofree”.

Lemma 2.8.31. A sequence of $R$-modules $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if $Z^* \xrightarrow{g^*} Y^* \xrightarrow{f^*} X^*$ is exact.

Proof. The “only if” part follows from the fact that the functor $M \mapsto M^*$ is exact. Next we show the “if” part.

Let us first show that $gf = 0$. Otherwise there exists $x \in X$ such that $gf(x) \neq 0$, so that by Lemma 2.8.29 there exists $h \in Z^*$ such that $hgf(x) \neq 0$, which implies $f^*g^*(h) = hgf \neq 0$, contradicting $f^*g^* = 0$.

If $\text{Im}(f) \supseteq \text{Ker}(g)$ does not hold, then applying Lemma 2.8.29 to $Y/\text{Im}(f)$ we get $i \in Y^*$ such that $i(\text{Im}(f)) = 0$ but $i(\text{Ker}(g)) \neq 0$. Then $f^*(i) = 0$ so that $i = g^*(h) = hg$, contradiction.

Theorem 2.8.32 (Lambek). A left $R$-module $M$ is flat if and only if $M^*$ is an injective right $R$-module.

Proof. This follows from the natural isomorphism $\text{Hom}_{\text{Mod-}R}( -, M^*) \simeq (- \otimes_R M)^*$ of functors $\text{Mod-}R \to \text{Ab}$ and the preceding lemma.

Proposition 2.8.33. A left $R$-module $M$ is flat if and only if for every right ideal $I \subseteq R$, the map $I \otimes_R M \to M$ is an injection (in other words, $I \otimes_R M \to IM$ is a bijection).

Proof. This follows from the natural isomorphism $\text{Hom}_{\text{Mod-}R}( -, M^*) \simeq (- \otimes_R M)^*$ together with Baer’s test.

Corollary 2.8.34. Let $R$ be a ring, let $M$ be a left $R$-module, and let $n \geq 0$ be an integer. Then $\text{fl.dim}(M) \leq n$ if and only if $\text{Tor}^R_{n+1}(R/I, M) = 0$ for every right ideal $I$ of $R$.

Proof. The “only if” part is clear. For the “if” part, take an exact sequence $0 \to N \to F_{n+1} \to \cdots \to F_0 \to M \to 0$ with $F_i$ flat. Then $\text{Tor}^R_1(R/I, N) \cong \text{Tor}^R_{n+1}(R/I, M) = 0$. Thus the map $I \otimes_R N \to N$ is an injection. It follows from the proposition that $N$ is flat.

Corollary 2.8.35. For any ring $R$, we have

$$w.\text{dim}(R) = \sup_I \text{fl.dim}(R/I) = \sup_J \text{fl.dim}(R/J),$$

where $I$ (resp. $J$) runs through left (resp. right) ideals of $R$.

Definition 2.8.36. We say that an $R$-module $M$ is finitely presented if there exists an exact sequence $R^n \to R^n \to M \to 0$ with $m, n \geq 0$.

Finitely presented $R$-modules are finitely generated. Conversely, $R$ is left Noetherian if and only if every finitely generated $R$-module is finitely presented [L1, Proposition 4.29].
Proposition 2.8.37. Let $M$ be a finitely generated $R$-module. Then $M$ is projective if and only if it is flat and finitely presented.

In particular, if $M$ is finitely presented, then $M$ is projective if and only if $M$ is flat. We refer the reader to [L1, Theorem 4.30] for a generalization.

Lemma 2.8.38. For left $R$-modules $X$ and $Y$, the homomorphism $Y^* \otimes_R X \to \text{Hom}_R(X,Y)^*$ carrying $g \otimes x$ to $f \mapsto gf(x)$ is an isomorphism whenever $X$ is finitely presented.

Proof. Since both functors $Y^* \otimes_R -$ and $\text{Hom}_R(-,Y)^*$ are right exact, we reduce to the trivial case where $X = R^n$.

Proof of Proposition 2.8.37. The “only if” part. It is clear that $M$ is flat. Consider an epimorphism $R^n \to M$. The kernel is a direct summand of $R^n$, hence finitely generated. Thus $M$ is finitely presented.

The “if” part. Since $(-)^* \otimes_R M \simeq \text{Hom}(M,-)^*$ is an exact functor, $\text{Hom}(M,-)^*$ is exact as well.

Theorem 2.8.39 (Auslander). Let $R$ be a left Noetherian ring. For any finitely generated left $R$-module $M$, we have $\text{fl.dim}(M) = \text{proj.dim}(M)$. Moreover, $\text{w.dim}(R) = \text{l.gl.dim}(R)$.

Proof. For the first assertion it suffices to show $\text{proj.dim}(M) \leq \text{fl.dim}(M) = n$. Consider an exact sequence $0 \to N \to F^{-n+1} \to \cdots \to F^0 \to M \to 0$ with $F^n = R^n$. Then $\text{fl.dim}(N) = 0$, namely that $N$ is flat. Since $N$ is finitely generated, it is projective. Thus $\text{proj.dim}(M) \leq n$. For the second assertion, it suffices to show $\text{l.gl.dim}(R) \leq \text{w.dim}(R)$. This follows from the first assertion and Corollary 2.5.22.

Corollary 2.8.40. Let $R$ be a left and right Noetherian ring. Then $\text{w.dim}(R) = \text{l.gl.dim}(R) = \text{r.gl.dim}(R)$.

Theorem 2.8.41. A ring $R$ has weak dimension zero if and only if for each $r \in R$, there exists $s \in R$ such that $rsr = r$.

We refer the reader to [L1, Theorem 4.21] for a proof. Rings satisfying the equivalent conditions of the theorem are called von Neumann regular rings. A von Neumann regular domain is a division ring. A ring is semisimple if and only if it is von Neumann regular and left (or right) Noetherian.

Example 2.8.42. Boolean rings ($r^2 = r$ for all $r \in R$) are von Neumann regular.

The following consequence of Lemma 2.8.38 will be used in the next section.

Proposition 2.8.43. Let $R$ be a left Noetherian ring and let $X$ be a finitely generated left $R$-module. For $Y \in D(R\text{-Mod})$, we have $Y^* \otimes_R X \simeq \text{RHom}_R(X,Y)^*$. In particular, $\text{Tor}_n^R(Y^*,X) \simeq \text{Ext}_R^n(X,Y)^*$ for all $n \in \mathbb{Z}$.

Proof. Replacing $X$ by a resolution by free $R$-modules of finite ranks, we are reduced to Lemma 2.8.38.
2.9 Homology and cohomology of groups

Standard resolution

Let $G$ be a group. Consider the trivial $G$-action on $\mathbb{Z}$. For any $G$-module $M$, we have

$$M^G \simeq \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M), \quad M_G = M/I_G M \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} M,$$

where $I_G = \text{Ker}(\mathbb{Z}G \to \mathbb{Z})$ (the map given by $\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g$) is the augmentation ideal. Thus

$$H^n(G, M) \simeq \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M), \quad H_n(G, M) \simeq \text{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M).$$

These groups can be computed using a projective resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$. Note that the isomorphism $\mathbb{Z}G \simeq (\mathbb{Z}G)^{\text{op}}$ carrying $g$ to $g^{-1}$ induces an isomorphism of categories between left $G$-modules and right $G$-modules.

Definition 2.9.1. The standard resolution of $\mathbb{Z}$ is the sequence

$$\cdots \to F^{-1} \to F^0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules defined as follows. For each $n \geq 0$, $F^{-n}$ is the free $\mathbb{Z}$-module on the set $G^{n+1} = \{(g_0, \ldots, g_n)\}$, with $G$-action defined by $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$. The differentials are defined by

$$d^{-n}(g_0, \ldots, g_n) = \sum_{i=0}^n (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n),$$

where $\hat{g}_i$ means removing $g_i$.

The sequence is clearly a complex of $\mathbb{Z}G$-modules. It is exact, since the underlying complex of $\mathbb{Z}$-modules is homotopy equivalent to zero: $\text{id} = dh + hd$, where $h^{-n}(g_0, \ldots, g_n) = (1, g_0, \ldots, g_n)$. Note that each $F^{-n}$ is a free $\mathbb{Z}G$-module. Thus $H^n(G, M)$ is the $n$-th cohomology group of the complex

$$0 \to C^0(G, M) \to C^1(G, M) \to \cdots,$$

and $H_n(G, M)$ is the $-n$-th cohomology group of the complex

$$\cdots \to C_1(G, M) \to C_0(G, M) \to 0,$$

where

$$C^n(G, M) = \text{Hom}_{\mathbb{Z}G}(F^{-n}, M), \quad C_n(G, M) = F^{-n} \otimes_{\mathbb{Z}G} M.$$

A basis of the free $\mathbb{Z}G$-module $F^{-n}$ is given by those elements of $G^{n+1}$ whose 0-th component is 1. It is convenient to adopt the bar notation:

$$[g_1|g_2|\ldots|g_n] = (1, g_1, g_1g_2, \ldots, g_1\ldots g_n).$$

We have

$$d^{-n}[g_1|g_2|\ldots|g_n] = g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1|\ldots|g_ig_{i+1}|\ldots|g_n] + (-1)^n [g_1|g_2|\ldots|g_{n-1}].$$

Thus $C^n(G, M)$ can be identified with the abelian group of maps $f: G^n \to M$, with differential given by

$$(-1)^n(d^{-n-1}f)(g_1, \ldots, g_n) = g_1f(g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i f(g_1, \ldots, g_ig_{i+1}, \ldots, g_n) + (-1)^n f(g_1, g_2, \ldots, g_{n-1}).$$
2.9. HOMOLOGY AND COHOMOLOGY OF GROUPS

$H^1$ and $H_1$

We have

$$H^1(G, M) = Z^1(G, M)/B^1(G, M),$$

where $Z^1(G, M)$ is the group of crossed homomorphisms (or derivations) $f : G \to M$, namely maps satisfying $f(gh) = f(g) + gf(h)$, and $B^1(G, M)$ is the group of principal crossed homomorphisms, namely maps of the form $g \mapsto gm - m$ for some $m \in M$. In particular, for a trivial $G$-module (namely, abelian group with trivial $G$-action) $A$, we have $H^1(G, A) \simeq \text{Hom}(G_{ab}, A)$, where $G_{ab} = G/[G, G]$ is the abelianization of $A$.

The short exact sequence

$$0 \to I_G \to ZG \to Z \to 0$$

induces a long exact sequence

$$0 \to H_1(G, M) \to I_G \otimes_{ZG} M \xrightarrow{\partial} M \to M_G \to 0,$$

where $f(\sum_{g \in G} a_g g) \otimes m = \sum_{g \in G} gm$. For a trivial $G$-module $A$, we have $f = 0$ so that $H_1(G, A) \simeq I_G \otimes_{ZG} A \simeq I_G \otimes_{ZG} Z \otimes Z A$. Applying $I_G \otimes_{ZG} \to (2.9.1)$, we get $I_G \otimes_{ZG} Z \simeq I_G/I_G^2$. Moreover, we have an isomorphism $I_G/I_G^2 \simeq G_{ab}$ carrying the class of $\sum a_g g$ to $\prod \bar{g}^m$, where $\bar{g}$ denotes the class of $g$ in $G_{ab}$. The inverse carries $\bar{g}$ to the class of $g - 1$, which is well-defined since $gh - 1 - (g - 1) - (h - 1) = (g - 1)(h - 1) \in I_G^2$. Therefore, $H_1(G, A) \simeq G_{ab} \otimes Z A$.

Universal coefficients and duality

**Theorem 2.9.2** (Universal coefficients). For any trivial $G$-module $A$, we have split short exact sequences

$$0 \to H_n(G) \otimes Z A \to H_n(G, A) \to \text{Tor}_1^Z(H_{n-1}(G), A) \to 0,$$

$$0 \to \text{Ext}^1_Z(H_{n-1}(G), A) \to H^n(G, A) \to \text{Hom}_Z(H_n(G), A) \to 0,$$

functorial in $A$. Here $H_n(G) = H_n(G, Z)$.

**Proof.** This follows from Künneth formulas over $Z$ and the isomorphisms

$$(Z \otimes_{ZG} Z) \otimes^L_Z A \simeq Z \otimes^L_{ZG} (Z \otimes^L_Z A) \simeq Z \otimes^L_{ZG} A,$$

$$R\text{Hom}_Z(Z \otimes^L_{ZG} Z, A) \simeq R\text{Hom}_{ZG}(Z, R\text{Hom}_Z(Z, A)) \simeq R\text{Hom}_{ZG}(Z, A).$$

\[
\Box
\]

**Remark 2.9.3.** This also follows from the interpretation of $H^n(G, A)$ and $H_n(G, A)$ as the cohomology and homology of any $K(G, 1)$-space.

**Theorem 2.9.4.** Let $G$ be a finite group. For any $G$-module $M$, we have

$$H_n(G, M^*) \simeq H^n(G, M)^*.$$
Here \( M^* = \text{Hom}(M, Q/\mathbb{Z}) \) is a right \( G \)-module. For any right \( G \)-module \( N \), we set \( H_n(G, N) = \text{Tor}^\mathbb{Z}_n(G, N) \).

**Proof.** By Proposition 2.8.43 applied to \( R = \mathbb{Z}G \) and \( X = \mathbb{Z} \), we have
\[
\text{Tor}^\mathbb{Z}_n(G^*, \mathbb{Z}) \simeq \text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, M^*).
\]

**Extensions and crossed extensions**

A sequence of homomorphisms of groups \( G' \to G \xrightarrow{g} G'' \) is said to be **exact** at \( G \) if \( \text{Im}(f) = \text{Ker}(g) \). Given a short exact sequence of groups
\[
0 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1,
\]
where \( M \) is an abelian group, conjugation in \( E \) induces an action of \( G \) on \( M \). We call \( E \) an **extension** of \( G \) by the \( G \)-module \( M \).

The extension splits if there exists a section of \( \pi \) that is a group homomorphism. In this case the extension can be identified with the semidirect product \( M \rtimes G \), with \( i \) and \( \pi \) given by the inclusion and projection. The underlying set of \( M \rtimes G \) is \( M \times G \), with group law given by \( (m, g)(n, h) = (m + gn, gh) \).

**Theorem 2.9.5.** Let \( G \) be a group and \( M \) a \( G \)-module. There is a canonical bijection between \( H^2(G, M) \) and the set of isomorphism classes of extensions of \( G \) by \( M \), carrying the class of \( f \in Z^2(G, M) \) to the class of the group on the set \( M \times G \) with group law given by \( (m, g)(n, h) = (m + gn + f(g, h), gh) \).

In particular, the bijection carries \( 0 \in H^2(G, M) \) to the class of split extensions. The proof is not hard. See for example [HS, Theorem 10.3].

**Example 2.9.6.** The class of the extension
\[
0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}
\]
in \( H^2(G, M) \) with \( G = \mathbb{Z}/n\mathbb{Z} \), \( M = \mathbb{Z}/n\mathbb{Z} \), and trivial \( G \)-action on \( M \), is given by the 2-cocycle \( G^2 \to M \) given by the table of carries \( (\bar{a}, \bar{b}) \mapsto ([\frac{a+b}{n}] \mod n) \), where \( 0 \leq a, b < n \) are representatives of \( \bar{a}, \bar{b} \in \mathbb{Z}/n\mathbb{Z} \).

**Definition 2.9.7.** A **crossed module** is a group homomorphism \( f: G \to H \), equipped with an action \( \alpha \) of \( H \) on \( G \), such that the following diagram commutes
\[
G \times G \xrightarrow{f \times \text{id}_G} H \times G \xrightarrow{\text{id}_H \times f} H \times H
\]
\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & H, \\
\downarrow & & \downarrow, \\
G & \xrightarrow{f} & H.
\end{array}
\]
Here the vertical arrows are given by conjugation \( (g, g') \mapsto gg'g^{-1} \).
Given a crossed module, \( \text{Im}(f) \) is a normal subgroup of \( H \), and \( \text{Ker}(f) \) is central in \( G \), equipped with an action of \( \text{Coker}(f) \). We get an exact sequence of groups

\[
0 \to \text{Ker}(f) \to G \xrightarrow{f} H \to \text{Coker}(f) \to 1.
\]

**Example 2.9.8.** For any group \( G \), the homomorphism \( G \to \text{Aut}(G) \), where \( \text{Aut}(G) \) denotes the group of automorphisms of \( G \), given by conjugation is a crossed module. It gives rise to the exact sequence

\[
0 \to Z(G) \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1,
\]

where \( Z(G) \) denotes the center of \( G \) and \( \text{Out}(G) \) denotes the group of outer automorphisms of \( G \).

**Definition 2.9.9.** Given a group \( G \), a \( G \)-module \( M \), and an integer \( n \geq 1 \), a crossed \( n \)-extension of \( G \) by \( M \) is an exact sequence of groups

\[
0 \to M \xrightarrow{d_0} G_{n-1} \xrightarrow{d_1} \cdots \xrightarrow{d_2} G_2 \xrightarrow{d_1} G_1 \xrightarrow{d_0} G \to 1,
\]

where \( d_1 \) is a crossed module, \( G_i \) is a \( G \)-module for \( i \geq 2 \), \( d_2 : G_2 \to \text{Ker}(d_1) \) is a homomorphism of \( G \)-modules, and \( d_i \) is a homomorphism of \( G \)-modules for \( i \geq 3 \).

**Theorem 2.9.10.** Let \( G \) be a group and \( M \) a \( G \)-module. For each integer \( n \geq 1 \), there is a canonical bijection between \( H^{n+1}(G, M) \) and the set of equivalence classes of \( n \)-extensions of \( G \) by \( M \).

The equivalence relation is defined similarly to Yoneda \( n \)-extensions.

The theorem was discovered independently by several people in the 1970s. We refer to [ML1] for a sketch of the history and references.

## 2.10 Spectral objects and spectral sequences

Let \( \hat{\mathbb{Z}} \) be the set \( \mathbb{Z} \cup \{ \pm \infty \} \), ordered by inclusion. Let \( C_2 \) be the partially ordered set of pairs \((p, q) \in \hat{\mathbb{Z}}^2, p \leq q \), corresponding to the category of morphisms of \( \hat{\mathbb{Z}} \). Let \( C_3 \) be the partially ordered set of triples \((p, q, r) \in \hat{\mathbb{Z}}^3, p \leq q \leq r \).

**Definition 2.10.1.** Let \( D \) be a triangulated category. A spectral object in \( D \) consists of a functor \( X : C_2 \to D \) and morphisms \( \delta(p, q, r) : X(q, r) \to X(p, q)[1] \), functorial in \((p, q, r) \in C_3 \), such that for each \((p, q, r) \),

\[
X(p, q) \to X(p, r) \to X(q, r) \xrightarrow{\delta(p,q,r)} X(p, q)[1]
\]

is a distinguished triangle in \( D \).

**Definition 2.10.2.** Let \( A \) be an abelian category. A spectral object in \( A \) consists of functors \( H^n : C_2 \to A, n \in \mathbb{Z} \), and morphisms \( \delta^n(p, q, r) : H^n(q, r) \to H^{n+1}(p, q) \), functorial in \((p, q, r) \in C_3 \), such that for each \((p, q, r) \), the sequence

\[
H^n(p, r) \to H^n(q, r) \xrightarrow{\delta^n(p,q,r)} H^{n+1}(p, q) \to H^{n+1}(p, r)
\]

is exact.
Example 2.10.3. Let $X$ be a complex in $\mathcal{A}$, equipped with an increasing filtration

$$0 = X(-\infty) \hookrightarrow \cdots \hookrightarrow X(p) \hookrightarrow X(p+1) \hookrightarrow X(\infty) = X.$$ 

Set $X(p, q) = X(q)/X(p)$. The short exact sequence

$$0 \to X(q)/X(p) \to X(r)/X(p) \to X(r)/X(q) \to 0$$

induces a distinguished triangle

$$X(p, q) \to X(p, r) \to X(q, r) \to X(p, q)[1]$$

in $D(\mathcal{A})$. We thus obtain a spectral object in $D(\mathcal{A})$.

For any complex $X$, the following spectral objects induced by truncation are particularly useful. The spectral object associated to the filtration $X(p) = \sigma\geq -p X$ is called the first spectral object of $X$, satisfying $X(-p-1, -p) \simeq X[p][-p]$. The spectral object associated to the filtration $X(p) = \tau\leq p X$ is called the second spectral object of $X$, satisfying $X(p-1, p) \simeq (H^p X)[p]$.

Note that the first spectral object of $X$ is functorial in $X \in C(\mathcal{A})$ and the second spectral object of $X$ is functorial in $X \in D(\mathcal{A})$.

Remark 2.10.4. Let $X$ be a spectral object in $\mathcal{D}$. Any triangulated functor $F: \mathcal{D} \to \mathcal{D}'$ induces a spectral object $F X$ in $\mathcal{D}'$. Any cohomological functor $H: \mathcal{D} \to \mathcal{A}$ induces a spectral object $H(X[n])$ in $\mathcal{A}$.

Given a spectral object $(H^n, \delta^n)$ in $\mathcal{A}$, consider the increasing filtration

$$F^q = F^q H^n(-\infty, \infty) = \text{Im}(H^n(-\infty, q) \to H^n(-\infty, \infty)).$$

We approximate $\text{gr}^q H^n(-\infty, \infty) = F^q/F^{q-1}$ by

$$E^{p,q}_{r+1} = \text{Im}(H^n(q-r, q) \to H^n(q-r, q+r-1))/\text{Im}(H^n(q-r, q-1) \to H^n(q-r, q+r-1))$$

for $r \geq 1$, where $p = n - q$. We have $E^{p,q}_2 \simeq H^n(q-1, q)$ and $E^{p,q}_{\infty} = \text{gr}^q H^n(-\infty, \infty)$.

One can show that $E^{p,q}_{r+1} \simeq \text{Im}(H^n(q-r, q) \to H^n(q-1, q+r-1))$. The commutative diagram

$$\begin{array}{ccc}
H^n(q-r, q) & \longrightarrow & H^n(q-1, q+r-1) \\
\downarrow \delta^n & & \downarrow \delta^n \\
H^{n+1}(q-2r, q-r) & \longrightarrow & H^{n+1}(q-r-1, q-1)
\end{array}$$

induces a morphism $d^{p,q}_{r+1}: E^{p,q}_{r+1} \to E^{p+r+1,q-r}_{r+1}$.

Definition 2.10.5. Let $a \in \mathbb{Z}$. A spectral sequence $(E^{p,q}_r)_{r \geq a}$ in $\mathcal{A}$ is a family of objects $E^{p,q}_r$ in $\mathcal{A}$ for $p, q \in \mathbb{Z}$ and $r \geq a$, equipped with differentials $d_r = d^{p,q}_r: E^{p,q}_r \to E^{p+r,q-r+1}_r$ such that $d_r^2 = 0$, and isomorphisms of $E^{p,q}_{r+1}$ with the cohomology of $E^{p-r,q+r-1}_r$ at $E^{p,q}_r$. For each $r$, the collection $(E^{p,q}_r, d_r)$ is sometimes called a page of the spectral sequence.

---

Our convention for the second spectral object differs from [V2 III.4.3.1] by a shift by 1.
Given a spectral sequence \((E^p,q_r)_{r \geq a}\) and objects \((H^n)_{n \in \mathbb{Z}}\) in \(\mathcal{A}\), an abutment, usually denoted by \(E^p,q_0 \Rightarrow H^n\), consists of an increasing filtration \(F^a H^n\) on each \(H^n\) and an identification of \(E^p,q_{\infty} = gr^q_F H^n = F^a H^n / F^{q-1} H^n\) with a subquotient of \(E^p,q_r\), compatible with the identification of \(E^p,q_{r+1}\) with a subquotient of \(E^p,q_r\). Here \(p = n - q\).

**Theorem 2.10.6.** Let \((H^n, \delta^n)\) be a spectral object in \(\mathcal{A}\). The construction above provides a spectral sequence with abutment

\[
E^p,q_2 = H^{p+q}(q-1, q) \Rightarrow H^n(-\infty, \infty).
\]

We refer to [V2, Sections II.4.2, II.4.3] for a proof (cf. [CE, Section XV.7]).

**Remark 2.10.7 (Page shift).** Given a spectral sequence \((E^p,q_r)_{r \geq a}\) and \(c \in \mathbb{Z}\), we can produce a spectral sequence \((E'^p,q_r)_{r \geq a-c}\) by \(E'^p,q_r = E^{p+cn,q-rn}\), where \(n = p + q\). Moreover, an abutment \(E^p,q_a \Rightarrow H^n\) is the same as an abutment \(E'^p,q_{a-c} \Rightarrow H^n\). Applying this to the spectral sequence in the theorem, we get a spectral sequence with abutment

\[
E'^p,q_1 = H^{p+q}(-p-1, -p) \Rightarrow H^n(-\infty, \infty).
\]

**Example 2.10.8.** Let \(F: D^*(\mathcal{A}) \to D(\mathcal{B})\) be a triangulated functor and let \(X \in C^*(\mathcal{A})\), where * is either empty or one of +, −, b. Then the first and second spectral objects of \(X\) induce, via \(F\) and the cohomological functor \(H^0: D(\mathcal{B}) \to \mathcal{B}\), spectral sequences with abutments:

\[
E'^p,q_1 = H^q F(X^p) \Rightarrow H^n(FX),
E'^p,q_2 = H^p F(H^q X) \Rightarrow H^n(FX).
\]

In the first spectral sequence, \(d^p,q_1 = H^q F(d^p)\).
Summary of properties of rings and modules

Properties of rings

field $\rightarrow$ division ring $\rightarrow$ semisimple $\rightarrow$ von Neumann regular

$\downarrow$ $\downarrow$ $\downarrow$

PID $\rightarrow$ PLID

Dedekind $\rightarrow$ left hereditary $\rightarrow$ left Noetherian $\rightarrow$ left hereditary

Prüfer $\rightarrow$ left semi-hereditary

commutative domain $\rightarrow$ domain

Properties of $R$-modules

free $\rightarrow$ projective $\rightarrow$ flat $\rightarrow$ torsionfree

$R$ PLID or $R = S[x_1, \ldots, x_n]$ $\rightarrow$ finitely presented $\rightarrow$ $R$ Prüfer

$R$ quasi-Frobenius $\rightarrow$ divisible

$R$ Dedekind or PLID

(R domain)

Here $S$ denotes a PID.
Bibliography


