Lectures on Homological Algebra

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Preface

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Chapter 1

Categories and functors

Very rough historical sketch

Homological algebra studies derived functors between

- categories of modules (since the 1940s, culminating in the 1956 book by Cartan and Eilenberg [CE]);
- abelian categories (Grothendieck's 1957 Tōhoku article [G]); and
- derived categories (Verdier's 1963 notes [V1] and 1967 thesis of *doctorat d'État* [V2] following ideas of Grothendieck).

1.1 Categories

Definition 1.1.1. A *category* $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Ob}(\mathcal{C})$, a collection of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (or simply $\mathrm{Hom}(X, Y)$ when no confusion arises) for every pair of objects (X, Y) of $\mathcal{C}$, and a *composition law*, namely a map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z),$$

denoted by $(f, g) \mapsto gf$ (or $g \circ f$), for every triple of objects (X, Y, Z) of $\mathcal{C}$. These data are subject to the following axioms:

- (associativity) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, we have $h(gf) = (hg)f$.
- (unit law) For every object X of $\mathcal{C}$, there exists an *identity morphism* $\mathrm{id}_X \in \mathrm{End}(X) := \mathrm{Hom}(X, X)$ such that $f\mathrm{id}_X = f$, $\mathrm{id}_X g = g$ for all $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, X)$.

The morphism id_X is clearly unique. A morphism $X \to X$ is called an *endomorphism*.

Remark 1.1.2. For convenience we usually assume that the Hom sets are disjoint. In other words, every morphism $f \in \mathrm{Hom}(X, Y)$ has a unique *source* X and a unique *target* Y.

Remark 1.1.3. Russell's paradox shows that not every collection is a set. Indeed, the collection R of sets S such that $S \notin S$ cannot be a set, for otherwise $R \in R$ if and only if $R \notin R$. To avoid the paradox, the conventional ZFC (Zermelo–Fraenkel + axiom of choice) set theory does not allow the existence of a set containing all sets or unrestricted comprehension. In category theory, however, it is convenient to introduce a collection of all sets in some sense. In NBG (von Neumann–Bernays–Gödel) set theory, which is an extension of ZFC set theory, one distinguishes between sets and proper classes. Another approach, which we adopt, is to assume the existence of an uncountable Grothendieck universe $\mathcal{U}$.[1] Elements of $\mathcal{U}$ are called *small sets*. The following table loosely summarizes the basic terminological differences of the two approaches.

NBG	class	set	proper class
ZFC + $\mathcal{U}$	set	small set	large set

We will mostly be interested in categories whose Hom sets are small, which are sometimes called *locally small* categories. A category $\mathcal{C}$ is called *small* if it is locally small and $\mathrm{Ob}(\mathcal{C})$ is small.

Example 1.1.4.

(1) Let R be a ring.

category	objects	morphisms
Set	small sets	maps
Top	small topological spaces	continuous maps
Grp	small groups	homomorphisms of groups
Ring	small rings	homomorphisms of rings
R-**Mod** (e.g. **Ab**)	small (left) R-modules	homomorphisms of R-modules

In all of the above examples, composition is given by composition of maps.

(2) Any set S can be regarded as a category by

$$\mathrm{Hom}(x,y) = \begin{cases} \{*\} & x = y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Such a category is said to be *discrete*.

(3) More generally, any partially ordered set $(S, \leq)$ can be regarded as a category by

$$\mathrm{Hom}(x,y) = \begin{cases} \{*\} & x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$

(4) Any monoid M can be regarded as a category BM with one object $*$ and $\mathrm{End}(*) = M$. Conversely, given any object X of a category $\mathcal{C}$, $\mathrm{End}(X)$ is a monoid.

Definition 1.1.5. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called an *isomorphism* if there exists a morphism $g\colon Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. The morphism g is unique and is called the *inverse* of f, denoted by f^{-1}.

[1]A Grothendieck universe $\mathcal{U}$ is a set satisfying the following conditions: $y \in x \in \mathcal{U}$ implies $y \in \mathcal{U}$; $x, y \in \mathcal{U}$ implies $\{x, y\} \in \mathcal{U}$; $x \in \mathcal{U}$ implies $P(x) \in \mathcal{U}$ where $P(x)$ is the power set of x; $x_i \in \mathcal{U}$, $i \in I \in \mathcal{U}$ implies $\bigcup_{i\in I} x_i \in \mathcal{U}$. TG (Tarski–Grothendieck) set theory is obtained from ZFC by adding Tarski's axiom, which states that for every set x, there exists a Grothendieck universe $\mathcal{U} \ni x$.

An isomorphism $X \to X$ is called an *automorphism.* We let $\mathrm{Aut}(X) \subseteq \mathrm{End}(X)$ denote the subset consisting of automorphisms.

Remark 1.1.6. The identity map id_X is an isomorphism. The collection of isomorphisms is stable under composition. In particular, $\mathrm{Aut}(X)$ is a group.

A category of which where every morphism is an isomorphism is called a *groupoid.* For example, given a group G, the category BG defined in Example 1.1.4 (3) is a groupoid.

Definition 1.1.7. Let $f\colon X \to Y$ be a morphism in a category $\mathcal{C}$. We say that f is a *monomorphism* if for every pair of morphisms $(g_1, g_2)\colon W \rightrightarrows X$ satisfying $fg_1 = fg_2$, we have $g_1 = g_2$. We say that f is an *epimorphism* if for every pair of morphisms $(h_1, h_2)\colon Y \rightrightarrows Z$ satisfying $h_1 f = h_2 f$, we have $h_1 = h_2$. In other words, f is a monomorphism if and only if the map $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ carrying g to fg is an injection; f is an epimorphism if and only if the map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ carrying h to hf is an injection.

We sometimes represent monomorphisms by $\hookrightarrow$ and epimorphisms by $\twoheadrightarrow$.

Remark 1.1.8. One can show that a morphism in **Set**, **Top**, **Grp**, or R-**Mod** is a monomorphism (resp. epimorphism) if and only if it is an injection (resp. surjection). See the Exercises. On the other hand, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in **Ring** that is not a surjection. One can show that a morphism $f\colon X \to Y$ in the category **HausTop** of small Hausdorff topological spaces is an epimorphism if and only if the image $f(X)$ is dense in Y.

Remark 1.1.9. An isomorphism is necessarily a monomorphism and an epimorphism. The converse does not hold in general. For example, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ in **Ring** is a monomorphism and an epimorphism, but not an isomorphism. Here is another example. In **Top**, the continuous map $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ carrying x to x, where $\mathbb{R}_{\mathrm{disc}}$ denotes the set $\mathbb{R}$ equipped with the discrete topology, is a monomorphism and an epimorphism, but not an isomorphism.

We leave the proof of the following lemma as an exercise.

Lemma 1.1.10. *Consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then*
1. *If f and g are monomorphisms, then gf is a monomorphism.*
2. *If f and g are epimorphisms, then gf is an epimorphism.*
3. *If gf is a monomorphism, then f is a monomorphism.*
4. *If gf if an epimorphism, then g is an epimorphism.*
5. *If gf is an isomorphism and either f is an epimorphism or g is a monomorphism, then g and f are isomorphisms.*

Remark 1.1.11. By Remark 1.1.6 and Lemma 1.1.10 (5), the collection of isomorphisms satisfies the *two-out-of-three* property: For any composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if two of the three morphisms f, g, and gf are isomorphisms, then so is the third one.

Definition 1.1.12. The opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

A morphism f of a category $\mathcal{C}$ is a monomorphism in $\mathcal{C}$ if and only if it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

1.2 Functors

Functors

Definition 1.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and, for every pair of objects (X, Y) of $\mathcal{C}$, a map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, compatible with composition and identity: $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F(gf) = F(g)F(f)$ for all morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Remark 1.2.2. Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$, we have the composite functor $GF\colon \mathcal{C} \to \mathcal{E}$. For any category $\mathcal{C}$, we have the identity functor $\mathrm{id}_{\mathcal{C}}$. We can thus organize small categories and functors into a category **Cat**.

Example 1.2.3. (1) We have forgetful functors **Top** $\to$ **Set** and

$$R\text{-}\mathbf{Mod} \to \mathbf{Ab} \to \mathbf{Grp} \to \mathbf{Set}.$$

(2) We have a functor **Set** $\to$ R-**Mod** carrying a set S to the free R-module $R^{(S)} = \bigoplus_{s\in S} Rs$.
(3) We have a functor $H_n\colon \mathbf{Top} \to \mathbf{Ab}$ carrying a topological space X to its n-th singular homology group $H_n^{\mathrm{sing}}(X)$.
(4) For any object X in a category $\mathcal{C}$ with small Hom sets, we have functors $\mathrm{Hom}(X, -)\colon \mathcal{C} \to \mathbf{Set}$ and $\mathrm{h}_{\mathcal{C}}(X) = \mathrm{Hom}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 1.2.4. A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Definition 1.2.5. Let $(\mathcal{C}_i)_{i\in I}$ be a family of categories. The product category $\mathcal{C} = \prod_{i\in I} \mathcal{C}_i$ is defined by $\mathrm{Ob}(\mathcal{C}) = \prod_{i\in I} \mathrm{Ob}(\mathcal{C}_i)$ and $\mathrm{Hom}_{\mathcal{C}}((X_i), (Y_i)) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i)$.

A functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is sometimes called a *bifunctor*.

Example 1.2.6. (1) For any category $\mathcal{C}$ with small Hom sets, we have a functor $\mathrm{Hom}(-, -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.
(2) Let R, S, and T be rings. We have functors

$$- \otimes_S -\colon (R, S)\text{-}\mathbf{Mod} \times (S, T)\text{-}\mathbf{Mod} \to (R, T)\text{-}\mathbf{Mod},$$
$$\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(-, -)\colon ((R, S)\text{-}\mathbf{Mod})^{\mathrm{op}} \times (R, T)\text{-}\mathbf{Mod} \to (S, T)\text{-}\mathbf{Mod},$$
$$\mathrm{Hom}_{\mathbf{Mod}\text{-}T}(-, -)\colon ((S, T)\text{-}\mathbf{Mod})^{\mathrm{op}} \times (R, T)\text{-}\mathbf{Mod} \to (R, S)\text{-}\mathbf{Mod}.$$

Here (R, S)-**Mod** denotes the category of small (R, S)-bimodules, which can be identified with $(R \otimes_{\mathbb{Z}} S^{op})$-**Mod**.

Natural transformations

Definition 1.2.7. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha\colon F \to G$ consists of morphisms $\alpha_X\colon FX \to GX$ in $\mathcal{D}$ for all objects X of $\mathcal{C}$, such that for every morphism $f\colon X \to Y$ of $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"] & GY. \end{tikzcd}$$

We denote the set of natural transformations $F \to G$ by $\mathrm{Nat}(F, G)$.

Example 1.2.8. Let $U\colon R\text{-}\mathbf{Mod} \to \mathbf{Set}$ be the forgetful functor carrying an R-module to its underlying set. Let $F\colon \mathbf{Set} \to R\text{-}\mathbf{Mod}$ be the free module functor. Then the injection $S \to R^{(S)}$ carrying s to $1 \cdot s$ defines a natural transformation $\alpha\colon \mathrm{id}_{\mathbf{Set}} \to UF$.

Remark 1.2.9. Given functors $F, G, H\colon \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$, we have the (vertically) composite natural transformation $\beta\alpha\colon F \to H$. Functors $\mathcal{C} \to \mathcal{D}$ and natural transformations form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Isomorphisms in this category are called *natural isomorphisms*[2]. A natural transformation α is a natural isomorphism if and only if α_X is an isomorphism for every object X of $\mathcal{C}$.

There is also a horizontal composition of natural transformations: Given a natural transformation $\alpha\colon F \to G$ between functors $\mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha'\colon F' \to G'$ between functors $\mathcal{D} \to \mathcal{E}$, we have $\alpha'\alpha\colon F'F \to G'G$ between functors $\mathcal{C} \to \mathcal{E}$. This composition satisfies various compatibilities. Small categories, functors, and natural transformations, together with horizontal and vertical compositions, form a "2-category".

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an *isomorphism of categories* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. A more useful notion is the following.

Definition 1.2.10. An *equivalence of categories* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that there exist a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{id}_{\mathcal{C}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{D}}$.[3] The functors F and G are then called *quasi-inverses* of each other.

Quasi-inverses of a functor F are unique up to natural isomorphisms.

Example 1.2.11. Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of X is defined as follows: The objects are points of X and a morphism from x to y is a homotopy equivalence class of paths from x to y in X. Composition is given by concatenation of paths.

If X is path-connected and simply connected, then $\Pi_1(X)$ is equivalent to $\{*\}$, but not isomorphic to $\{*\}$ unless X is a singleton.

Remark 1.2.12. If $F \to F'$ is a natural isomorphism of functors, then F is an equivalence of categories if and only if F' is. Given a composable pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, if two of the three functors F, G, and GF are equivalences of categories, then so is the third one. For example, if F is an equivalence and K is a quasi-inverse of GF, then FK is a quasi-inverse of G.

[2]Some authors call them *natural equivalences*.

[3]Some authors write $\simeq$ for equivalences and $\cong$ for isomorphisms. We will write $\simeq$ for isomorphisms and state equivalences verbally.

Faithful functors, full functors

Definition 1.2.13. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful (resp. full, resp. fully faithful) if the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ is an injection (resp. surjection, resp. bijection) for all $X,Y \in \mathrm{Ob}(\mathcal{C})$.

Lemma 1.2.14. *Let $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful functor.*

(1) Let $f\colon X \to Y$ be a morphism of $\mathcal{C}$ such that Ff is an isomorphism. Then f is an isomorphism.

(2) Let X and Y be objects of $\mathcal{C}$ such that $FX \simeq FY$. Then $X \simeq Y$.

Proof. (1) Let g' be an inverse of Ff and let $g\colon Y \to X$ be such that $Fg = g'$. Then g is an inverse of f.

(2) Let $f'\colon FX \to FY$ be an isomorphism and let $f\colon X \to Y$ be such that $Ff = f'$. By (1), f is an isomorphism. □

Definition 1.2.15. Let $\mathcal{C}$ be a category. A *subcategory* of $\mathcal{C}$ is a category $\mathcal{C}_0$ such that $\mathrm{Ob}(\mathcal{C}_0) \subseteq \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}_0}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \mathrm{Ob}(\mathcal{C}_0)$, and the inclusion $\mathcal{C}_0 \to \mathcal{C}$ is a functor. The inclusion functor is necessarily faithful. A *full subcategory* is a subcategory such that the inclusion functor is *fully faithful*.

To specify a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$, it suffices to say which objects belong to $\mathcal{C}_0$. One speaks of the full subcategory *spanned by* (or consisting of) its objects.

Example 1.2.16. The category **Ab** is a full subcategory of **Grp**. The forgetful functor **Grp** → **Set** is faithful, but not fully faithful.

Definition 1.2.17. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an isomorphism $FX \simeq Y$.

Proposition 1.2.18. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. Let F be an equivalence of categories and let G be a quasi-inverse. Since $FG \simeq \mathrm{id}$, F is essentially surjective. Since $GF \simeq \mathrm{id}$, G is essentially surjective. For objects X, X' of $\mathcal{C}$, since the composition

$$(GF)_{X,X'}\colon \mathrm{Hom}_{\mathcal{C}}(X,X') \xrightarrow{F_{X,X'}} \mathrm{Hom}_{\mathcal{D}}(FX,FX') \xrightarrow{G_{FX,FX'}} \mathrm{Hom}_{\mathcal{C}}(GFX,GFX')$$

is an isomorphism, $F_{X,X'}$ is an injection. Similarly, for objects Y,Y' of $\mathcal{D}$, $F_{GY,GY'}$ is a surjection. Since G is essentially surjective, it follows that $F_{X,X'}$ is a surjection.

For the converse, we will prove the following more explicit version. □

Proposition 1.2.19. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. For each object Y of $\mathcal{D}$, let X_Y be an object of $\mathcal{C}$ and $\epsilon_Y\colon FX_Y \to Y$ an isomorphism. Then there exists a unique quasi-inverse $G\colon \mathcal{D} \to \mathcal{C}$ of F such that $GY = X_Y$ for every object Y of $\mathcal{D}$ and that $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.*

Proof. By the full faithfulness of F, for any morphism $g\colon Y \to Y'$ in $\mathcal{D}$, there exists a unique morphism $Gg\colon GY \to GY'$ rendering the following square commutative:

$$
\begin{tikzcd}
FGY \arrow[r, "\epsilon_Y"] \arrow[d, "FGg"'] & Y \arrow[d, "g"] \\
FGY' \arrow[r, "\epsilon_{Y'}"] & Y'.
\end{tikzcd}
\tag{1.2.1}
$$

This proves the uniqueness of G. For the existence, it is easy to check that the G defined above is a functor. Clearly ϵ is an natural isomorphism. For every object X of $\mathcal{C}$, $\epsilon_{FX}\colon FGFX \to FX$ equals $F\eta_X$ for a unique morphism $\eta_X\colon GFX \to X$, which is an isomorphism by Lemma 1.2.14. From (1.2.1), one deduces the commutative square

$$
\begin{tikzcd}
GFX \arrow[r, "\eta_X"] \arrow[d, "GFf"'] & X \arrow[d, "f"] \\
GFX' \arrow[r, "\eta_{X'}"] & X'
\end{tikzcd}
$$

for every morphism $f\colon X \to X'$ in $\mathcal{C}$. Thus $\eta\colon GF \to \mathrm{id}_{\mathcal{C}}$ is a natural isomorphism and G is a quasi-inverse of F. □

Corollary 1.2.20. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then F induces an equivalence of categories $\mathcal{C} \to \mathcal{D}_0$, where $\mathcal{D}_0$ is the full subcategory of $\mathcal{D}$ spanned by the image of F.*

Corollary 1.2.21. *For any category $\mathcal{C}$, there exists a full subcategory $\mathcal{C}_0$ such that the inclusion functor $\mathcal{C}_0 \to \mathcal{C}$ is an equivalence of categories and isomorphic objects in $\mathcal{C}_0$ are equal.*

Proof. By the axiom of choice, we can pick a representative in each isomorphism class of objects $\mathcal{C}$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by the representatives. The inclusion functor $\mathcal{C}_0 \to \mathcal{C}$ is fully faithful and essentially surjective, and hence an equivalence of categories by Proposition 1.2.18. □

Yoneda's lemma and representable functors

Let $\mathcal{C}$ be a category with small Hom sets. For every object X of $\mathcal{C}$, consider the functor $\mathrm{h}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Lemma 1.2.22 (Yoneda)**.** *For every functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, the map*

$$\phi\colon \mathrm{Nat}(\mathrm{h}_{\mathcal{C}}(X), F) \to F(X)$$

given by $\phi(\alpha) = \alpha_X(\mathrm{id}_X)$ is a bijection.

We leave it as an exercise to state a dual of Yoneda's lemma.

Proof. We construct the inverse $\psi\colon F(X) \to \mathrm{Nat}(\mathrm{h}_{\mathcal{C}}(X), F)$ by $\psi(x)_Y(f) = F(f)(x)$ for $f\colon Y \to X$. We have $(\phi\psi)(x) = \psi(x)_X(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = x$. Moreover, $(\psi\phi)(\alpha)_Y(f) = F(f)(\phi(\alpha)) = F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Y(h_{\mathcal{C}}(f)(\mathrm{id}_X)) = \alpha_Y(f)$. □

Note that $\mathrm{h}_{\mathcal{C}}(X)$ is functorial in X, in the sense that we have a functor $\mathrm{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$.

Corollary 1.2.23. *The functor* $\mathrm{h}_{\mathcal{C}}$ *is fully faithful.*

This functor is called the *Yoneda embedding.*

Proof. Indeed, the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Nat}(\mathrm{h}_{\mathcal{C}}(X), \mathrm{h}_{\mathcal{C}}(Y))$ given by $\mathrm{h}_{\mathcal{C}}$ coincides with the bijection ψ constructed in the proof of the lemma for $F = \mathrm{h}_{\mathcal{C}}(Y)$. □

Applying Lemma 1.2.14, we get the following.

Corollary 1.2.24. *(1) Let* $f\colon X \to Y$ *be a morphism such that* $\mathrm{h}_{\mathcal{C}}(f)\colon \mathrm{h}_{\mathcal{C}}(X) \to \mathrm{h}_{\mathcal{C}}(Y)$ *is a natural isomorphism. Then* f *is an isomorphism.*
(2) Let X, Y *be objects such that* $\mathrm{h}_{\mathcal{C}}(X) \simeq \mathrm{h}_{\mathcal{C}}(Y)$*. Then* $X \simeq Y$*.*

Definition 1.2.25. We say that a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *represented* by an object X of $\mathcal{C}$ if there exists a natural isomorphism $F \simeq \mathrm{h}_{\mathcal{C}}(X)$. We say that F is *representable* if it is represented by some object X of $\mathcal{C}$.

By Corollary 1.2.23 and Proposition 1.2.18, $h_{\mathcal{C}}$ induces an equivalence of categories from $\mathcal{C}$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ spanned by representable functors.

1.3 Universal constructions

Initial objects, final objects, zero objects

Definition 1.3.1. Let $\mathcal{C}$ be a category. An object X of $\mathcal{C}$ is called an *initial* object if, for every object Y of $\mathcal{C}$, there exists precisely one morphism $X \to Y$. An object Y of $\mathcal{C}$ is called a *final* (or terminal) object if, for every object X of $\mathcal{C}$, there exists precisely one morphism $X \to Y$.

Remark 1.3.2. An initial object of $\mathcal{C}$ is a final object of $\mathcal{C}^{\mathrm{op}}$ and a final object of $\mathcal{C}$ is an initial object of $\mathcal{C}^{\mathrm{op}}$.

Proposition 1.3.3. *If* X_1 *and* X_2 *are initial objects of* $\mathcal{C}$*, then there exists a unique isomorphism between them. If* Y_1 *and* Y_2 *are final objects of* $\mathcal{C}$*, then there exists a unique isomorphism between them.*

Proof. By definition there exists a unique morphism $f\colon X_1 \to X_2$ and a unique morphism $f'\colon X_2 \to X_1$. By the uniqueness of morphisms $X_1 \to X_1$, we have $f'f = \mathrm{id}_{X_1}$, and similarly $ff' = \mathrm{id}_{X_2}$. Thus f is an isomorphism. The case of final objects follows by duality. □

Example 1.3.4. Let R be a ring.

category	initial object	final object
Set	$\emptyset$	$\{*\}$
Top		
Grp	$\{1\}$	$\{1\}$
R-**Mod** (e.g. **Ab**)	$\{0\}$	$\{0\}$
Ring$^{\text{nu}}$	$\{0\}$	$\{0\}$
Ring	$\mathbb{Z}$	$\{0\}$
Field	none	none
$(S, \leq)$	least element (if any)	greatest element (if any)

Here **Ring**$^{\text{nu}}$ denotes the category of small nonunital rings (also known as rngs).

Definition 1.3.5. If an object is both initial and final, it is called a *zero* object.

Remark 1.3.6. If $\mathcal{C}$ admits a zero object, then for every pair of objects X, Y, there exists a unique morphism $X \to Y$, called the *zero morphism*, that factors through a zero object. Zero objects and zero morphisms are often denoted by 0.

Products, coproducts

Definition 1.3.7. Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A *product* of $(X_i)_{i\in I}$ is an object P of $\mathcal{C}$ equipped with morphisms $p_i \colon P \to X_i$, $i \in I$, called *projections*, satisfying the following universal property: for each object Q of $\mathcal{C}$ equipped with morphisms $q_i \colon Q \to X_i$, $i \in I$, there exists a unique morphism $q \colon Q \to P$ such that $q_i = p_i q$. A *coproduct* of $(X_i)_{i\in I}$ is an object U of $\mathcal{C}$ equipped with morphisms $u_i \colon X_i \to U$, $i \in I$, satisfying the following universal property: for each object V of $\mathcal{C}$ equipped with morphisms $v_i \colon X_i \to V$, $i \in I$, there exists a unique morphism $v \colon U \to V$ such that $v_i = v u_i$.

Remark 1.3.8. A product in $\mathcal{C}$ is a coproduct in $\mathcal{C}^{\text{op}}$ and a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{\text{op}}$.

Remark 1.3.9. For $I = \emptyset$, a product is a final object and a coproduct is an initial object.

Proposition 1.3.10. *The product of $(X_i)_{i\in I}$, if it exists, is unique up to unique isomorphism. More precisely, if $(P, (p_i))$ and $(P', (p'_i))$ are products of (X_i), then there exists a unique isomorphism $f \colon P \to P'$ such that $p_i = f p'_i$ for all $i \in I$.*

Proof. Indeed, by the universal property of product, we have a unique morphism $f \colon P \to P'$ such that $p_i = f p'_i$ and a unique isomorphism $f' \colon P' \to P$ such that $p'_i = f' p_i$ for all $i \in I$. It follows that $p_i = f' f p_i$ for all $i \in I$, so that $f' f = \mathrm{id}_P$ by the universal property (applied to $Q = P$). Similarly, $f f' = \mathrm{id}_{P'}$. Therefore, f is an isomorphism. $\square$

Notation 1.3.11. We let $\prod_{i\in I} X_i$ denote the underlying object of a product of $(X_i)_{i\in I}$ if it exists. We let $\coprod_{i\in I} X_i$ denote the underlying object of a coproduct of $(X_i)_{i\in I}$ if it exists.

We speak of finite (resp. small, etc.) products/coproducts if the indexing set I is finite (resp. small, etc.).

Example 1.3.12. Let R be a ring.

<table>
<tr><td>category</td><td>small coproduct</td><td>small product</td></tr>
<tr><td>Set</td><td rowspan="2">disjoint union</td><td rowspan="3">usual product</td></tr>
<tr><td>Top</td></tr>
<tr><td>R-Mod (e.g. Ab)</td><td>direct sum</td></tr>
</table>

In the case of **Top**, the product $P = \prod_{i \in I} X_i$ is the set-theoretic product equipped with the coarsest topology (sometimes called the Tychonoff topology) such that the projections $P \to X_i$ are continuous. The product $U = \coprod_{i \in I} X_i$ is the disjoint union equipped with the finest topology such that the inclusions $X_i \to U$ are continuous.

In the case of R-**Mod**, recall that the direct sum $\bigoplus_{i \in I} M_i$ is the R-submodule of $\prod_{i \in I} M_i$ consisting of elements $(m_i)_{i \in I}$ such that $m_i = 0$ for all but finitely many i.

In the categories **Grp**, **Ring**, **CRing**, small products are usual products. Here **CRing** is the category of small commutative rings. In **CRing**, the coproduct of a pair of rings is tensor product.

Example 1.3.13. In the category associated to a partially ordered set $(S, \leq)$, product means infimum and coproduct means supremum. In particular, if $\leq$ is a *total* order, then $(S, \leq)$ admits products of pairs of objects and coproducts of pairs of objects.

Remark 1.3.14. Let $(X_i)_{i \in I}$ be a family of objects of $\mathcal{C}$ and let $I = \coprod_{j \in J} I_j$ be a partition. If $P_j = \prod_{i \in I_j} X_i$ exists for each j, and $P = \prod_{j \in J} P_j$ exists, then P is a product of $(X_i)_{i \in I}$. In particular, a category admitting products of pairs of objects admits finite nonempty products.

Remark 1.3.15. The universal property for product can be summarized as a bijection

$$\mathrm{Hom}_{\mathcal{C}}(Q, \prod_{i \in I} X_i) \simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Q, X_i).$$

We defined $\prod_{i \in I} X_i$ by spelling out the functor $\mathrm{Hom}(-, \prod_{i \in I} X_i)$ it represents.

Over-categories and under-categories

Let $\mathcal{C}$ be a category and let Y be an object of $\mathcal{C}$. The category $\mathcal{C}_{/Y}$ of objects of $\mathcal{C}$ over Y is defined as follows. An object of $\mathcal{C}_{/Y}$ is a pair (X, f), where X is an object of $\mathcal{C}$ and $f \colon X \to Y$ is a morphism of $\mathcal{C}$. A morphism $(X, f) \to (X', f')$ is a morphism $h \colon X \to X'$ such that $f = f'h$, i.e. the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y. \end{tikzcd}$$

Composition is defined in an obvious way.

Dually, the category $\mathcal{C}_{Y/}$ of objects of $\mathcal{C}$ under Y is defined as follows. An object of $\mathcal{C}_{Y/}$ is a pair (X, f), where X is an object of $\mathcal{C}$ and $f\colon Y \to X$ is a morphism of $\mathcal{C}$. A morphism $(X, f) \to (X', f')$ is a morphism $h\colon X \to X'$ such that $hf = f'$. We have an isomorphism of categories $(\mathcal{C}_{/Y})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{Y/}$.

More generally, let $F\colon I \to \mathcal{C}$ be a functor. The category $\mathcal{C}_{/F}$ of objects of $\mathcal{C}$ over F (or *cones to the base* F) is defined as follows. An object of $\mathcal{C}_{/F}$ is an object X of $\mathcal{C}$ equipped with morphisms $p_i\colon X \to F(i)$, $i \in \mathrm{Ob}(I)$ such that $p_j = F(a)p_i$ for all morphisms $a\colon i \to j$. A morphism $(X, (p_i)) \to (Y, (q_i))$ is a morphism $f\colon X \to Y$ such that $p_i f = q_i$.

Dually, the category $\mathcal{C}_{F/}$ of objects of $\mathcal{C}$ under F (or *cones from the base* F) is defined as follows. An object of $\mathcal{C}_{F/}$ is an object X of $\mathcal{C}$ equipped with morphisms $p_i\colon F(i) \to X$, $i \in \mathrm{Ob}(I)$ such that $F(a)p_j = p_i$ for all morphisms $a\colon i \to j$. A morphism $(X, (p_i)) \to (Y, (q_i))$ is a morphism $f\colon X \to Y$ such that $fp_i = q_i$. We have an isomorphism of categories $(\mathcal{C}_{/F})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{F^{\mathrm{op}}/}$, where $F^{\mathrm{op}}\colon I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Limits, colimits

Given a set I, we may regard a family of objects $(X_i)_{i\in I}$ of $\mathcal{C}$ as a functor $F\colon I \to \mathcal{C}$. Then a product of $(X_i)_{i\in I}$ is the same as a final object of $\mathcal{C}_{/F}$ and a coproduct is the same as an initial object of $\mathcal{C}_{F/}$. More generally, we have the following notion.

Definition 1.3.16. Let I and $\mathcal{C}$ be categories and let $F\colon I \to \mathcal{C}$ be a functor. A *limit* (also called projective limit) of F is a final object of $\mathcal{C}_{/F}$ and a *colimit* (also called inductive limit) of F is an initial object of $\mathcal{C}_{F/}$.

We speak of finite (resp. small, etc.) limits/colimits if the indexing category I is finite (resp. small, etc.).

Limits and colimits are unique up to unique isomorphisms.

Remark 1.3.17. Let us spell out the definition of limit. A limit of F is an object L of $\mathcal{C}$ equipped with morphisms $p_i\colon L \to F(i)$, $i \in \mathrm{Ob}(I)$ such that $p_j = F(f)p_i$ for all morphisms $f\colon i \to j$ and satisfying the following universal property: For every object M of $\mathcal{C}$ equipped with morphisms $q_i\colon M \to F(i)$, $i \in \mathrm{Ob}(I)$ satisfying $q_j = F(f)q_i$ for all morphisms $f\colon i \to j$, there exists a unique morphism $a\colon M \to L$ such that $q_i = p_i a$ for all i. We leave it as an exercise to spell out the definition of colimit.

Notation 1.3.18. We let $\lim F$ or $\lim_{i\in I} F(i)$ denote the underlying object of a limit of F if it exists. We let $\operatorname{colim} F$ or $\operatorname{colim}_{i\in I} F(i)$ denote the underlying object of a colimit of F if it exists. (Other notation: $\varprojlim$ for limit and $\varinjlim$ for colimit.)

Example 1.3.19. As already remarked, for I discrete, a limit indexed by I is a product and a colimit indexed by I is a coproduct.

Example 1.3.20. Let $I = (\bullet \rightrightarrows \bullet)$. Here each $\bullet$ denotes an object and each arrow denotes a morphism that is not an identity. A functor $F\colon I \to \mathcal{C}$ is represented by a pair of morphisms $f, g\colon X \to Y$ in $\mathcal{C}$ with the same source and the same target. A limit of F is called an *equalizer* of the pair and the underlying object is denoted by $\mathrm{eq}(f, g)$. Let us spell out the definition. An equalizer of (f, g) is an object E of $\mathcal{C}$

equipped with a morphism $p\colon E \to X$ such that $fp = gp$ and satisfying the following universal property: For every object E' of $\mathcal{C}$ equipped with a morphism $q\colon E' \to X$ of $\mathcal{C}$ such that $pq = gq$, there exists a unique morphism $a\colon E' \to E$ such that $q = pa$. A colimit of F is called a *coequalizer* and the underlying object is denoted by $\operatorname{coeq}(f, g)$. If $\mathcal{C}$ admits a zero object and g is the zero morphism, these are called respectively *kernel* and *cokernel* of f: $\ker(f) = \operatorname{eq}(f, 0)$, $\operatorname{coker}(f) = \operatorname{coeq}(f, 0)$.

Example 1.3.21. For $I = (\bullet \to \bullet \leftarrow \bullet)$, $F\colon I \to \mathcal{C}$ is given by a pair of morphisms $f\colon A \to B$, $g\colon C \to B$ with the same target in $\mathcal{C}$. A limit of F is called a *pullback* of (f, g) or a fiber product of A and C above B and the underlying object is denoted by $A \times_B C$. Let us spell out the definition. A pullback of (f, g) is an object D of $\mathcal{C}$ equipped with morphisms $a\colon D \to A$, $c\colon D \to C$ such that $fa = gc$ and satisfying the following universal property: For every object E of $\mathcal{C}$ equipped with morphisms $a'\colon E \to A$, $c'\colon E \to C$ such that $fa' = gc'$, there exists a unique morphism $e\colon E \to D$ such that $a' = ae$ and $c' = ce$. The universal property can be visualized as follows

$$\begin{tikzcd} E \arrow[dr, dashed, "e"] \arrow[drr, "c'"] \arrow[ddr, "a'"'] & & \\ & D \arrow[r, "c"'] \arrow[d, "a"] & C \arrow[d, "g"] \\ & A \arrow[r, "f"] & B. \end{tikzcd}$$

In this case, the square above is said to be *Cartesian*.

Dually, for $I = (\bullet \leftarrow \bullet \to \bullet)$, a colimit indexed by I is called a *pushout*. The underlying object B of the colimit of $(A \leftarrow D \to C)$ is denoted by $A \amalg_D C$ and the square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is said to be *coCartesian*.

Remark 1.3.22. A morphism $f\colon X \to Y$ is a monomorphism if and only if the following is a pullback square

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Example 1.3.23. The category **Set** admits small limits. For a functor $F\colon I \to \mathbf{Set}$, $\lim F$ is represented by the subset $L \subseteq \prod_{i \in \operatorname{Ob}(I)} F(i)$ consisting of elements (x_i) such that $F(f)(x_i) = x_j$ for every morphism $f\colon i \to j$ is small, whenever L. The same holds for limits in **Grp**, R-**Mod**, and **Ring**.

For example, for $I = (\bullet \rightrightarrows \bullet)$ and $f, g\colon X \to Y$ in **Set**, $\operatorname{eq}(f, g)$ can be identified with the subset $\{x \in X \mid f(x) = g(x)\}$ of X.

Example 1.3.24. The category **Set** admits small colimits. For a functor $F\colon I \to \mathbf{Set}$, $\operatorname{colim} F$ is represented by the quotient

$$Q = \left(\coprod_{i\in \mathrm{Ob}(I)} F(i)\right) / \sim$$

by the equivalence relation $\sim$ generated by $x \sim F(f)(x)$ for $f\colon i \to j$ and $x \in F(i)$, whenever Q is small.

Similarly, the category R-**Mod** admits small colimits. For a functor $F\colon I \to \mathbf{Set}$, $\operatorname{colim} F$ is represented by the quotient

$$Q = \left(\bigoplus_{i\in \mathrm{Ob}(I)} F(i)\right) /M$$

by the R-submodule M generated by $x - F(f)(x)$ for $f\colon i \to j$ and $x \in F(i)$, whenever Q is small.

Definition 1.3.25. Let $S\colon I \to \mathcal{C}$ be a functor. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ *preserves* (or commutes with) a limit $a\colon \Delta X \to S$ of S, if $\Delta(FX) = F^I(\Delta X) \xrightarrow{F^I(a)} F^I(S)$ is a limit of $F^I(S) = F \circ S$. Here $F^I = F \circ -\colon \mathcal{C}^I \to \mathcal{D}^I$. We say $F\colon \mathcal{C} \to \mathcal{D}$ *preserves limits* if it preserves all limits that exist in $\mathcal{C}$.

The preservation of limit can be written as $F(\lim_{i\in I} S(i)) \simeq \lim_{i\in I} F(S(i))$.

Proposition 1.3.26. *Let $\mathcal{C}$ be a category with small* Hom *sets. For every object X of $\mathcal{C}$, the functor* $\mathrm{Hom}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathbf{Set}$ *preserves limits (that exist in $\mathcal{C}$). Dually, the functor* $\mathrm{Hom}_{\mathcal{C}}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *preserves limits (that exist in $\mathcal{C}^{\mathrm{op}}$).*

Proof. Let $F\colon I \to \mathcal{C}$ be a functor. For the first assertion, it suffices to show that the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim F) \to \lim \mathrm{Hom}_{\mathcal{C}}(X, F-)$$

is a bijection. Under the description of the limit in the right-hand side in Example 1.3.23, the map sends a to $(p_i \circ a)_{i\in\mathrm{Ob}(I)}$, where $p_i\colon \lim F \to F(i)$ is the canonical morphism. The map is a bijection by the universal property for $\lim F$.

The last assertion follows from the first one applied to $\mathcal{C}^{\mathrm{op}}$. □

Warning 1.3.27. None of the forgetful functors

$$\mathbf{Grp} \to \mathbf{Set}, \quad \mathbf{Ab} \to \mathbf{Set}, \quad \mathbf{Ab} \to \mathbf{Grp}$$

preserve finite coproducts.

Comma categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. For an object Y of $\mathcal{D}$, we let $(F \downarrow Y)$ denote the category defined as follows. An object of $(F \downarrow Y)$ is a pair (X, f), where X is an object of $\mathcal{C}$ and $f\colon FX \to Y$ is a morphism of $\mathcal{D}$. A morphism $(X, f) \to (X', f')$ is a morphism $h\colon X \to X'$ such that $f = f'(Fh)$, i.e. the following diagram commutes

$$\begin{tikzcd} FX \arrow[r, "Fh"] \arrow[dr, "f"'] & FX' \arrow[d, "f'"] \\ & Y. \end{tikzcd}$$

Dually, we let $(Y \downarrow F)$ denote the category of pairs (X, f), where X is an object of $\mathcal{C}$ and $f\colon Y \to FX$ is a morphism of $\mathcal{D}$. A morphism $(X, f) \to (X', f')$ is a morphism $h\colon X \to X'$ such that $f' = (Fh)f$. We have an isomorphism of categories $(F \downarrow Y)^{\mathrm{op}} \simeq (Y \downarrow F^{\mathrm{op}})$, where $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Example 1.3.28. Let $\mathcal{C}$ and I be categories and let $\Delta\colon \mathcal{C} \to \mathcal{C}^I := \mathrm{Fun}(I, \mathcal{C})$ be the "diagonal" functor carrying X to the constant functor of value X. Then we have obvious isomorphisms of categories $(\Delta \downarrow F) \simeq \mathcal{C}_{/F}$ and $(F \downarrow \Delta) \simeq \mathcal{C}_{F/}$. Thus comma categories generalize over-categories and under-categories.

Universal constructions

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and let Y be an object of $\mathcal{D}$. A *universal arrow* from U to Y is a final object of $(F \downarrow Y)$. A universal arrow from Y to U is an initial object of $(Y \downarrow F)$.

Remark 1.3.29. Note that $(X_0, \epsilon\colon FX_0 \to Y)$ is a final object of $(F \downarrow Y)$ if and only if the map $\mathrm{Hom}_{\mathcal{C}}(X, X_0) \to \mathrm{Hom}_{\mathcal{D}}(FX, Y)$ carrying f to the composite $FX \xrightarrow{Ff} FX_0 \xrightarrow{\epsilon} Y$ is a bijection for all X.

Example 1.3.30. Let $U\colon \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor and let S be a small set. The free group FS with basis S, equipped with the map $i\colon S \to UFS$ satisfies the following property: for every small group G equipped with a map $f\colon S \to UG$, there exists a unique homomorphism $h\colon FS \to G$ such that $f = (Uh)i$. Thus (FS, i) is an initial object of $(S \downarrow U)$.

1.4 Adjunction

Adjunction

Definition 1.4.1 (Kan)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* is a triple (F, G, ϕ), where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are functors, and ϕ is a natural isomorphism $\phi_{XY}\colon \mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GY)$. We then say that F is *left adjoint* to G, G is *right adjoint* to F, and (F, G) is a pair o adjoint functors, and we sometimes write $\phi\colon F \dashv G$.

If $\mathcal{C}$ and $\mathcal{D}$ have small Hom sets, then ϕ is a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$.

Example 1.4.2. The free group functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ is left adjoint to the forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$.

Example 1.4.3. Let I be a category. If $\mathcal{C}$ admits limits indexed by I, then, by Remark 1.3.29, we have a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim F) \simeq \mathrm{Hom}_{\mathcal{C}^I}(\Delta X, F)$$

natural in X and F, exhibiting the limit functor $\lim\colon \mathcal{C}^I \to \mathcal{C}$ as a right adjoint to the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^I$. Dually, if $\mathcal{C}$ admits colimits indexed by I, then the colimit functor $\operatorname{colim}\colon \mathcal{C}^I \to \mathcal{C}$ is left adjoint to the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^I$.

Example 1.4.4. Let X, Y, Z be small sets. Then we have a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(X \times Y, Z) \simeq \mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{Hom}_{\mathbf{Set}}(Y, Z))$$

natural in X, Y, Z. Thus $- \times Y \dashv \mathrm{Hom}_{\mathbf{Set}}(Y, -)$.

Example 1.4.5. Let R, S, T be rings and consider small bimodules ${}_RM_S$, ${}_SN_T$, and ${}_RP_T$. We have isomorphisms of abelian groups

$$\mathrm{Hom}_{(R,T)\text{-}\mathbf{Mod}}(M \otimes_S N, P) \simeq \mathrm{Hom}_{(R,S)\text{-}\mathbf{Mod}}(M, \mathrm{Hom}_{\mathbf{Mod}\text{-}T}(N, P)),$$
$$\mathrm{Hom}_{(R,T)\text{-}\mathbf{Mod}}(M \otimes_S N, P) \simeq \mathrm{Hom}_{(S,T)\text{-}\mathbf{Mod}}(N, \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, P)),$$

natural in M, N, P. Thus $-\otimes_S N \dashv \mathrm{Hom}_{\mathbf{Mod}\text{-}T}(N, -)$ and $M \otimes_S - \dashv \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, -)$.

Remark 1.4.6. Let $\phi\colon F \dashv G$ be an adjunction. Then ϕ induces $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$.

Proposition 1.4.7. *Let $\phi\colon F \dashv G$. Then G is determined by F up to natural isomorphism.*

Proof. Let $\phi'\colon F \dashv G'$. Consider the natural isomorphism $\phi'^{-1} \circ \phi\colon \mathrm{Hom}_{\mathcal{D}}(X, GY) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X, G'Y)$. By Yoneda's lemma, this is given by an isomorphism $GY \to G'Y$, which is natural in Y by the naturalness of ϕ and ϕ'. □

The following proposition shows that the functoriality of an adjoint is automatic.

Proposition 1.4.8. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ have small* Hom *sets. Then F admits a right adjoint if and only if for every object Y of $\mathcal{D}$, the functor $h_{\mathcal{D}}(Y) \circ F = \mathrm{Hom}_{\mathcal{C}}(F-, Y)$ is representable.*

Proof. We can construct an adjunction $\phi\colon F \dashv G$ as follows. For every object Y of $\mathcal{D}$, choose an object GY of $\mathcal{C}$ and an isomorphism $\phi\colon h_{\mathcal{D}}(Y) \circ F \xrightarrow{\sim} h_{\mathcal{C}}(GY)$. For every morphism $Y \to Y'$, we get a morphism $GY \to GY'$ in $\mathcal{C}$ by Yoneda's lemma. □

We leave it to the reader to state duals of the preceding propositions.

Proposition 1.4.9. *Let $F\colon \mathcal{C} \to \mathcal{D}$, $F'\colon \mathcal{D} \to \mathcal{E}$, $G\colon \mathcal{D} \to \mathcal{C}$, $G'\colon \mathcal{E} \to \mathcal{D}$ be functors and let $\phi\colon F \dashv G$, $\phi'\colon F' \dashv G'$ be adjunctions. Then $\phi\phi'\colon F'F \dashv GG'$.*

Proof. Indeed we have

$$\mathrm{Hom}_{\mathcal{E}}(F'FX, Y) \xrightarrow[\sim]{\phi'_{FX,Y}} \mathrm{Hom}_{\mathcal{D}}(FX, G'Y) \xrightarrow[\sim]{\phi_{X,G'Y}} \mathrm{Hom}_{\mathcal{C}}(X, GG'Y).$$

□

Unit, counit

The naturalness of ϕ means

$$\phi(b \circ f \circ Fa) = Gb \circ \phi(f) \circ a, \tag{1.4.1}$$

for all $a\colon X' \to X$, $f\colon FX \to Y$, $b\colon Y \to Y'$. Let $\eta_X = \phi(\mathrm{id}_{FX})\colon X \to GFX$ and let $\epsilon_Y = \phi^{-1}(\mathrm{id}_{GY})\colon FGY \to Y$. By (1.4.1), $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ are natural transformations. We call η the *unit* and ϵ the *counit*. Note that (1.4.1) implies that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

are identity transformations: $\epsilon F \circ F\eta = \mathrm{id}_F$, $G\epsilon \circ \eta G = \mathrm{id}_G$. Indeed, $\phi(\epsilon_{FX} \circ F\eta_X) = \phi(\epsilon_{FX}) \circ \eta_X = \eta_X = \phi(\mathrm{id}_{FX})$ and the second relation is proved similarly. Note that (1.4.1) also implies that ϕ is determined by η by the rule $\phi(f) = Gf \circ \eta_X$ for $f\colon FX \to Y$. Moreover, ϕ^{-1} is determined by ϵ by the rule $\phi^{-1}(g) = \epsilon_Y \circ Fg$ for $g\colon X \to GY$.

Example 1.4.10. Let $U\colon \mathbf{Ab} \to \mathbf{Set}$ be the forgetful functor and let $F\colon \mathbf{Set} \to \mathbf{Ab}$ be the functor carrying S to $\mathbb{Z}^{(S)} = \bigoplus_{s\in S} \mathbb{Z}a_s$. The unit $S \to UFS$ carries s to a_s. The counit $FUA \to A$ carries $\sum_{s\in A} n_s a_s$ to $\sum_{s\in A} n_s s$.

Proposition 1.4.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction (F, G, ϕ) is uniquely determined by each of the following data:*

(1) Functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ and natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ be such that $\epsilon F \circ F\eta = \mathrm{id}_F$, $G\epsilon \circ \eta G = \mathrm{id}_G$.

(2) A functor $F\colon \mathcal{C} \to \mathcal{D}$, and for every object Y of $\mathcal{D}$, a final object (GY, ϵ_Y) of $(F \downarrow Y)$;

(3) A functor $G\colon \mathcal{D} \to \mathcal{C}$, and for every object X of $\mathcal{C}$, an initial object (FX, η_X) of $(X \downarrow G)$;

Part (2) is in some sense a restatement of Proposition 1.4.8.

Proof. For (1), note we have seen that $\phi(f) = Gf \circ \eta_X$ is uniquely determined. Clearly ϕ a natural transformation. Put $\psi(g) = \epsilon_Y \circ Fg$. Then

$$\phi\psi(g) = \phi(\epsilon_Y \circ Fg) = G\epsilon_Y \circ GFg \circ \eta_X = G\epsilon_Y \circ \eta_{GY} \circ g = g.$$

Similarly, $\psi\phi(f) = f$. Thus ϕ is a natural isomorphism.

(2) and (3) are dual to each other. We only treat (3). For any morphism $a\colon X \to X'$ in $\mathcal{C}$, there exists a unique morphism $Fa\colon FX \to FX'$ in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "a"'] & GFX \arrow[d, "GFa"] \\ X' \arrow[r, "\eta_{X'}"] & GFX' \end{tikzcd}$$

commutes. It is easy to check that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. The above commutativity means that $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ is a natural transformation. Then $\phi(f) = Gf \circ \eta_X$ is uniquely determined. Clearly ϕ is a natural transformation, and is a natural isomorphism by the universal property. $\square$

Proposition 1.4.12. *Let $\phi\colon F \dashv G$. Then*
(1) F is fully faithful if and only if the unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ is a natural isomorphism.
(2) G is fully faithful if and only if the counit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.

Proof. By (1.4.1), for $f\colon X \to X'$, we have $\phi(Ff) = \phi(\mathrm{id}_{FX'} \circ f) = \phi(\mathrm{id}_{FX'}) \circ f = \eta_{X'} \circ f$. In other words, the composite

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(FX, FX') \underset{\sim}{\xrightarrow{\phi}} \mathrm{Hom}_{\mathcal{C}}(X, GFX')$$

is induced by $\eta_{X'}$. Then (1) follows from Yoneda's lemma. We obtain (2) by duality. □

Corollary 1.4.13. *Let $\phi\colon F \dashv G$. Then the following conditions are equivalent:*
(1) F is an equivalence of categories.
(2) G is an equivalence of categories.
(3) F and G are fully faithful.
(4) The unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ are natural isomorphisms.
Under these conditions, F and G are quasi-inverse to each other.

Proof. If F is an equivalence, then, by the proposition, $\mathrm{id}_{\mathcal{C}} \simeq GF$, so that G is also an equivalence. By duality, (1) $\Longleftarrow$ (2). It is clear that (4) $\Longrightarrow$ (1)+(2) $\Longrightarrow$ (3). By the proposition, (3) $\Longrightarrow$ (4). □

Remark 1.4.14. If F is an equivalence of categories and G is a quasi-inverse to F, then G is both right adjoint to F and left adjoint to F.

Adjunction and (co)limits

Proposition 1.4.15. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$. Then*
(1) $F^J\colon \mathcal{C}^J \to \mathcal{D}^J$ is left adjoint to $G^J\colon \mathcal{D}^J \to \mathcal{C}^J$ for any category J;
(2) G preserves limits (that exist in $\mathcal{D}$) and F preserves colimits (that exist in $\mathcal{C}$).

Proof. (1) follows from the determination of adjunction by unit and counit (Proposition 1.4.11 (1)). For (2), by duality it suffices to show the first assertion. Let $S\colon J \to \mathcal{D}$ be a functor such that $\lim S$ exists. Consider the commutative square

(1.4.2)
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "F"'] & \mathcal{C}^J \arrow[d, "F^J"] \\ \mathcal{D} \arrow[r, "\Delta"] & \mathcal{D}^J. \end{tikzcd}$$

The canonical morphism $G\lim S \to \lim G^J S$ is an isomorphism, because it induces a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, G\lim S) \simeq \mathrm{Hom}_{\mathcal{D}}(FX, \lim S) \simeq \mathrm{Hom}_{\mathcal{D}^J}(\Delta FX, S) = \mathrm{Hom}_{\mathcal{D}^J}(F^J \Delta X, S) \\ \simeq \mathrm{Hom}_{\mathcal{C}^J}(\Delta X, G^J S) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \lim G^J S) \end{aligned}$$

for every X. □

In the case where $\mathcal{C}$ and $\mathcal{D}$ admit limits indexed by J, part (2) of the proposition can be paraphrased as follows: (1.4.2) induces by taking right adjoints a square

$$\begin{tikzcd} \mathcal{C} & \mathcal{C}^J \arrow[l, "\lim"'] \\ \mathcal{D} \arrow[u, "G"] & \mathcal{D}^J \arrow[l, "\lim"] \arrow[u, "G^J"'] \end{tikzcd}$$

which commutes up to natural isomorphism.

Example 1.4.16. Let $\mathcal{C}$, P, I be categories.

(1) The functor $\Delta_I\colon \mathcal{C}^P \to (\mathcal{C}^P)^I$ can be identified with the functor $(\Delta_I)^P\colon \mathcal{C}^P \to (\mathcal{C}^I)^P$ via the canonical isomorphism $(\mathcal{C}^P)^I \simeq (\mathcal{C}^I)^P$. Thus, if $\mathcal{C}$ admits limits indexed by I, then $\mathcal{C}^P$ admits limits indexed by I and $\lim_I\colon (\mathcal{C}^P)^I \to \mathcal{C}^P$ can be identified with $(\lim_I)^P\colon (\mathcal{C}^I)^P \to \mathcal{C}^P$. In this case, limits indexed by I can be computed pointwise: The evaluation functor $E_p\colon \mathcal{C}^P \to \mathcal{C}$ at any object p of P preserves limits indexed by I. For a generalization to the case where $\mathcal{C}$ does not admit all limits indexed by I, see Proposition 1.4.17 below.

(2) Limits preserve limits: If $\mathcal{C}$ admits limits indexed by P, the limit functor $\lim_P\colon \mathcal{C}^P \to \mathcal{C}$ is right adjoint to Δ_P and hence preserves limits.

(3) If $\mathcal{C}$ admits limits indexed by P and I, we have a diagram

$$\begin{tikzcd} \mathcal{C} & & \mathcal{C}^P \arrow[ll, "\lim_P"'] \\ \mathcal{C}^I \arrow[u, "\lim_I"] & & \mathcal{C}^{P\times I} \arrow[ll, "(\lim_P)^I"] \arrow[u, "(\lim_I)^P"'] \arrow[ull, "\lim_{P\times I}"'] \end{tikzcd}$$

which commutes up to natural isomorphisms. Here we have identified $(\mathcal{C}^P)^I$ and $(\mathcal{C}^I)^P$ with $\mathcal{C}^{P\times I}$ via the canonical isomorphisms.

Proposition 1.4.17. *Let $\mathcal{C}$, I, P be categories and let $F\colon I \to \mathcal{C}^P$ be a functor such that for each object p of P, $F_p = E_p \circ F\colon I \to \mathcal{C}$ admits a limit $\tau_p\colon \Delta L_p \to F_p$. Here $E_p\colon \mathcal{C}^P \to \mathcal{C}$ denotes the evaluation functor at p carrying G to $G(p)$. Then there exists a unique functor $L\colon P \to \mathcal{C}$ such that $L(p) = L_p$ and $p \mapsto \tau_p$ gives a natural transformation $\tau\colon \Delta L \to F$. Moreover, this τ exhibits L as a limit of F.*

Proof. This follows easily from the universal properties of limits. We leave the details to the reader. □

Remark 1.4.18. Example 1.4.16 also holds for colimits. In particular, colimits preserve colimits.

Limits and colimits do not commute with each other in general. For example, in **Set**, for nonempty sets X, X', Y, Y', the canonical map $(X \times X')\coprod(Y \times Y') \to (X \coprod Y) \times (X' \coprod Y')$ is not a bijection. See however Propositions 1.6.26 and 1.6.27 for cases where limits and colimits do commute with each other.

The rest of this section is not used elsewhere in these notes. In favorable cases one can give necessary and sufficient conditions for the existence of adjoints. Let us first give a criterion for the existence of an initial object.

Theorem 1.4.19. *Let $\mathcal{D}$ be a category with small* Hom *sets and admitting small limits. Then $\mathcal{D}$ has an initial object if and only if it satisfies the following* Solution Set Condition*: there exists a small set S of objects of $\mathcal{D}$ that is* weakly initial *in the sense that for every object x of $\mathcal{D}$, there exist $s \in S$ and a morphism $s \to x$.*

Note that the condition is a set-theoretic one that is automatically satisfied if $\mathcal{D}$ is small (by taking S to be $\mathrm{Ob}(\mathcal{D})$).

Proof. (Copied from [ML2, Theorem X.2.1]) The "only if" part is clear: if i is an initial object, then $S = \{i\}$ satisfies the Solution Set Condition. Let us show the "if" part. We let S also denote the full subcategory of $\mathcal{D}$ spanned by S. Let $F\colon S \to \mathcal{D}$ be the inclusion. We claim that $i = \lim F$ is an initial object of $\mathcal{D}$. Choose, for every object x of $\mathcal{D}$, a morphism $f_x\colon i \to x$ that factorizes through the projection $p\colon i \to s$ to some $s \in S$. We get a cone $\Delta i \to \mathrm{id}_{\mathcal{D}}$. Indeed, for every morphism $x \to x'$, we have a commutative diagram

$$\begin{tikzcd}
& & i \arrow[d, "p"] \arrow[ddl, "p"'] \arrow[ddr, "p"] \arrow[dddll, "f_x"'] \arrow[dddrr, "f_{x'}"] & & \\
& & s'' \arrow[d] & & \\
& s \arrow[dl] & x'' \arrow[l] \arrow[r] & s' \arrow[dr] & \\
x \arrow[rrrr] & & & & x',
\end{tikzcd}$$

where x'' is the fiber product of s and s' over x'. We may assume that for all $s \in S$, $f_s\colon i \to s$ is the projection. We conclude by the following lemma. □

Lemma 1.4.20. *Let $F\colon S \to \mathcal{D}$ be a functor and let $c\colon \Delta i \to \mathrm{id}_{\mathcal{D}}$ be a cone such that $cF\colon \Delta i \to F$ is a limiting cone. Then i is initial.*

Proof. Since c is a cone, the diagram

$$\begin{tikzcd}
& i \arrow[dl, "c_i"'] \arrow[dr, "c_{Fs}"] & \\
i \arrow[rr, "c_{Fs}"] & & Fs
\end{tikzcd}$$

commutes for every $s \in \mathrm{Ob}(S)$. Since cF is limiting, this implies $c_i = \mathrm{id}_i$. Since c is a cone, for every morphism $i \to x$, the diagram

$$\begin{tikzcd}
& i \arrow[dl, "c_i"'] \arrow[dr, "c_x"] & \\
i \arrow[rr] & & x
\end{tikzcd}$$

commutes, so that c_x is the unique morphism $i \to x$. □

Theorem 1.4.21 (Freyd Adjoint Functor Theorem)**.** *Let $\mathcal{D}$ be a category with small* Hom *sets and admitting small limits. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor. Then G admits a left adjoint if and only if G preserves small limits and satisfies the Solution Set Condition: for each object X of $\mathcal{C}$, there exists a small set of objects S of $(X \downarrow G)$ that is weakly initial (namely, for every object x of $(X \downarrow G)$, there exist $s \in S$ and a morphism $s \to x$ in $(X \downarrow G)$).*

Proof. Recall from Proposition 1.4.11 that G admits a left adjoint if and only if $(X \downarrow G)$ admits an initial object for every object X of $\mathcal{C}$. The "only if" part then follows from Proposition 1.4.15. To show the "if" part, we apply Theorem 1.4.19. It suffices to check, under the assumption that G preserves small limits, that $(X \downarrow G)$ admits small limits. Let I be a small category and let $F\colon I \to (X \downarrow G)$ be a functor. We write Fi as $(Y_i, f_i\colon X \to GY_i)$. Let Y be the limit of Y_i in $\mathcal{D}$. Since G preserves small limits, (f_i) determines a morphism $f\colon X \to GY$. It is easy to check that (X, f) is the limit of F. □

1.5 Additive categories

Additive categories

Recall that a *magma* is a set equipped with a binary operation. The magma is said to be *unital* if it has an identity element.

Proposition 1.5.1. *Let $\mathcal{A}$ be a category with each* $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ *equipped with a structure of unital magma such that composition is bilinear. Let A_1 and A_2 be objects of $\mathcal{A}$. Then the following conditions are equivalent.*

(1) $A_1 \times A_2$ exists.

(2) $A_1 \amalg A_2$ exists.

Under these assumptions, the morphism $\phi_{A_1,A_2}\colon A_1 \amalg A_2 \to A_1 \times A_2$ described by the matrix $\begin{pmatrix} \mathrm{id}_{A_1} & 0 \\ 0 & \mathrm{id}_{A_2} \end{pmatrix}$ is an isomorphism. Moreover, if $Y \times Y$ and $Y \times Y \times Y$ exist, then $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ *is a commutative monoid, and for $f, f'\colon X \to Y$, $f + f'$ is given by the composition*

$$X \xrightarrow{(f,f')} Y \times Y \xrightarrow{\phi_{YY}^{-1}} Y \amalg Y \xrightarrow{(\mathrm{id}_Y,\mathrm{id}_Y)} Y. \tag{1.5.1}$$

We denote the operation on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ by $+$ and the identity element by $0 = 0_{XY}$. The bilinearity of composition means the following: for $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$, we have (a) $g(f+f') = gf+gf'$, $(g+g')f = gf+g'f$; (b) $g0_{XY} = 0_{XZ}$, $0_{YZ}f = 0_{XZ}$. Condition (b) follows from (a) if $\mathrm{Hom}_{\mathcal{A}}(X,Z)$ is a group. If $\mathcal{A}$ admits a zero object, it follows from (b) that the zero morphism $z_{XY}\colon X \to Y$ (that factors through every zero object) is the unit of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. Indeed, $z_{XY} = 0_{YY}z_{XY} = 0_{XY}$.

Proof. By duality we may assume that (1) holds. Let $i_1 = (\mathrm{id}_{A_1}, 0)\colon A_1 \to A_1 \times A_2$ and let $i_2 = (0, \mathrm{id}_{A_2})\colon A_2 \to A_1 \times A_2$. We show that $(A_1 \times A_2, i_1, i_2)$ exhibits $A_1 \times A_2$ as a coproduct of A_1 and A_2. Let B be an object of $\mathcal{A}$ equipped with

$h_1\colon A_1 \to B$ and $h_2\colon A_2 \to B$. We put $h = \langle h_1, h_2\rangle = h_1p_1 + h_2p_2\colon A_1 \times A_2 \to B$. Here $p_1\colon A_1 \times A_2 \to A_1$ and $p_2\colon A_1 \times A_2 \to A_2$ are the projections. Then $hi_1 = h_1p_1i_1 + h_2p_2i_1 = h_1 + 0 = h_1$ and similarly $hi_2 = h_2$. If $h'\colon A_1 \times A_2 \to B$ is a morphism such that $h'i_1 = h_1$, $h'i_2 = h_2$. Then

$$h' = h'(i_1p_1 + i_2p_2) = h_1p_1 + h_2p_2 = h$$

by Lemma 1.5.2 below. Therefore, (2) and the second assertion hold.

Assume that $Y \times Y$ exists. Then in (1.5.1), $(\mathrm{id}_Y, \mathrm{id}_Y) = (p_1 + p_2)\phi_{YY}$. Thus

$$f + f' = (p_1 + p_2)(f, f') = (\mathrm{id}_Y, \mathrm{id}_Y)\phi_{YY}^{-1}(f, f')$$

is given by (1.5.1). The diagram

$$\begin{tikzcd}
X \arrow[r, "{(f,f')}"] \arrow[dr, "{(f',f)}"'] & Y \times Y \arrow[r, "\phi_{YY}^{-1}"] \arrow[d, "{(p_2,p_1)}", "\simeq"'] & Y \amalg Y \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "\simeq", "{(i_2,i_1)}"'] & Y \\
& Y \times Y \arrow[r, "\phi_{YY}^{-1}"] & Y \amalg Y \arrow[ur, "{(\mathrm{id},\mathrm{id})}"'] &
\end{tikzcd}$$

commutes, which implies $f + f' = f' + f$. Here the square in the middle commutes since the following square commutes

$$\begin{tikzcd}
Y \times Y \arrow[d, "{(p_2,p_1)}", "\simeq"'] & Y \amalg Y \arrow[l, "\phi_{YY}"'] \arrow[d, "\simeq", "{(i_2,i_1)}"'] \\
Y \times Y & Y \amalg Y. \arrow[l, "\phi_{YY}"']
\end{tikzcd}$$

Assume moreover that $Y \times Y \times Y$ exists. For $f, f', f''\colon X \to Y$, $f + (f' + f'')$ is given by the composition

$$\begin{tikzcd}
X \arrow[r, "{(f,f'+f'')}"] \arrow[d, "{(f,(f',f''))}"'] & Y \times Y \arrow[r, "\phi_{YY}^{-1}"] & Y \amalg Y \arrow[r] & Y \\
Y \times (Y \times Y) \arrow[r, "\mathrm{id}_Y \amalg \phi_{YY}^{-1}"] & Y \times (Y \amalg Y) \arrow[r, "\phi_{Y,Y\amalg Y}^{-1}"] \arrow[u] & Y \amalg (Y \amalg Y) \arrow[u] &
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
Y \times (Y \times Y) \arrow[r, "\mathrm{id}_Y \times \phi_{YY}^{-1}"] \arrow[d, "\simeq"'] & Y \times (Y \amalg Y) \arrow[r, "\phi_{Y,Y\amalg Y}^{-1}"] & Y \amalg (Y \amalg Y) \arrow[d, "\simeq"] \\
(Y \times Y) \times Y \arrow[r, "\phi_{YY}^{-1} \times \mathrm{id}_Y"] & (Y \amalg Y) \times Y \arrow[r, "\phi_{Y\amalg Y,Y}^{-1}"] & (Y \amalg Y) \amalg Y
\end{tikzcd}$$

commutes, because the inverse of the composition of the upper horizontal arrows and the inverse of the composition of the lower horizontal arrows are both given by the matrix $\begin{pmatrix} \mathrm{id}_Y & 0 & 0 \\ 0 & \mathrm{id}_Y & 0 \\ 0 & 0 & \mathrm{id}_Y \end{pmatrix}$. Therefore, $f + (f' + f'') = (f + f') + f''$. □

Lemma 1.5.2. *Under the above notation,* $i_1p_1 + i_2p_2 = \mathrm{id}_{A_1 \times A_2}$.

Proof. We have $p_1(i_1p_1 + i_2p_2) = p_1i_1p_1 + p_1i_2p_2 = p_1 + 0 = p_1$ and similarly $p_2(i_1p_1 + i_2p_2) = p_2$. Therefore, $i_1p_1 + i_2p_2 = \mathrm{id}_{A_1\times A_2}$. $\square$

Remark 1.5.3. Under the assumptions (1) and (2) of Proposition 1.5.1, the composition of $C \xrightarrow{(a,b)} A_1 \times A_2 \xrightarrow{\langle c,d\rangle} B$ is $\langle c, d\rangle(a, b) = ca+db$. In other words, composition is given by matrix multiplication, with (a, b) considered as a column vector and $\langle c, d\rangle$ considered as a row vector.

Proposition 1.5.4. *Let $\mathcal{A}$ be a category admitting a zero object, finite products, and finite coproducts satisfying*

(*) The morphism $\phi_{YY} \colon Y \amalg Y \to Y \times Y$ described by the matrix $\begin{pmatrix} \mathrm{id}_Y & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix}$ is an isomorphism for every object Y of $\mathcal{A}$.

Then there exists a unique way to equip every $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a unital magma such that composition is bilinear. Moreover, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a commutative monoid.

Here 0 in the description of ϕ_{YY} denotes the zero morphism (that factors through every zero object).

Proof. For morphisms $f, f' \colon X \to Y$, we define $f + f'$ to be the composition

$$X \xrightarrow{(f,f')} Y \times Y \xrightarrow{\phi_{YY}^{-1}} Y \amalg Y \xrightarrow{(\mathrm{id}_Y,\mathrm{id}_Y)} Y.$$

The diagram

$$\begin{tikzcd}
X \arrow[r, "{(f,0)}"] \arrow[dr, "f"'] & Y \times Y \arrow[r, "\phi_{YY}^{-1}"] & Y \amalg Y \arrow[dr, "{(\mathrm{id}_Y,\mathrm{id}_Y)}"] & \\
& Y \arrow[u, "{(\mathrm{id}_Y,0)}"] \arrow[r, "{(\mathrm{id}_Y,0)^{-1}}"] & Y \amalg 0 \arrow[u, "\mathrm{id}_Y\amalg 0"] \arrow[r, "{(\mathrm{id}_Y,0)}"] & Y
\end{tikzcd}$$

commutes, so that $f + 0 = f$. Similarly $0 + f = f$. This construction equips $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a unital magma. It is clear from the construction that $(g + g')f = gf + g'f$. The diagram

$$\begin{tikzcd}
X \arrow[r, "{(f,f')}"] \arrow[dr, "{(gf,gf')}"'] & Y \times Y \arrow[r, "\phi_{YY}^{-1}"] \arrow[d, "g\times g"] & Y \amalg Y \arrow[r, "{(\mathrm{id}_Y,\mathrm{id}_Y)}"] \arrow[d, "g\amalg g"] & Y \arrow[d, "g"] \\
& Z \times Z \arrow[r, "\phi_{ZZ}^{-1}"] & Z \amalg Z \arrow[r, "{(\mathrm{id}_Z,\mathrm{id}_Z)}"] & Z
\end{tikzcd}$$

commutes, so that $g(f + f') = gf + gf'$. Moreover, $0f = 0$ and $g0 = 0$. Thus the composition law on $\mathcal{A}$ is bilinear.

The uniqueness and the fact that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a commutative monoid follows from Proposition 1.5.1. $\square$

In a category satisfying the above assumptions, coproducts are also called direct sums and we often write $\oplus$ instead of $\amalg$.

Remark 1.5.5. Let $\mathcal{A}$ be a category satisfying the assumptions of Proposition 1.5.4. Let A, A_1, A_2 be objects of A equipped with morphisms $p_j\colon A \to A_j$, $i_j\colon A_j \to A$, $i = 1, 2$, satisfying $p_j i_j = \mathrm{id}_{A_j}$, $i = 1, 2$ and

$$i_1 p_1 + i_2 p_2 = \mathrm{id}_A. \tag{1.5.2}$$

Then A can be identified with the direct sum of A_1 and A_2, and p_j and i_j can be identified with the canonical morphisms. More precisely, the morphisms $(p_1, p_2)\colon A \to A_1 \oplus A_2$ and $(i_1, i_2)\colon A_1 \oplus A_2 \to A$ are inverse to each other. Indeed, multiplying (1.5.2) by p_1 on the left and i_2 on the right, we get $p_1 i_2 = 0$. Similarly, $p_2 i_1 = 0$.

Definition 1.5.6. An *additive* category is a category $\mathcal{A}$ admitting a zero object, finite products, finite coproducts, satisfying $(*)$ above[4], and such that the commutative monoids $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are abelian groups.

Remark 1.5.7. By Proposition 1.5.1, if $\mathcal{A}$ is a category with each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ equipped with a structure of abelian group such that composition is bilinear[5], and such that $\mathcal{A}$ admits a zero object, finite products (or finite coproducts), then $\mathcal{A}$ is an additive category.

Example 1.5.8. Let R be a ring. Then R-**Mod** is an additive category. Indeed, R-**Mod** admits finite products, the Hom sets are naturally equipped with structures of abelian groups and composition is bilinear.

Example 1.5.9. Let $\mathcal{B}$ be an additive category. Let $\mathcal{A} \subseteq \mathcal{B}$ be a full subcategory such that for A and A' in $\mathcal{A}$, the direct sum $A \oplus A'$ in $\mathcal{B}$ is isomorphic to an object of $\mathcal{A}$. Then $\mathcal{A}$ is an additive category by Lemma 1.5.10 below. In particular, the full subcategory R-**mod** of R-**Mod** spanned by finitely generated R-modules is an additive category. Similarly, the full subcategory of R-**Mod** spanned by free left R-modules is also an additive category.

Lemma 1.5.10. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$ and let $F\colon I \to \mathcal{B}$ be a functor. If $p\colon \Delta X \to F$ exhibits X as a limit of F in $\mathcal{C}$ with X in $\mathcal{B}$, then p exhibits X as a limit of F in $\mathcal{B}$.*

Proof. This follows easily from the universal properties of limits. We leave the details to the reader. □

Warning 1.5.11. The converse of the lemma is false. For example, the inclusion $\mathbb{Z}_{\le -1} \subseteq \mathbb{Z}_{\le 0}$ does not preserve the final object. Here $\mathbb{Z}$ is equipped with the usual order.

Example 1.5.12. Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}^{\mathrm{op}}$ is an additive category.

Example 1.5.13. Let $\mathcal{A}$ be an additive category and let P be a category. Then the functor category $\mathcal{A}^P$ is an additive category. For $X, Y\colon P \to \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}^P}(X, Y)$ is a subgroup of $\prod_{p \in \mathrm{Ob}(P)} \mathrm{Hom}_{\mathcal{A}}(X_p, Y_p)$.

[4]A category $\mathcal{A}$ admitting a zero object, finite products, finite coproducts, and satisfying $(*)$ above is called a *semiaddtive* category.

[5]A category $\mathcal{A}$ with each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ equipped with a structure of abelian group such that composition is bilinear is called a *preadditive category*, or a category *enriched* over $(\mathbf{Ab}, \otimes)$.

Additive functors

Proposition 1.5.14. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then the following conditions are equivalent:*

(1) F preserves products of pairs of objects;

(2) F preserves coproducts of pairs of objects;

(3) for every pair of objects A, A' of $\mathcal{A}$, the map $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ induced by F is a homomorphism.

The proposition holds more generally for functors between categories satisfying Proposition 1.5.4 (admitting zero objects, finite products, finite coproducts, and satisfying $(*)$).

Proof. Let A_1 and A_2 be objects of $\mathcal{A}$ and let $p_j\colon A_1 \times A_2 \to A_j$, $j = 1, 2$ be the projections.

(1) $\Longrightarrow$ (2). Let us first show that F carries zero objects to zero objects. Note that any zero object 0 is the product of 0 and 0, with id_0 as the projections. By (1), $F(0)$ is the product of $F(0)$ and $F(0)$, with $\mathrm{id}_{F(0)}$ as the projections, so that $F(0)$ is a zero object by Lemma 1.5.15 below. It follows that F carries zero morphisms to zero morphisms. Now let $\phi\colon A_1 \amalg A_2 \xrightarrow{\sim} A_1 \times A_2$ be the isomorphism described by the matrix $\begin{pmatrix} \mathrm{id}_{A_1} & 0 \\ 0 & \mathrm{id}_{A_2} \end{pmatrix}$ and let $\iota_j\colon A_j \to A_1 \amalg A_2$, $j = 1, 2$ be the canonical morphisms. Then the composite

$$F(A_1) \amalg F(A_2) \xrightarrow{(F\iota_1, F\iota_2)} F(A_1 \amalg A_2) \xrightarrow[\sim]{F(\phi)} F(A_1 \times A_2) \xrightarrow[\sim]{(Fp_1, Fp_2)} F(A_1) \times F(A_2)$$

is

$$\begin{pmatrix} F(\mathrm{id}_{A_1}) & F(0) \\ F(0) & F(\mathrm{id}_{A_2}) \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{F(A_1)} & 0 \\ 0 & \mathrm{id}_{F(A_2)} \end{pmatrix},$$

which is an isomorphism. Therefore, $(F\iota_1, F\iota_2)$ is an isomorphism.

By duality, we have (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (3). Let $f, g\colon A \to B$. Then $f + g = \langle \mathrm{id}, \mathrm{id}\rangle (f, g)$, so that $F(f + g) = \langle \mathrm{id}, \mathrm{id}\rangle (Ff, Fg) = Ff + Fg$.

(3) $\Longrightarrow$ (1). We must show that $(Fp_1, Fp_2)\colon F(A_1 \times A_2) \to FA_1 \times FA_2$ is an isomorphism. Let us check that $F(i_1)q_1 + F(i_2)q_2$ is an inverse to (Fp_1, Fp_2), where $q_j\colon FA_1 \times FA_2 \to FA_j$, $j = 1, 2$ are the projections and $i_j\colon A_j \to A_1 \times A_2$, $j = 1, 2$ are the canonical morphisms. We have

$$(Fp_1)(F(i_1)q_1 + F(i_2)q_2) = F(\mathrm{id})q_1 + F(0)q_2 = q_1,$$
$$(Fp_2)(F(i_1)q_1 + F(i_2)q_2) = F(0)q_1 + F(\mathrm{id})q_2 = q_2.$$

Thus

$$(Fp_1, Fp_2)(F(i_1)q_1 + F(i_2)q_2) = (q_1, q_2) = \mathrm{id}.$$

Moreover,

$$\begin{aligned}(F(i_1)q_1 + F(i_2)q_2)(Fp_1, Fp_2) &= F(i_1)q_1(Fp_1, Fp_2) + F(i_2)q_2(Fp_1, Fp_2) \\ &= F(i_1)F(p_1) + F(i_2)F(p_2) = F(i_1p_1 + i_2p_2) = F(\mathrm{id}) = \mathrm{id}.\end{aligned}$$

Here we used Lemma 1.5.2. □

Lemma 1.5.15. *Let A and B be objects of a category $\mathcal{C}$ with a zero object.*

(1) B is a zero object if and only if $\mathrm{id}_B = 0$.

(2) If $A \times B$ exists, with the projection $p\colon A \times B \to A$ being an isomorphism, then B is a zero object.

Proof. (1) Indeed, $\mathrm{id}_B = 0$ is equivalent to the assertion that the morphisms $0 \to B$ and $B \to 0$ are inverses of each other.

(2) Let $q\colon A \times B \to B$ be the projection. Consider the morphisms $0\colon B \to A$ and $\mathrm{id}_B\colon B \to B$. There exists a unique morphism $f = (0, \mathrm{id}_B)\colon B \to A \times B$ such that $pf = 0$, $qf = \mathrm{id}_B$. It follows that $f = 0$, $\mathrm{id}_B = 0$, so that B is a zero object by (1). □

Remark 1.5.16. Even in an additive category, $A \times B \simeq A$ does *not* imply $B \simeq 0$ in general. For example, if $A = \mathbb{Z}^{(S)}$ is the free abelian group with an infinite basis S, then any bijection $S \amalg S \simeq S$ induces an isomorphism $A \oplus A \simeq A$ in the category **Ab**.

Definition 1.5.17. We say that a functor $F\colon \mathcal{A} \to \mathcal{B}$ between additive categories is *additive* if it satisfies the conditions of Proposition 1.5.14. We say that a subcategory $\mathcal{A}$ of an additive category $\mathcal{B}$ is an *additive subcategory* if $\mathcal{A}$ is additive and the inclusion functor is additive.

If F is a functor between additive categories admitting a left or right adjoint, then F is additive. A composition of additive functors is additive. The term "additive subcategory" needs to be used with caution, as a subcategory of an additive category can be an additive category without being an additive subcategory.

Remark 1.5.18. Let $\mathcal{B}$ be an abelian category. A full subcategory $\mathcal{A}$ of $\mathcal{B}$ is an additive subcategory if and only if $\mathcal{A}$ admits a zero object 0 of $\mathcal{B}$ and for A and A' in $\mathcal{A}$, the direct sum $A \oplus A'$ in $\mathcal{B}$ is isomorphic to an object of $\mathcal{A}$.

Example 1.5.19. Let $\mathcal{A}$ be an additive category and let $F\colon P \to Q$ be a functor. Then the functor $\mathcal{A}^Q \to \mathcal{A}^P$ induced by F is additive.

Example 1.5.20. Let $\mathcal{A}$ be an additive category. The functor $- \oplus -\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is additive. It follows that the functor $\mathcal{A} \to \mathcal{A}$ given by $A \mapsto A \oplus A$ is additive. Let B be an object of $\mathcal{A}$. The functor $- \oplus B\colon \mathcal{A} \to \mathcal{A}$ is *not* additive unless $B = 0$.

Example 1.5.21. Let R, S, T be rings. The functor $- \otimes_S -\colon (R,S)$-**Mod** $\times$ (S,T)-**Mod** $\to (R,T)$-**Mod** *not* additive in general. By contrast, it is additive in each variable. That is, given ${}_RM_S$ and ${}_SN_T$, $M \otimes_S -\colon (S,T)$-**Mod** $\to (R,T)$-**Mod** and $- \otimes_S N\colon (R,S)$-**Mod** $\to (R,T)$-**Mod** are additive functors. The functor

$$(S,S)\text{-}\mathbf{Mod} \to (S,S)\text{-}\mathbf{Mod}, \quad A \mapsto A \otimes_S A$$

is *not* additive unless $S = 0$ (assuming S small).

Example 1.5.22. Let $\mathcal{A}$ be an additive category with small Hom sets. The functor $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$ is additive in each variable.

1.6 Abelian categories

Kernels and cokernels

In any category, the equalizer $e\colon E \to X$ of a pair of morphisms $(f,g)\colon X \rightrightarrows Y$, whenever it exists, is always a monomorphism. Indeed, if $(a,b)\colon A \rightrightarrows E$ are morphisms such that $ea = eb$, then $a = b$ by the universal property of the equalizer. Dually, a cokernel, whenever it exists, is an epimorphism.

Lemma 1.6.1.

(1) In a category with a zero object, every monomorphism $f\colon X \to Y$ has zero kernel, and every epimorphism has zero cokernel.

(2) In an additive category, every morphism of zero kernel $f\colon X \to Y$ is a monomorphism and every morphism of zero cokernel is an epimorphism.

Proof. We prove the assertions on monomorphisms and those on epimorphisms follow by duality.

(1) We check that $0 \xrightarrow{0_{0\to X}} X$ satisfies the universal property of a kernel of f. We have $f0_{0\to X} = 0$. Let $g\colon Z \to X$ be a morphism such that $fg = 0_{Z\to Y}$. Since g is a monomorphism, we have $g = 0_{Z\to X}$.

(2) Let $(g,h)\colon Z \to X$ be morphisms such that $fg = fh$. Then $f(g-h) = 0$. It follows that $g - h = 0$, so that $g = h$. □

In an additive category, the equalizer of (f,g) is the kernel of $f - g$ and the coequalizer of (f,g) is the cokernel of $f - g$.

Remark 1.6.2. Let $F\colon I \to \mathcal{C}$ be a functor. Assume that the products $A = \prod_{i\in \mathrm{Ob}(i)} F(i)$ and $B = \prod_{f\colon i\to j} F(j)$ (f running through morphisms of I) exist. Then $\lim F$ can be identified with $\mathrm{eq}(a,b)$, where $a,b\colon A \to B$ are such that $a_f\colon A \to F(j)$ is the projection and $b_f\colon A \to F(i) \xrightarrow{F(f)} F(j)$ is the composition of the projection with $F(f)$.

It follows that a category $\mathcal{C}$ admits small (resp. finite) limits if and only if $\mathcal{C}$ admits equalizers and small (resp. finite) products. Dually, $\mathcal{C}$ admits small (resp. finite) colimits if and only if $\mathcal{C}$ admits coequalizers and small (resp. finite) coproducts. Similar statements hold for preservation of limits and colimits.

An additive category admitting kernels and cokernels admits all finite products and finite coproducts.

Example 1.6.3. Let R be a ring. The additive category R-**Mod** admits finite kernels and cokernels. Indeed, for a morphism $f\colon A \to B$, $\ker(f) = f^{-1}(0)$ and $\mathrm{coker}(f) = B/\mathrm{im}(f)$, where $\mathrm{im}(f)$ denotes the image of f.

Abelian categories

Definition 1.6.4. Let $\mathcal{A}$ be an additive category admitting kernels and cokernels and let $f\colon A \to B$ be a morphism. We define the *coimage* and *image* of f to be $\mathrm{coim}(f) = \mathrm{coker}(g)$, $\mathrm{im}(f) = \ker(h)$, where $g\colon \ker(f) \to A$ and $h\colon B \to \mathrm{coker}(f)$ are the canonical morphisms.

In the above situation, every morphism $f\colon A \to B$ factors uniquely into

$$A \twoheadrightarrow \mathrm{coim}(f) \to \mathrm{im}(f) \hookrightarrow B.$$

Definition 1.6.5. An *abelian category* is an additive category $\mathcal{A}$ satisfying the following axioms:

(AB1) $\mathcal{A}$ admits kernels and cokernels.

(AB2) For each morphism $f\colon A \to B$, the morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

The axioms were introduced by Grothendieck in 1955 (see his seminal 1957 Tōhoku paper [G]). The notion was introduced independently in Buchsbaum's 1954 thesis (under the name of "exact category"[6], see [B] and [CE, Appendix]).

Example 1.6.6. By the first isomorphism theorem, R-**Mod** is an abelian category for every ring R. The full subcategory of R-**Mod** consisting of Noetherian (resp. Artinian) R-modules is stable under subobjects and quotients, and hence is an abelian category. The category R-**mod** of finitely-generated R-modules is an abelian category if and only if R is left Noetherian. Indeed, if I is a left ideal that is not finitely generated, then the morphism $A \to A/I$ has no kernel in R-**mod**.

Example 1.6.7. Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}^{\mathrm{op}}$ is an abelian category.

Example 1.6.8. Let $\mathcal{A}$ be an abelian category and let P be a category. Then the functor category $\mathcal{A}^P$ is an abelian category.

Example 1.6.9. A *topological abelian group* is defined to be an abelian group equipped with a topology such that the group law and $a \mapsto -a$ are continuous. The category of topological abelian groups (where the morphisms are continuous homomorphisms) is an additive category admitting kernels and cokernels, but does not satisfy (AB2). For example, let f be the map $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ carrying x to x. Then $\mathrm{coim}(f) = \mathbb{R}_{\mathrm{disc}}$ and $\mathrm{im}(f) = \mathbb{R}$.

Remark 1.6.10. The following properties follow from (AB2) and Lemma 1.6.1:

(1) If a morphism is both a monomorphism and an epimorphism, then it is an isomorphism.
(2) Every monomorphism is the kernel of its cokernel.
(3) Every epimorphism is the cokernel of its kernel.
(4) Every morphism $f\colon A \to B$ can be decomposed into

$$A \xrightarrow{g} \mathrm{im}(f) \xrightarrow{h} B,$$

where g is an epimorphism and h is a monomorphism.

[6]This terminology is no longer in use. In modern usage, *exact category* refers to a more general notion introduced by Quillen.

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in an abelian category such that $gf = 0$. Then the sequence decomposes uniquely into

$$\begin{tikzcd}
A \arrow[d, two heads] \arrow[ddr, "f"] & & C \\
\mathrm{im}(f) \arrow[dr, hook] \arrow[dd, hook] & & \mathrm{im}(g) \arrow[u, hook] \\
& B \arrow[uur, "g"] \arrow[ur, two heads] \arrow[dr, two heads] & \\
\ker(g) \arrow[ur, hook] & & \mathrm{coker}(f) \arrow[uu, two heads]
\end{tikzcd}$$

Definition 1.6.11. We say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is *exact at B* if $gf = 0$ and the morphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism (or equivalently, $\mathrm{coker}(f) \to \mathrm{im}(g)$ is an isomorphism). We say that a sequence $A^0 \to A^1 \to \cdots \to A^n$ is *exact* if it is exact at each A^i, $1 \leq i \leq n-1$.

Example 1.6.12. (1) A sequence $0 \to A \to 0$ is exact if and only if A is a zero object.

(2) A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism. Dually, a sequence $A \xrightarrow{f} B \to 0$ is exact if and only if f is an epimorphism.

(3) A sequence $0 \to A \xrightarrow{f} B \to 0$ is exact if and only if f is an isomorphism.

(4) A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if f is the kernel of g. Dually, a sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if g is the cokernel of f.

(5) A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence if and only if g is the cokernel of f and f is the kernel of g. Such a sequence is called a *short exact sequence.*

Remark 1.6.13. Every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ decomposes into a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads] & & B \arrow[rr, "g"] \arrow[dr, two heads] & & C \\
& \mathrm{im}(f) \arrow[ur, hook] & & \mathrm{im}(g) \arrow[ur, hook] &
\end{tikzcd}$$

where $0 \to \mathrm{im}(f) \to B \to \mathrm{im}(g) \to 0$ is a short exact sequence.

Yoneda embedding for additive categories

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. We let $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{Fun}(\mathcal{A}, \mathcal{B})$ denote the full subcategory spanned by additive functors. Note that $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B})$ is an abelian category if $\mathcal{B}$ is an abelian category.

Lemma 1.6.14. *The forgetful functor induces a fully faithful functor* $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathbf{Ab}) \to \mathrm{Fun}(\mathcal{A}, \mathbf{Set})$.

Proof. The functor is faithful because the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$ is faithful. It remains to show that for additive functors F and F', every natural transformation $\alpha\colon UF \to UF'$ lifts to a natural transformation $F \to F'$. Indeed, the group structure of FX is induced by the map $\langle \mathrm{id}_{FX}, \mathrm{id}_{FX}\rangle\colon FX \times FX \to FX$, which is the composite

$$FX \times FX \xrightarrow[\sim]{(Fp_1,Fp_2)^{-1}} F(X \times X) \xrightarrow{F\langle \mathrm{id}_X,\mathrm{id}_X\rangle} FX.$$

Here $p_1, p_2\colon X \times X \to X$ are the two projections. The diagram

$$\begin{tikzcd}
FX \times FX \arrow[r, "(Fp_1,Fp_2)^{-1}"] \arrow[d, "\alpha_X\times\alpha_X"] & F(X \times X) \arrow[r, "F\langle \mathrm{id}_X,\mathrm{id}_X\rangle"] \arrow[d, "\alpha_{X\times X}"] & FX \arrow[d, "\alpha_X"] \\
F'X \times F'X \arrow[r, "(F'p_1,F'p_2)^{-1}"] & F'(X \times X) \arrow[r, "F'\langle \mathrm{id}_X,\mathrm{id}_X\rangle"] & F'X
\end{tikzcd}$$

commutes. In other words, α_X is a group homomorphism. □

Remark 1.6.15. Let $\mathcal{A}$ be an additive category with small Hom sets. Then the Yoneda embedding can be lifted to an additive functor $\mathcal{A} \to \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ carrying X to $\mathrm{Hom}_{\mathcal{A}}(-, X)$, which is fully faithful by the above lemma.

Exact functors

Definition 1.6.16. Let $\mathcal{C}$ be a category admitting finite limits (resp. finite colimits). We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left exact* (resp. *right exact*) if it preserves finite limits (resp. finite colimits). For $\mathcal{C}$ admitting finite limits and finite colimits, we say that F is *exact* if it is both left exact and right exact.

A left exact functor between abelian categories is additive. The same holds for right exact functor. A left adjoint functor between abelian categories is right exact. A right adjoint functor between abelian categories is left exact.

Proposition 1.6.17. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:*
(1) F is left exact.
(2) F preserves kernels (or equivalently, for every exact sequence $0 \to X \to Y \to Z$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$).
(3) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Obvious.

(2) $\Longrightarrow$ (1). This follows from Remark 1.6.2 and the assumption that F preserves finite products.

(3) $\Longrightarrow$ (2). We decompose the sequence into a short exact sequence $0 \to X \to Y \xrightarrow{g} Z' \to 0$ and a monomorphism $Z' \xrightarrow{i} Z$. The latter extends to a short exact sequence $0 \to Z' \xrightarrow{i} Z \to Z''$. By (3),

$$0 \to FX \to FY \xrightarrow{Fg} FZ' \to 0,$$
$$0 \to FZ' \xrightarrow{Fi} FZ \to FZ'' \to 0$$

are exact. In particular, $Fi\colon FZ' \to FZ$ is a monomorphism. It follows that $0 \to FX \to FY \xrightarrow{(Fi)(Fg)} FZ$ is exact. □

Corollary 1.6.18. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:*

(1) F is exact.

(2) For every exact sequence $X \to Y \to Z$ in $\mathcal{A}$, $FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$.

(3) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ \to 0$ is a short exact sequence in $\mathcal{B}$.

(4) F is left exact and preserves epimorphisms.

(5) F is right exact and preserves monomorphisms.

Proof. (1) $\implies$ (2) $\implies$ (3). Obvious.

(3) $\implies$ (1). This follows from the proposition.

(1) $\implies$ (4) $\implies$ (3). Obvious.

(1) $\implies$ (5) $\implies$ (3). Obvious. □

Example 1.6.19. Let $\mathcal{A}$ be an abelian category and let $F\colon P \to Q$ be a functor. Then the functor $\mathcal{A}^Q \to \mathcal{A}^P$ induced by F is exact.

Example 1.6.20. Let $\mathcal{A}$ be an abelian category with small Hom sets. Then the functor $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$ is left exact in each variable.

Example 1.6.21. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. If a functor $F\colon \mathcal{A} \to \mathcal{B}$ admits a left (resp. right) adjoint, then F is left (resp. right) exact.

(1) If an abelian category $\mathcal{A}$ admits limits (resp. colimits) indexed by I, then $\lim\colon \mathcal{A}^I \to \mathcal{A}$ (resp. $\mathrm{colim}\colon \mathcal{A}^I \to \mathcal{A}$) is left (resp. right) exact. In particular, for $I = (\bullet \to \bullet)$, $\ker\colon \mathcal{A}^I \to \mathcal{A}$ is left exact and $\mathrm{coker}\colon \mathcal{A}^I \to \mathcal{A}$ is right exact. For I finite and discrete, the product functor $\mathcal{A}^I \to \mathcal{A}$ is exact.

(2) Let R, S, T be rings and consider small bimodules ${}_RM_S$, ${}_SN_T$. Then $-\otimes_S N$ and $M \otimes_S -$ are right exact.

Theorem 1.6.22 (Freyd–Mitchell)**.** *Let $\mathcal{A}$ be a small abelian category. Then there exists a small ring R and a fully faithful exact functor $F\colon \mathcal{A} \to R\text{-}\mathbf{Mod}$.*

We refer the reader to [KS2, Theorem 9.6.10] for a proof of the theorem, which is beyond the scope of these lectures. Let us briefly indicate some ingredients used in the proof. Applying the Yoneda embedding to $\mathcal{A}^{\mathrm{op}}$, we get a fully faithful functor $i\colon \mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ carrying X to $\mathrm{Hom}_{\mathcal{A}}(X,-)$, where $\mathrm{Pro}(\mathcal{A}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A},\mathbf{Ab})^{\mathrm{op}}$ denotes the full subcategory spanned by left exact functors $\mathcal{A} \to \mathbf{Ab}$.[7] One shows that $\mathrm{Pro}(\mathcal{A})$ is an abelian category and i is an exact functor. We take $R = \mathrm{End}_{\mathcal{A}}(G)^{\mathrm{op}}$ for a suitable projective (see the next section) object G of $\mathrm{Pro}(\mathcal{A})$ and we take F to be the composite of i and the exact functor $\mathrm{Pro}(\mathcal{A}) \to R\text{-}\mathbf{Mod}$ carrying H to $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(G,H)$.

[7] More generally, for any category $\mathcal{C}$, the category $\mathrm{Pro}(\mathcal{C})$ of pro-objects of $\mathcal{C}$ is the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathbf{Set})^{\mathrm{op}}$ spanned by small cofiltered limits of the image of the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C},\mathbf{Set})^{\mathrm{op}}$.

Filtered colimits

Definition 1.6.23. A nonempty category I is said to be *filtered* if
(1) For objects i, j in I, there exist morphisms $i \to k$, $j \to k$ in I; and
(2) For morphisms $u, v \colon i \rightrightarrows j$ in I, there exists a morphism $w \colon j \to k$ such that $wu = wv$.

A category I is said to be *cofiltered* if I^{op} is filtered.

Example 1.6.24. A partially ordered set I is filtered if and only if for all $i, j \in I$, there exists $k \in I$ such that $i \le k$ and $j \le k$.
(1) A totally ordered set is filtered.
(2) Let S be a set. The set I_S of finite subsets of S, ordered by inclusion, is filtered. Indeed, if T and T' are finite subsets of S, then so is $T \cup T'$. Given a family of objects $(X_s)_{s\in S}$ in a category $\mathcal{C}$ admitting finite coproducts, $\coprod_{s\in S} X_s$ can be identified with $\mathrm{colim}_{T\in I_S} \coprod_{t\in T} X_t$. Thus a category admitting finite coproducts and small filtered colimits admits small coproducts.

Remark 1.6.25. Recall from Example 1.3.24 and small colimits exist in the category **Set**. Filtered colimits can be described more explicitly. Let I be a filtered category and let $F \colon I \to \mathbf{Set}$ be a functor. Then $\mathrm{colim}\, F$ is represented by

$$Q = \left(\coprod_{i\in \mathrm{Ob}(I)} F(i) \right) / \sim$$

whenever Q is small. Here for $x \in F(i)$ and $y \in F(j)$, $x \sim y$ if and only if there exist morphisms $u \colon i \to k$ and $v \colon j \to k$ in I such that $F(u)(x) = F(v)(y)$.

The underlying sets of filtered colimits in **Grp**, R-**Mod**, **Ring** admit the same description. For example, in the case of R-**Mod**, for $x \in F(i)$ and $y \in F(j)$, $[x]+[y]$ is defined to be $F(u)x+F(v)y$, where $u \colon i \to k$ and $v \colon j \to k$. The forgetful functors **Ring** $\to$ **Ab**, R-**Mod** $\to$ **Grp**, **Grp** $\to$ **Set** commute with small filtered colimits. Compare with Warning 1.3.27.

Proposition 1.6.26. *Small filtered colimits in* R-**Mod** *are exact. In other words, for any small filtered category* I*, the functor* $\mathrm{colim} \colon R\text{-}\mathbf{Mod}^I \to R\text{-}\mathbf{Mod}$ *is exact.*

Proof. Since colim is a left adjoint functor, it is right exact. It remains to check that colim preserves monomorphisms. Let $f \colon F \to G$ be a monomorphism in $R\text{-}\mathbf{Mod}^I$. Let $[x] \in \ker(\mathrm{colim}\, f)$ be the equivalence class of $x \in F(i)$. Then $[f_i(x)] = 0$, so that there exists $u \colon i \to j$ such that $f_j F(u)(x) = G(u) f_i(x) = 0$. Since f_j is a monomorphism, we have $F(u)(x) = 0$, so that $[x] = 0$. ☐

One can also deduce the above from the following.

Proposition 1.6.27. *Small filtered colimits in* **Set** *are exact. In other words, for any small filtered category* I*, any finite category* J*, and any functor* $F \colon I \times J \to \mathbf{Set}$*, the map*

$$\operatorname*{colim}_{i\in I} \lim_{j\in J} F(i,j) \to \lim_{j\in J} \operatorname*{colim}_{i\in I} F(i,j)$$

is a bijection.

It follows from the proposition that the same holds for **Grp** and **Ring**.

Proof. Since any finite limit is an equalizer of finite products (Remark 1.6.2), it suffices to show that colim_I preserves equalizers and finite products.

Let $(f,g)\colon G \rightrightarrows H$ be morphisms in $\mathbf{Set}^I$. We show that the map

$$\phi\colon \operatorname*{colim}_{i\in I} \mathrm{eq}(f(i),g(i)) \to \mathrm{eq}(\mathrm{colim}\, f, \mathrm{colim}\, g)$$

is bijective. Let $[x]$ and $[y]$ be elements of the left-hand side, $x \in \mathrm{eq}(f(i),g(i))$, $y \in \mathrm{eq}(f(j),g(j))$ such that $\phi([x]) = \phi([y])$. Then there exist $u\colon i \to k$, $v\colon j \to k$ such that $G(u)(x) = G(v)(y)$. Thus $[x] = [y]$ in $\mathrm{colim}_{i\in I}\mathrm{eq}(f(i),g(i))$. This proves that ϕ is injective (in fact this is equivalent to the fact that colim preserves monomorphisms, which can be proved similarly to Proposition 1.6.26).

Consider an element $[x]$ of the right-hand side, equivalence class of an element x of $G(i)$ such that $f(i)(x) \sim g(i)(x)$. In other words, there exists $u,v\colon i \rightrightarrows j$, such that $H(u)(f(i)(x)) = H(v)(g(i)(x))$. Since I is filtered, there exists $w\colon j \to k$ such that $wu = wv$. Since

$$f(i)G(wu)(x) = H(wu)f(i)(x) = H(wv)(g(j)(x)) = g(k)H(wv)(x),$$

we have $G(wu)(x) \in \mathrm{eq}(f(k),g(k))$. Then $\phi([G(wu)(x)]) = [x]$. This proves that ϕ is surjective.

Similarly, one proves that colim_I preserves finite products. $\square$

Sheaves

Let X be a small topological space. Let $\mathrm{Open}(X)$ be the set of open subsets of X, ordered by inclusion.

Definition 1.6.28. A *sheaf of abelian groups* on X is a functor $\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$ satisfying the following gluing condition: for every open covering $(U_i)_{i\in I}$ of an open subset U, the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_i)$ induce a bijection from $\mathcal{F}(U)$ to the equalizer of the maps

$$\prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{F}(U_i\cap U_j), \tag{1.6.1}$$

induced by the restriction maps $\mathcal{F}(U_i) \to \mathcal{F}(U_i\cap U_j)$ and $\mathcal{F}(U_j) \to \mathcal{F}(U_i\cap U_j)$. The category $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathbf{Ab})$ is called the category of *presheaves*. The category $\mathrm{Shv}(X)$ of sheaves of abelian groups on X is the full subcategory spanned by sheaves of abelian groups on X.

The category $\mathrm{Shv}(X)$ is stable under small limits in $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathbf{Ab})$). In particular, $\mathrm{Shv}(X)$ admits small limits. The inclusion functor ι admits a left adjoint a, called the *sheafification* functor, with $(a\mathcal{F})(U)$ given by the colimit indexed by $\mathrm{Cov}(U)$ of the equalizer of (1.6.1). Here $\mathrm{Cov}(U)$ denotes the set of open coverings of U, with $\mathcal{U} \le \mathcal{U}'$ if $\mathcal{U}'$ refines $\mathcal{U}$. Since $\mathrm{Cov}(U)$ is filtered, a is exact. The category $\mathrm{Shv}(X)$ also admits small colimits, given by $\mathrm{colim}_i\, \mathcal{F}_i = a\,\mathrm{colim}_i\, \iota\mathcal{F}_i$. It is clear that $\mathrm{Shv}(X)$ is an abelian category (the exactness of a is used in (AB2)).

Example 1.6.29. Let X be a complex manifold. We have an epimorphism $\exp\colon \mathcal{O} \to \mathcal{O}^\times$ of sheaves on X, where $\mathcal{O}(U)$ is the additive group of holomorphic functions on U and $\mathcal{O}^\times$ is the multiplicative group of nowhere-vanishing holomorphic functions on U. As a morphism of presheaves, exp is not an epimorphism in general (for example if $X = \mathbb{C} - \{0\}$, then the function $z \mapsto z$ is not in the image of exp). However, for any $f \in \mathcal{O}^\times(U)$, and any point $x \in U$, there exists an open neighborhood $x \in V \subseteq U$ such that $f|_V$ is in the image of exp.

Diagram lemmas

Let $\mathcal{A}$ be an abelian category.

Proposition 1.6.30 (Snake lemma)**.** *Consider a commutative diagram in $\mathcal{A}$ with exact rows*

(1.6.2)
$$\begin{tikzcd} & X' \arrow[r] \arrow[d, "u'"'] & X \arrow[r] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y''. \end{tikzcd}$$

There exists a unique morphism $v\colon X \times_{X''} \ker(u'') \to Y'$ making the diagram

$$\begin{tikzcd} X \times_{X''} \ker(u'') \arrow[r, "p_1"] \arrow[d, "v"'] & X \arrow[d, "u"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

commute and a unique morphism $\delta\colon \ker(u'') \to \operatorname{coker}(u')$ making the diagram

$$\begin{tikzcd} X \times_{X''} \ker(u'') \arrow[r, "p_2"] \arrow[d, "v"'] & \ker(u'') \arrow[d, "\delta"] \\ Y' \arrow[r] & \operatorname{coker}(u') \end{tikzcd}$$

commute. Here p_1 and p_2 are the projections. Moreover, the sequence

(1.6.3)
$$\ker(u') \to \ker(u) \to \ker(u'') \xrightarrow{\delta} \operatorname{coker}(u') \to \operatorname{coker}(u) \to \operatorname{coker}(u'')$$

is exact.

Proof. By the Freyd–Mitchell theorem, we may work in a module category and take elements. Let $a \in \ker(u'')$ and let $b \in X$ be a preimage. Then the image of $u(b)$ in Y'' is $u''(a) = 0$, so that $u(b)$ is the image of $c \in Y'$. We define $\delta(a)$ to be the class of c in $\operatorname{coker}(u')$, as shown by the diagram:

$$\begin{tikzcd} & & a \arrow[d, mapsto] \\ & b \arrow[r, mapsto] \arrow[d, mapsto] & a \arrow[d, mapsto] \\ c \arrow[r, mapsto] \arrow[d, mapsto] & u(b) \arrow[r, mapsto] & 0 \\ \delta(a) & & \end{tikzcd}$$

It is easy to check that the assertions of the proposition. □

For a direct (but tedious) proof of the snake lemma without diagram-chasing, we refer to [KS2, Lemma 12.1.1]. It is also possible to give a proof by diagram-chasing yet not relying on the Freyd–Mitchell theorem, by introducing a notion of members as substitutes for elements [ML2, Section VIII.4].

Remark 1.6.31. If the upper row of (1.6.2) is a short exact sequence, then the sequence (1.6.3) extends to an exact sequence $0 \to \ker(u') \to \ker(u)$. Dually, if the lower row of (1.6.2) is a short exact sequence, then the sequence (1.6.3) extends to an exact sequence $\operatorname{coker}(u) \to \operatorname{coker}(u'') \to 0$.

Corollary 1.6.32. *Under the assumptions of Proposition 1.6.30,*

(1) if u' and u'' are monomorphisms, then u is a monomorphism;
(2) if u' and u'' are epimorphisms, then u is an epimorphism;
(3) if u' and u'' are isomorphisms, then u is an isomorphism;
(4) if u' is an epimorphism and u is a monomorphism, then u'' is a monomorphism.
(5) if u'' is a monomorphism and u is an epimorphism, then u' is an epimorphism.

Remark 1.6.33. In case (3) of the corollary, the two rows of 1.6.2 are short exact sequences. Note that given short exact sequences $0 \to X' \to X \to X'' \to 0$, $0 \to Y' \to Y \to Y'' \to 0$, with $X' \simeq Y'$ and $X'' \simeq Y''$, it is in general *not* true that $X \simeq Y$. The existence of a morphism $X \to Y$ compatible with the isomorphisms $X' \simeq Y'$ and $X'' \simeq Y''$ is crucial to the conclusion of case (3).

Corollary 1.6.34. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be a short exact sequence in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) f admits a retraction: there exists $r\colon X \to X'$ such that $rf = \mathrm{id}_{X'}$.
(2) g admits a section: there exists $s\colon X'' \to X$ such that $gs = \mathrm{id}_{X''}$.
(3) The sequence is isomorphic (as an object of $\mathcal{A}^{\bullet\to\bullet\to\bullet}$) to the short exact sequence $0 \to X' \xrightarrow{i} X' \oplus X'' \xrightarrow{p} X'' \to 0$, where i and p are the canonical morphisms.

Proof. (3) $\Longrightarrow$ (1). Clear.

(1) $\Longrightarrow$ (3). We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, "{(r,g)}"] & X'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X' \arrow[r] & X' \oplus X'' \arrow[r] & X'' \arrow[r] & 0. \end{tikzcd}$$

By Corollary 1.6.32, (r, g) is an isomorphism.

(2) $\Longleftrightarrow$ (3). This follows by duality. □

Definition 1.6.35. A short exact sequence satisfying the above equivalent conditions is said to be *split*.

Remark 1.6.36. Any additive functor preserves split short exact sequences.

Corollary 1.6.37 (Four lemma)**.** *Consider a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d, "u^0"'] & X^1 \arrow[r] \arrow[d, "u^1"] & X^2 \arrow[r] \arrow[d, "u^2"] & X^3 \arrow[d, "u^3"] \\ Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3 \end{tikzcd}$$

with exact rows.

(1) *If u^0 is an epimorphism, u^1 and u^3 are monomorphisms, then u^2 is a monomorphism.*

(2) *If u^3 is a monomorphism, u^0 and u^2 are epimorphisms, then u^1 is an epimorphism.*

Proof. The diagram can be decomposed into commutative diagrams with exact rows

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d, "u^0"'] & A^1 \arrow[r] \arrow[d, "v^1"] & 0 \\ Y^0 \arrow[r] & B^1 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & A^1 \arrow[r] \arrow[d, "v^1"'] & X^1 \arrow[r] \arrow[d, "u^1"] & A^2 \arrow[r] \arrow[d, "v^2"] & 0 \\ 0 \arrow[r] & B^1 \arrow[r] & Y^1 \arrow[r] & B^2 \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A^2 \arrow[r] \arrow[d, "v^2"'] & X^2 \arrow[r] \arrow[d, "u^2"] & A^3 \arrow[r] \arrow[d, "v^3"] & 0 \\ 0 \arrow[r] & B^2 \arrow[r] & Y^2 \arrow[r] & B^3 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & A^3 \arrow[r] \arrow[d, "v^3"'] & X^3 \arrow[d, "u^3"] \\ 0 \arrow[r] & B^3 \arrow[r] & Y^3 \end{tikzcd}$$

Since u^0 is an epimorphism, so is v^1. Since u^3 is a monomorphism, so is v^3. In the case of (1), v^2 is a monomorphism and hence so is u^2 by Corollary 1.6.32. Assertion (2) follows by duality. □

Corollary 1.6.38 (Five lemma)**.** *Consider a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d, "u^0"'] & X^1 \arrow[r] \arrow[d, "u^1"] & X^2 \arrow[r] \arrow[d, "u^2"] & X^3 \arrow[r] \arrow[d, "u^3"] & X^4 \arrow[d, "u^4"] \\ Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3 \arrow[r] & Y^4 \end{tikzcd}$$

with exact rows. If u^0 is an epimorphism, u^4 is a monomorphism, and u^1, u^3 are isomorphisms, then u^2 is an isomorphism.

Proof. By the four lemma, u^2 is a monomorphism and an epimorphism. □

Corollary 1.6.39 (Nine lemma)**.** *Consider a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns. If the top two rows or the bottom two rows are exacts, then all the three rows are exact.

The diagram in the nine lemma (commutative with exact rows and columns) is called a nine-diagram.

Proof. It suffices to apply the snake lemma to the top two rows in the first case, and the bottom two rows in the second case. □

Remark 1.6.40. If the top and bottom rows are exact, then the middle row is *not* exact in general. (The claim of [ML2, Section VIII.4, Exercise 5 (c)] is mistaken.) Indeed, if $f\colon Y \to X$ is a nonzero morphism, then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "\mathrm{id}_X"] \arrow[d, "(\mathrm{id}_X,0)"] & X \arrow[r] \arrow[d, "\mathrm{id}_X"] & 0 \\
0 \arrow[r] & Y \arrow[r, "(f,\mathrm{id}_Y)"] \arrow[d, "\mathrm{id}_Y"'] & X \oplus Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r, "\mathrm{id}_Y"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

provides a counterexample. Here p_X and p_Y are projections. By contrast, if, moreover, the composition of $Y' \to Y \to Y''$ is zero, then the middle row *is* exact by Proposition 2.1.13.

Remark 1.6.41. Every commutative square with exact rows and columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
Y \arrow[r] & Y'' \arrow[r] & 0
\end{tikzcd}$$

extends to a nine diagram uniquely up to isomorphism. A commutative square with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

extends to a nine diagram if and only if the square is a pullback (Exercise).

1.7 Projective and injective objects

Definition 1.7.1. Let $\mathcal{C}$ be a category. An object P of $\mathcal{C}$ is said to be *projective* if given a morphism f and an epimorphism u in $\mathcal{C}$ as shown in the diagram

(1.7.1)
$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\ X \arrow[r, "u"] & Y, \end{tikzcd}$$

there exists g rendering the diagram commutative. Dually, an object I of $\mathcal{C}$ is said to be *injective* if given a morphism f and a monomorphism u in $\mathcal{C}$ as shown in the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "g"] \\ I, \end{tikzcd}$$

there exists g rendering the diagram commutative.

Remark 1.7.2. An object I is injective in $\mathcal{C}$ if and only if it is projective in $\mathcal{C}^{\mathrm{op}}$.

Note that we do *not* require uniqueness of the dotted arrow. By definition, an object P of $\mathcal{C}$ is projective if and only if for every epimorphism $u\colon X \to Y$ in $\mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is a surjection. Dually, an object I of $\mathcal{C}$ is injective if and only if for every monomorphism $u\colon X \to Y$ in $\mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(Y, I) \to \mathrm{Hom}_{\mathcal{C}}(X, I)$ is a surjection. We obtain the following.

Proposition 1.7.3. *Let $\mathcal{A}$ be an abelian category with small* Hom *sets. An object P is projective if and only if the functor* $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$ *is exact. An object I is injective if and only if the functor* $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ *is exact.*

Proposition 1.7.4. *Let $\mathcal{A}$ be an abelian category. An object P is projective if and only if every epimorphism $f\colon M \to P$ admits a section. An object I is injective if and only if every monomorphism $g\colon I \to M$ admits a retraction.*

The "only if" parts hold in any category.

Proof. By duality, it suffices to prove the first assertion. If P is projective, applying the definition to the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, equal] \\ M \arrow[r, "f"] & P, \end{tikzcd}$$

we obtain a section of f. Conversely, given the diagram (1.7.1), we form the pullback square

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "p"'] & P \arrow[d, "f"] \\ X \arrow[r, "u"] & Y. \end{tikzcd}$$

Since v is an epimorphism (Exercise), it admits a section $s\colon P \to M$ and we take $g = ps$. □

Corollary 1.7.5. *Let $\mathcal{A}$ be an abelian category. The following conditions are equivalent:*
(1) Every object of $\mathcal{A}$ is projective.
(2) Every object of $\mathcal{A}$ is injective.
(3) Every short exact sequence in $\mathcal{A}$ is split.

Note that $\mathcal{A}$ satisfies the above conditions if and only if $\mathcal{A}^{\mathrm{op}}$ does. Compare with Example 1.7.9 below.

Remark 1.7.6. If $\mathcal{A}$ is a category satisfying the conditions of Corollary 1.7.5, then any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is exact.

Example 1.7.7. Recall that a ring R is called *semisimple* if it satisfies the following equivalent conditions:
(1) The (left) R-module R is semisimple;
(2) Every (left) R-module is semisimple;
(3) (Artin–Wedderburn) R is isomorphic to a finite product of matrix rings $M_n(D)$ over division rings D.

Note that by Condition (3), R is semisimple if and only if R^{op} is semisimple. Recall that an R-module M is called *semisimple* if every submodule is a direct summand. Thus, by Condition (2), that R is a semisimple ring is further equivalent to the conditions of Corollary 1.7.5 for $\mathcal{A} = R$-**Mod**.

Proposition 1.7.8. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.*
(1) If F admits a right adjoint G that carries epimorphisms to epimorphisms, then F carries projective objects to projective objects.
(2) If F admits a left adjoint G that carries monomorphisms to monomorphisms, then F carries injective objects to injective objects.

In particular, if F is a functor between abelian categories admitting an *exact* right (resp. left) adjoint, then F carries projective (resp. injective) objects to projective (resp. injective) objects.

Proof. By duality it suffices to prove (1). Let P be a projective object of $\mathcal{C}$ and let $u\colon X \to Y$ be an epimorphism in $\mathcal{D}$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(FP, X) \arrow[r, "u\circ-"] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathcal{C}}(FP, Y) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{\mathcal{D}}(P, GX) \arrow[r, "Gu\circ-"] & \mathrm{Hom}_{\mathcal{D}}(P, GY). \end{tikzcd}$$

By assumption, Gu is an epimorphism, so that the lower row is a surjection. It follows that the upper row is also a surjection. □

Example 1.7.9. In **Set**, every object is projective. Every nonempty set is injective. The empty set is *not* injective. ([HS, Proposition II.10.1] is inaccurate.)

Example 1.7.10. Let R be a ring. The free module functor $F\colon \mathbf{Set} \to R\text{-}\mathbf{Mod}$ is a right adjoint to the forgetful functor $U\colon R\text{-}\mathbf{Mod} \to \mathbf{Set}$. Since U carries epimorphisms to epimorphisms, F carries projective objects to projective objects. Thus every free module is projective.

Note that every R-module is the quotient of a free R-module. Indeed, the adjunction map $FUM \to M$ is clearly surjective. Thus every R-module is the quotient of a projective R-module. We will see later that the dual of this statement also holds.

Remark 1.7.11. Consider a family of objects $(X_i)_{i\in I}$ of a category $\mathcal{C}$.
(1) If each X_i is projective and the coproduct $\coprod_{i\in I} X_i$ exists, then the coproduct is projective.
(2) Dually, if each X_i is injective and the product $\prod_{i\in I} X_i$ exists, then the product is injective.

Remark 1.7.12. An object Y equipped with morphisms $s\colon Y \to X$ and $r\colon X \to Y$ such that $rs = \mathrm{id}_Y$ is called a *retract* of X. A retract of a projective (resp. injective) object is projective (resp. injective). In particular, a direct summand of a projective (resp. injective) object in an additive category is projective (resp. injective).

1.8 Projective and injective modules

Let R be a ring. As before, we will concentrate of left R-modules. Right R-modules can be identified with left R^{op}-modules. This duality is not to be confused with the duality between a category and its opposite: The opposite of R-**Mod** is not equivalent to a module category in general[8].

Projective modules

Proposition 1.8.1. *Let P be an R-module. The following conditions are equivalent:*
(1) P is projective.
(2) P is a direct summand of some free R-module.

[8]We have seen that in R-**Mod** small filtered colimits are exact, but limits indexed by I^{op} with I small and filtered are not exact in general.

Proof. (1) $\Longrightarrow$ (2). Let F be a free R-module equipped with a surjective homomorphism $f\colon F \to P$. By Proposition 1.7.4, the short exact sequence $0 \to \ker(f) \to F \to P \to 0$ splits.

(2) $\Longrightarrow$ (1). This follows from Remarks 1.7.11 and Example 1.7.10. □

The same proof gives the following.

Proposition 1.8.2. *Let P be an R-module. The following conditions are equivalent:*

(1) P is finitely-generated and projective.

(2) P is a direct summand of some R^n.

Example 1.8.3. The $(\mathbb{Z}/6\mathbb{Z})$-module $\mathbb{Z}/2\mathbb{Z}$ is projective, but not free. The $(\mathbb{Z}/4\mathbb{Z})$-module $\mathbb{Z}/2\mathbb{Z}$ is not projective.

Definition 1.8.4. (1) A ring R is said to be *left hereditary* if every left ideal of R is projective as an R-module.

(2) A ring R is said to be *left semi-hereditary* if every finitely generated left ideal of R is projective as an R-module.

Similarly one defines right hereditary rings and right semi-hereditary rings using right ideals. A ring R is a right hereditary (resp. semi-hereditary) ring if and only if R^{op} is a left hereditary (resp. semi-hereditary) ring. For commutative rings, we drop the words "left" and "right".

Remark 1.8.5. Recall that a ring is left Noetherian if and only if every left ideal is finitely generated. Thus a left Noetherian left semi-hereditary ring is left hereditary.

Theorem 1.8.6 (Kaplansky)**.** *Let R be a left hereditary ring. Then any submodule P of a free R-module $F = \bigoplus_{\alpha\in I} Re_\alpha$ is isomorphic to a direct sum of left ideals of R; in particular, P is a projective module.*

Proof. Choose a well-order on I. For each $\alpha \in I$, let

$$F_{<\alpha} = \sum_{\beta<\alpha} Re_\beta, \quad F_{\leq\alpha} = \sum_{\beta\leq\alpha} Re_\beta.$$

Let $P_{<\alpha} = P \cap F_{<\alpha}$ and $P_{\leq\alpha} = P \cap F_{\leq\alpha}$. Consider the homomorphism

$$f_\alpha\colon P_{\leq\alpha} \subseteq F \to R$$

carrying $\sum_{\beta\in I} r_\beta e_\beta$ to r_α. We have $\ker(f_\alpha) = P_{<\alpha}$. Since $\operatorname{im}(f_\alpha)$ is a left ideal of R, it is projective, so that we have

$$P_{\leq\alpha} = P_{<\alpha} \oplus Q_\alpha,$$

where Q_α is an R-submodule of $P_{\leq\alpha}$ such that $f|_{Q_\alpha}$ induces an isomorphism $Q_\alpha \xrightarrow{\sim} \operatorname{im}(f_\alpha)$.

Let us show $P = \bigoplus_{\alpha\in I} Q_\alpha$. Suppose we have $a_{\alpha_1} + \cdots + a_{\alpha_n} = 0$ with $a_{\alpha_i} \in Q_{\alpha_i}$ for $i = 1, \dots, n$. We may assume that $\alpha_1 < \cdots < \alpha_n$. Then $a_{\alpha_1}, \dots, a_{\alpha_{n-1}} \in P_{<\alpha_n}$, so that $a_{\alpha_n} \in P_{<\alpha_n} \cap Q_{\alpha_n} = 0$. By induction we have $a_{\alpha_i} = 0$ for all i.

It remains to show that $P = \sum_{\alpha\in I} Q_\alpha$. Assume the contrary. Since $P = \bigcup_{\alpha\in I} P_{\leq\alpha}$, there exists a smallest $\beta \in I$ and $a \in P_{\leq\beta}$ such that $a \notin Q = \sum_{\alpha\in I} Q_\alpha$. Write $a = b + c$, where $b \in P_{<\beta}$ and $c \in Q_\beta$. We have $b \in P_{\leq\gamma}$ for some $\gamma < \beta$. By the minimality of β, we have $b \in Q$. Then $a = b + c \in Q$, Contradiction. □

Corollary 1.8.7. *Let R be a left hereditary ring. An R-module is projective if and only if it is a submodule of a free R-module.*

Proof. The "if" part follows from the theorem. The "only if" part follows from Proposition 1.8.1. □

Corollary 1.8.8. *Let R be a ring. The following conditions are equivalent:*
(1) R is left hereditary;
(2) Submodules of free R-modules are projective;
(3) Submodules of projective R-modules are projective.

Proof. By Theorem 1.8.6, (1) $\Longrightarrow$ (2). By definition, (2) $\Longrightarrow$ (1). By Proposition 1.8.1, (2) $\Longleftrightarrow$ (3). □

For left semi-hereditary rings, we have the following variant of Theorem 1.8.6.

Theorem 1.8.9. *Let R be a left semi-hereditary ring. Then any finitely generated submodule P of a free R-module $F = \bigoplus_{\alpha\in I} Re_\alpha$ is isomorphic to a finite direct sum of finitely generated left ideals of R; in particular, P is a projective module.*

Proof. There exists a finite subset $J \subseteq I$ such that P is contained in $F_J = \bigoplus_{\alpha\in J} Re_\alpha$. Up to replacing F by F_J, we may assume that I is finite. In this case, we can repeat the proof of Theorem 1.8.6, noting that $\mathrm{im}(f_\alpha)$ is finitely generated. □

Corollary 1.8.10 (Albrecht)**.** *Let R be a ring. The following conditions are equivalent:*
(1) R is left semi-hereditary;
(2) Finitely generated submodules of free R-modules are projective;
(3) Finitely generated submodules of projective R-modules are projective.

Proof. Similar to the proof of Corollary 1.8.8. □

Warning 1.8.11. Following [L1], in these lectures a *domain* is a nonzero ring, not necessarily commutative, where 0 is the only zero-divisor.

Definition 1.8.12. (1) A hereditary commutative domain is called a *Dedekind domain* (or *Dedekind ring*).
(2) A *PLID* (principal left ideal domain) is a domain in which every left ideal is principal. Similarly one defines PRID using right ideals.
(3) A commutative PLID is called a *PID* (principal ideal domain).
(4) A semi-hereditary commutative domain is called a *Prüfer domain*.
(5) A *Bézout domain* is a commutative domain in which every finitely generated left ideal is principal.

Some authors exclude fields from the definition of Dedekind domain.

Remark 1.8.13. (1) A principal left ideal of a domain is free. Thus a PLID is left hereditary. Similarly, a Bézout domain is Prüfer. An ideal I of a commutative domain is principal if and only if it is free.
(2) PIDs are exactly Noetherian Bézout domains.

Corollary 1.8.14. *Let R be PLID. Then any submodule of a free R-module is free. Moreover, an R-module is free if and only if it is projective.*

Proof. The first assertion follows from Kaplansky's theorem. The second assertion follows from the first one and Proposition 1.8.1. □

Definition 1.8.15. An ideal I of a commutative domain R is called *invertible* if there exists an R-submodule of M of the quotient field K of R such that $IM = R$.

The condition implies that $1 = \sum_{i=1}^n a_i q_i$ with $a_i \in I$, $q_i \in M$. Then any $b \in I$ satisfies $b = \sum_{i=1}^n a_i q_i b$ with $q_i b \in IM = R$. Thus invertible ideals are finitely generated.

Proposition 1.8.16. *A nonzero ideal I of a commutative domain R is invertible if and only if it is projective as an R-module.*

Thus a Dedekind domain is a commutative domain of which every nonzero ideal is invertible.

Proof. Let I be an invertible ideal with $IM = R$. Then $1 = \sum_{i=1}^n a_i q_i$ with $a_i \in I$, $q_i \in M$. Consider the free R-module $F = \bigoplus_{i=1}^n Re_i$. Let $f\colon F \to I$ be the homomorphism such that $f(e_i) = a_i$. Then $s\colon I \to F$ given by $s(a) = \sum_{i=1}^n aq_i e_i$ is a section of f. Thus I is a direct summand of F, and hence projective.

Conversely, let I be an ideal of R that is projective as an R-module. Then there exists a free R-module $F = \bigoplus_{\alpha\in J} Re_\alpha$ and homomorphisms $f\colon F \to I$ and $s\colon I \to F$ with $fs = \mathrm{id}_I$. Put $a_\alpha = f(e_\alpha)$. Let $a \in I$ with $a \neq 0$. By the following lemma, $s(a) = \sum_{\alpha\in J} aq_\alpha e_\alpha$, where $q_\alpha \in K = \mathrm{Frac}(R)$ (zero for all but finitely many $\alpha \in J$) satisfies $q_\alpha I \subseteq R$. Take $M = \sum_{\alpha\in J} Rq_\alpha$. Then $IM \subseteq R$. Moreover, $a = fs(a) = \sum_{\alpha\in J} aq_\alpha a_\alpha$, so that $1 = \sum_{\alpha\in J} a_\alpha q_\alpha \in IM$. It follows that $IM = R$. □

Lemma 1.8.17. *Let $s\colon I \to R$ be a homomorphism of R-modules. Then there exists $q \in K = \mathrm{Frac}(R)$ such that $s(a) = qa$ for all $a \in I$.*

Proof. For $a, b \in I$, $bs(a) = s(ab) = as(b)$. Thus $q = s(a)/a$ does not depend on the choice of $a \in I$, $a \neq 0$. □

Corollary 1.8.18. *An ideal of a Prüfer domain is projective if and only if it is finitely generated.*

Proof. The "if" part is the definition and the only if part follows from the proposition. □

Corollary 1.8.19. *Dedekind domains are exactly Noetherian Prüfer domains.*

We refer the reader to standard textbooks on commutative algebra for other characterizations of Dedekind domains.

Example 1.8.20. Every field is a PID. More generally, every division ring is a PLID.

Example 1.8.21. The ring of rational integers $\mathbb{Z}$ is a PID. For any field k, the polynomial ring $k[x]$ and the ring $k[[x]]$ of formal power series are PIDs. For any division ring D, $D[x]$ is a PLID (and a PRID).

Example 1.8.22. The ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ is a PID. $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, but not a PID: the ideal $(2, 1 + \sqrt{-5})$ is not principal. As an R-module, this ideal is projective but not free.

More generally, for a square-free integer $a \neq 0, 1$, $R = \mathbb{Z}[\sqrt{a}]$ is a Dedekind domain if and only if $a \equiv 2, 3 \pmod 4$. The only negative square-free integers a for which $R = \mathbb{Z}[\sqrt{a}]$ is a PID are $a = -1$ and $a = -2$. The last statement was part of Gauss's class number problems for quadratic fields and was proven by Landau in 1902. It is not known whether there are infinitely many positive square-free integers a for which $R = \mathbb{Z}[\sqrt{a}]$ is a PID.

Example 1.8.23. Every semisimple ring R is left (and right) hereditary.

Example 1.8.24. If R and S are Morita equivalent rings, namely if the categories R-**Mod** and S-**Mod** are equivalent, then R is left (resp. right) hereditary if and only if S is left (resp. right) hereditary. In particular, if R is a Dedekind domain, then $M_n(R)$ is left (and right) hereditary.

Example 1.8.25. A commutative domain R such that for every $0 \neq x \in \mathrm{Frac}(R)$ either $x \in R$ or $x^{-1} \in R$ is called a *valuation ring*. A valuation ring is a Bézout domain. For any field k, the valuation ring $\bigcup_{n=1}^{\infty} k[[x^{1/n}]]$ is not Noetherian.

Example 1.8.26. The free algebra $R = k\langle X_i \rangle_{i \in I}$ over a field k generated by a set I of variables is a left (and right) hereditary domain (in fact any left ideal of R is free [C, Corollary II.4.3]), but not left (or right) Noetherian for $\#I > 1$.

Example 1.8.27. A ring R is said to be *von Neumann regular* if for each r, there exists $s \in R$ such that $rsr = r$; *Boolean* if $r^2 = r$ for all $r \in R$. Boolean rings are von Neumann regular (by taking $s = 1$). Countable von Neumann regular rings are hereditary [L1, Example 2.32 (e)]. The countable Boolean ring $R = \{f \colon \mathbb{N} \to \mathbb{F}_2 \mid f^{-1}(0) \text{ or } f^{-1}(1) \text{ is finite}\}$ is hereditary, but not Noetherian.

Example 1.8.28. The commutative domain $\mathbb{Z}[\sqrt{-3}]$ is *not* a Dedekind domain. Indeed, the ideal $(2, 1+\sqrt{-3})$ is not invertible. The commutative domain $\mathbb{Z}[x_1, \dots, x_n]$ is *not* a Dedekind domain for $n \geq 1$. Indeed the ideal $(2, x_1)$ is not invertible. The commutative domain $k[x_1, \dots, x_n]$ is *not* a Dedekind domain for $n \geq 2$. Indeed, the ideal (x_1, x_2) is not invertible. These rings are not semi-hereditary. However, we have the following deep result.

Theorem 1.8.29 (Quillen, Suslin)**.** *Let R be a PID and let $S = R[x_1, \dots, x_n]$. Then every projective S-module is free.*

The theorem was proved independently by Quillen and Suslin in 1976. The question (for R a field and finitely-generated modules) was first raised by Serre. See [L2] for an exposition.

Remark 1.8.30. Kaplansky and later Small constructed examples of right hereditary rings that are not left hereditary. Small's example is $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. See [L1, Section 2F].

Remark 1.8.31. Recall that a direct sum of projective modules is projective. On the other hand, an infinite product of projective modules is not necessarily projective. For example, the product $\mathbb{Z}^{\mathbb{N}}$ of countably many copies of $\mathbb{Z}$ is not a projective $\mathbb{Z}$-module (Baer, Exercise).

Injective modules

Theorem 1.8.32 (Baer's test)**.** *Let R be a ring. An R-module I is injective if and only if for every left ideal A, every homomorphism $A \to I$ extends to a homomorphism $R \to I$:*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\ I. & \end{tikzcd}$$

Proof. The "only if" part follows from the definition. To show the "if" part, consider an injective homomorphism $u\colon X \to Z$ and a homomorphism $f\colon X \to I$. To simplify notation, we consider u as the inclusion of a submodule. We look at the set S of pairs (Y, g), where $X \subseteq Y \subseteq Z$ and $g\colon Y \to I$ extends f. We equip S with the following order: $(Y, g) \leq (Y', g')$ if and only if $Y \subseteq Y'$ and g' extends g. Every chain $\{(Y_\alpha, g_\alpha)\}$ in S admits the upper bound (Y, g), where $Y = \bigcup_\alpha Y_\alpha$ and $g\colon Y \to I$ the unique homomorphism extending all the g_α. By Zorn's lemma, there exists a maximal element (Y_0, g_0) in S. It suffices to show that $Y_0 = Z$. Assume the contrary. Then there exists $z \in Z$ such that $z \notin Y_0$. Consider the left ideal A of R consisting of $r \in R$ such that $rz \in Y_0$. By assumption, the homomorphism $h_0\colon A \to I$ given by $h_0(r) = g_0(rz)$ extends to a homomorphism $h_1\colon R \to I$. Then g_0 extends to $g_1\colon Y_1 = Y_0 + Rz \to I$ by $g_1(y + rz) = g_0(y) + h_1(r)$. It is easy to check that g_1 is well-defined. Then $(Y_0, g_0) < (Y_1, g_1)$, contradicting the maximality of (Y_0, g_0). □

Definition 1.8.33. Let R be a domain. We say that an R-module D is *divisible* if for every $d \in D$ and every nonzero $r \in R$, there exists $c \in D$ such that $rc = d$.

Note that we do not require the uniqueness of c.

Remark 1.8.34. Any quotient of a divisible R-module is divisible. Any direct sum of divisible R-modules is divisible. Any product of divisible R-modules is divisible.

Proposition 1.8.35. *Let R be a domain. Then every injective R-module is divisible. Moreover, if R is a PLID or a Dedekind domain, then an R-module is injective if and only if it is divisible.*

Proof. Let I be an injective R-module. Let $d \in I$ and $r \in R$, $r \neq 0$. Since R is a domain, the homomorphism of R-modules $m\colon R \to R$ defined by $m(s) = sr$, is injective. Let $f\colon R \to I$ be the homomorphism carrying 1 to d. Then there exists $g\colon R \to I$ such that $f = gm$. Then $d = f(1) = gm(1) = g(r) = rg(1)$.

Assume that R is a PLID and let D be a divisible R-module. We apply Baer's test. Consider a left ideal A of R and a homomorphism $f\colon A \to D$. Since R is a PLID, A is a principal left ideal: $A = Ra$. Since D is divisible, there exists $c \in D$ such that $f(a) = ac$. Then f extends to $g\colon R \to D$ given by $g(r) = rc$.

Assume now that R is a Dedekind domain and let D be a divisible R-module. We apply Baer's test. Consider a nonzero ideal A of R and a homomorphism $f\colon A \to D$. Since R is a Dedekind domain, A is invertible: $1 = \sum_{i=1}^n a_i q_i$ with $a_i \in A$, $q_i \in K = \mathrm{Frac}(R)$, $q_i A \subseteq R$. Since D is divisible, there exists $c_i \in D$ such that $a_i c_i = f(a_i)$. Then, for all $a \in A$,

$$f(a) = f(\sum_{i=1}^n a_i q_i a) = \sum_{i=1}^n q_i a f(a_i) = a \sum_{i=1}^n q_i a_i c_i.$$

Thus f extends to $g\colon R \to D$ given by $g(r) = rc$ where $c = \sum_{i=1}^n q_i a_i c_i$. □

Corollary 1.8.36. *If R is a PLID or a Dedekind domain, then quotients of injective R-modules are injective and direct sums of injective R-modules are injective.*

Example 1.8.37. Let $R = \mathbb{Z}$. Then $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible and hence injective. The following modules are *not* divisible or injective: $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$.

The following dual of Corollary 1.8.8 generalizes the first assertion of Corollary 1.8.36.

Proposition 1.8.38 (Cartan–Eilenberg)**.** *A ring R is left hereditary if and only if quotients of injective R-modules are injective.*

We will deduce this later from general facts on homological dimensions. It is not hard to give a direct proof. See for example [CE, Theorem I.5.4] or [L1, Theorem 3.22].

Corollary 1.8.39. *Let R be a domain. If every divisible R-module is injective, then R is left hereditary.*

Example 1.8.40. Let $R = \mathbb{Z}[x]$ and let $K = \mathbb{Q}(x)$ be the fraction field of R. Then the R-module $M = K/R$ is divisible but not injective. Indeed, the homomorphism $A = 2R + xR \to M$ carrying 2 to 0 and x to the class of $1/2$ does not extend to a homomorphism $R \to M$.

Remark 1.8.41. Recall that a product of injective R-modules is injective. On the other hand, a direct sum of injective R-module is not necessarily injective. In fact, a ring R is left Noetherian if and only if direct sums of injective R-modules are injective. The "only if" part follows easily from Baer's test (Exercise)[9] The "if" part is a theorem of Bass and Papp (see [L1, Theorem 3.46] for a proof and other equivalent conditions). Thus, if R is a domain such that every divisible R-module is injective, then R is left Noetherian.

We refer the reader to [L1, Section 3C] for a more general discussion on the relation between injectivity and divisibility when R is not necessarily a domain.

[9] This part was known to Cartan and Eilenberg [CE, Exercise VII.8].

Enough injective modules

Proposition 1.8.42. *Any $\mathbb{Z}$-module (i.e. abelian group) M can be embedded into a divisible, and hence injective, $\mathbb{Z}$-module.*

Proof. We have $M = F/H$ with F free. Embedding F into a $\mathbb{Q}$-vector space V, we get $M \subseteq V/H$. Since V is divisible as a $\mathbb{Z}$-module, V/H is divisible. □

Remark 1.8.43. We will see later that every $\mathbb{Z}$-module can be embedded into a product of $\mathbb{Q}/\mathbb{Z}$ (such a product is sometimes called "cofree").

Remark 1.8.44. Let $R \to S$ be a ring homomorphism. The functor

$$\mathrm{Hom}_R(S, -) \colon R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$$

is a right adjoint to the restriction of scalars functor S-$\mathbf{Mod} \to R$-$\mathbf{Mod}$, which is exact. It follows that $\mathrm{Hom}_R(S, -)$ carries injective R-modules to injective S-modules.

Proposition 1.8.45. *Let R be a ring. Any R-module M can be embedded into an injective R-module.*

Proof. We embed the underlying $\mathbb{Z}$-module of M into an injective $\mathbb{Z}$-module I. Then $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module and we have injective homomorphisms

$$M \simeq \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I).$$

□

Self-injective rings

Definition 1.8.46. We say that a ring R is *left self-injective* if the left R-module R is injective.

Similarly one defines right self-injective rings. There are left self-injective rings that are not right self-injective [L1, Example 3.74B]. However we have the following theorem.

Theorem 1.8.47 (Faith–Walker)**.** *Let R be a ring. The following conditions are equivalent.*

(1) R is left Noetherian and left self-injective;

(2) Every projective R-module is injective;

(3) Every injective R-module is projective.

Moreover, R satisfies the above conditions if and only if R^{op} satisfies the above conditions.

By Remark 1.8.41, we have (1) $\Longrightarrow$ (2) and (2) + (3) $\Longrightarrow$ (1) (the latter follows from the theorem of Bass and Papp). The other parts of Theorem 1.8.47 are harder. We refer the reader to [F, Chapter 24] for a proof.

Definition 1.8.48. A ring R is called *quasi-Frobenius* if it satisfies Condition (1) of the above theorem.

Definition 1.8.49. Let k be a field. A *Frobenius k-algebra A* is a finite-dimensional k-algebra equipped with a k-linear form $\mathrm{tr}\colon A \to k$ such that the pairing

$$\begin{aligned} A \times A &\to k \\ (a, b) &\mapsto \mathrm{tr}(ab) \end{aligned}$$

is nondegenerate.

The nondegeneracy means that the homomorphism of A-modules

$$\begin{aligned} A &\to \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(A, k) \\ b &\mapsto (a \mapsto \mathrm{tr}(ab)) \end{aligned}$$

is an injection, and hence an isomorphism for dimension reasons. By Remark 1.8.44, $\mathrm{Hom}_{k\text{-}\mathbf{Mod}}(A, k)$ is an injective R-module. Thus A is a self-injective ring. Moreover, since $\dim_k(A)$ is finite, A is an Artinian ring. We have proved the following.

Proposition 1.8.50. *Every Frobenius k-algebra A is a quasi-Frobenius ring.*

Example 1.8.51. For any field k and any finite group G, the group algebra $k[G]$ is a Frobenius k-algebra, with trace map given by $\sum_{g \in G} a_g g \mapsto a_1$. In particular, $k[G]$ is a quasi-Frobenius ring. This fact is especially useful when the characteristic of k divides the order of G (otherwise $k[G]$ is a semisimple ring by Maschke's theorem).

Proposition 1.8.52. *Let R be a PLID and let $a \in R$ such that $Ra = aR$ and $a \neq 0$. Then R/aR is quasi-Frobenius. In particular, the quotient of any PID by a proper ideal is quasi-Frobenius.*

Proof. The ring R/Ra is clearly left Noetherian. We apply Baer's test to show that it is left self-injective. Let A be an ideal of R/Ra and let $h\colon A \to R/Ra$ be a homomorphism. Since R is a PLID, A has the form Rb/Ra. Set $h(\bar{b}) = \bar{r}$ for some $r \in R$, where $\bar{r}$ denotes the image of r in R/Ra. We have $Ra \subseteq Rb$, so that $a = cb$ for some nonzero $c \in R$. We have $\bar{0} = h(\bar{a}) = h(\bar{c}\bar{b}) = \bar{c}\bar{r}$. Since $Ra = aR$, it follows that $cr = as = cbs$ for some $s \in R$. Canceling c, we get $r = bs$, so $h(\bar{b}) = \bar{b}\bar{s}$. Thus h extends to the homomorphism $R/Ra \to R/Ra$ carrying $\bar{1}$ to $\bar{s}$. $\square$

Example 1.8.53. For $m \neq 0$, the ring $\mathbb{Z}/m\mathbb{Z}$ is quasi-Frobenius. For any field k and any nonzero $f \in k[x]$, the ring $k[x]/(f)$ is quasi-Frobenius.

1.9 Flat modules

Let R be a ring. For any left R-module L and right R-module M, the functors $L \otimes_R -$ and $- \otimes_R M$ are left adjoint functors, and hence commute with small colimits. In particular, they are right exact.

Definition 1.9.1. A left R-module M is said to be *flat* if the functor

$$- \otimes_R M \colon \mathbf{Mod}\text{-}R \to \mathbf{Ab}$$

is exact. A right R-module L is said to be *flat* if the functor $L \otimes_R -$ is exact.

Remark 1.9.2. A right R-module is flat if and only if the corresponding left R^{op}-module is flat.

Remark 1.9.3. Flat R-modules are stable under direct sums, direct summands, and filtered colimits. To see the stability under filtered colimits, let I be a filtered category and let $M\colon I \to R\text{-}\mathbf{Mod}$ be a functor such that $M(i)$ is flat for every object i of I. Since filtered colimits in $\mathbf{Ab}$ are exact, the functor $- \otimes \operatorname{colim}_{i\in I} M(i) \simeq \operatorname{colim}_{i\in I}(- \otimes M(i))$ is exact. In other words, $\operatorname{colim} M$ is flat.

Lemma 1.9.4. *Every projective R-module is flat.*

Proof. The R-module R is flat as $- \otimes_R R$ is the identity functor. Free R-modules are direct sums of copies of the R-module R, and hence are flat. Every projective R-module is a direct summand of a free R-module, and hence is flat. □

The converse does not hold. Indeed, projective R-modules are not stable under filtered colimits.

Example 1.9.5. $\mathbb{Q} = \operatorname{colim}_{n\in\mathbb{N}^\times} \frac{1}{n}\mathbb{Z}$, where $\mathbb{N}^\times$ denotes the set of positive integers, ordered by divisibility, is filtered colimit of free $\mathbb{Z}$-modules and hence a flat $\mathbb{Z}$-module. On the other hand, $\mathbb{Q}$ is not a free $\mathbb{Z}$-module, or, equivalently, not a projective $\mathbb{Z}$-module.

We will later show that flatness is equivalent to projectivity under a finiteness condition, which implies the following.

Theorem 1.9.6 (Lazard, Govorov)**.** *Every flat R-module is a filtered colimit of free R-modules.*

We refer the reader to [L1, Theorem 4.34] for a proof.

Torsion-free modules

Definition 1.9.7. Let R be a ring and let M be a left R-module. We say that M is *torsion-free* if $rm = 0$ for $r \in R$ and $m \in M$ implies that r is a left zero-divisor or $m = 0$. (In the case where R is a domain, the condition is that $rm = 0$ implies $r = 0$ or $m = 0$.)

Remark 1.9.8. Any submodule of a torsion-free R-module is torsion-free. Any product of torsion-free R-modules is torsion-free. It follows that any direct sum of torsion-free R-modules is torsion-free.

Proposition 1.9.9. *Any flat R-module M is torsion-free.*

Proof. It suffices to show that for any $r \in R$ which is not a left zero-divisor, the map $g\colon M \to M$ carrying m to rm is an injection. Consider the homomorphism of right R-modules $f\colon R \to R$ carrying x to rx, which is an injection. By the flatness of M, the map $f \otimes_R M\colon R \otimes_R M \to R \otimes_R M$, which can be identified with g, is an injection. □

Lemma 1.9.10. *Let R be a commutative domain and let $K = \mathrm{Frac}(R)$. For every torsion-free R-module M, the map $f\colon M \simeq M \otimes_R R \to M \otimes_R K$ induced by the inclusion $R \subseteq K$ is an injection.*

Proof. Let $S = R\backslash\{0\}$. Consider the module of fractions $S^{-1}M := (S \times M)/\sim$, where $\sim$ is the equivalence relation defined as follows: $(s,m) \sim (s',m')$ if and only if there exists $t \in S$ such that $t(sm' - s'm) = 0$. The equivalence class of (s,m) is denoted by m/s. The R-bilinear map $M \times K \to S^{-1}M$ sending $(m, a/b)$ to ma/b, where $a, b \in R$, induces a homomorphism $g\colon M \otimes_R K \to S^{-1}M$. The map gf is an injection: $m/1 = 0$ in $S^{-1}M$ if and only if $m = 0$. Thus f is an injection. □

Remark 1.9.11. The homomorphism $M \otimes_R K \to S^{-1}M$ in the above proof is in fact an isomorphism. See [AM, Proposition 3.5].

Lemma 1.9.12. *Let R be a commutative domain. Every finitely generated torsion-free R-module M can be embedded into R^n for some n.*

Proof. Let K be the fraction field of R. Then $M \simeq M \otimes_R R \hookrightarrow M \otimes_R K \simeq K^n$ for some n. Since M is finitely generated, there exists a common denominator $r \in R$ of the coordinates of the image of M in K^n. Then the image of M is contained in $r^{-1}R^n \simeq R^n$. □

In the case of commutative domains, the preceding lemma allows us to restate Corollary 1.8.10 (1) $\Longleftrightarrow$ (2) as follows.

Proposition 1.9.13. *A commutative domain R is a Prüfer domain if and only if every finitely generated torsion-free R-module is projective.*

Corollary 1.9.14. *Let R be a Prüfer domain (e.g. a Dedekind domain). Every torsion-free R-module M is flat.*

Proof. Since M is a filtered colimit of its finitely generated submodules, we may assume M finitely generated. In this case, M is projective by the proposition. □

Remark 1.9.15. Conversely, by a theorem of Chase, a ring R such that every torsion-free R-module is flat is left semi-hereditary. See [L1, Theorem 4.67] for details.

Example 1.9.16. The $\mathbb{Z}$-module $\mathbb{Z}^{\mathbb{N}}$ is torsion-free and hence flat.

For $n \geq 2$, the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not torsion-free or flat.

Flatness and change of rings

Lemma 1.9.17. *Let R and S be rings and let M be an (R,S)-bimodule.*

(1) If M is flat as a right S-module, then for any injective R-module I, $\mathrm{Hom}_R(M,I)$ is an injective S-module.
(2) If M is projective as a left R-module, then for any projective S-module P, $M \otimes_S P$ is a projective R-module.

Proof. The functor

$$\mathrm{Hom}_R(M, -)\colon R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$$

is a right adjoint to functor $M \otimes_S -$.

(1) By assumption, $M \otimes_S -$ is exact. It follows that $\mathrm{Hom}_R(M, -)$ carries injective R-modules to injective S-modules.

(2) By assumption, $\mathrm{Hom}_R(M, -)$ is exact. It follows that $M \otimes_S -$ carries projective S-modules to projective R-modules. (This also follows from the characterization of projective modules as direct summands of free modules.) ☐

Example 1.9.18. Let $R \to S$ be a ring homomorphism and take $M = {}_RS_S$. Then ${}_RS \otimes_S -$ can be identified with restriction of scalars.

(1) We recover Remark 1.8.44.
(2) If ${}_RS$ is projective, then any projective S-module P is a projective R-module by restriction of scalars.

Example 1.9.19. Let $S \to R$ be a ring homomorphism and take $M = {}_RR_S$. Then $\mathrm{Hom}_R(R_S, -)$ can be identified with restriction of scalars.

(1) If R_S is flat, then, any injective R-module I is an injective S-module by restriction of scalars.
(2) For any projective S-module P, $R \otimes_S P$ is a projective R-module.

For a summary of implications among the properties of rings and modules discussed in these lectures, see page 139.

Chapter 2

Derived categories and derived functors

Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. For any short exact sequence
$$0 \to X \to Y \to Z \to 0$$
in $\mathcal{A}$, we have, by the left exactness of F, an exact sequence
$$0 \to FX \to FY \to FZ$$
in $\mathcal{B}$. Under suitable conditions, we can define additive functors $R^nF\colon \mathcal{A} \to \mathcal{B}$, $i \geq 1$, called the *right derived functors* of F, such that the exact sequence in $\mathcal{B}$ extends to a long exact sequence
$$\begin{aligned}0 \to FX \to FY \to FZ \to R^1FX \to R^1FY \to R^1FZ \to \cdots \\ \to R^nFX \to R^nFY \to R^nFZ \to \cdots .\end{aligned}$$
Roughly speaking, the right derived functors measure the lack of right exactness of F. The functors can be assembled into one single functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ between *derived categories*.

2.1 Complexes

Complexes

Let $\mathcal{A}$ be an additive category.

Definition 2.1.1. A *(cochain) complex* in $\mathcal{A}$ consists of $X = (X^n, d_X^n)_{n\in\mathbb{Z}}$, where X^n is an object of $\mathcal{A}$, $d_X^n\colon X^n \to X^{n+1}$ is a morphism of $\mathcal{A}$ (called *differential*) such that for any n, $d_X^{n+1}d_X^n = 0$. The index n in X^n is called the *degree*. A *(cochain) morphism* of complexes $X \to Y$ is a collection of morphisms $(f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n\colon X^n \to Y^n$ in $\mathcal{A}$ such that $d_Y^n f^n = f^{n+1}d_X^n$. We let $C(\mathcal{A})$ denote the category of complexes in $\mathcal{A}$.

Remark 2.1.2. A *chain complex* in $\mathcal{A}$ consists of $X = (X_n, d_n)_{n\in\mathbb{Z}}$, where X_n is an object of $\mathcal{A}$, $d_n\colon X_n \to X_{n-1}$ is a morphism of $\mathcal{A}$ such that for any n, $d_{n-1}d_n = 0$. A chain complex can be regarded as a cochain complex via the formulas $X^{-n} = X_n$ and $d^{-n} = d_n$, and vice versa. Unless otherwise stated, complex always means cochain complex in these lectures.

Note that $C(\mathcal{A})$ is isomorphic to the full subcategory of $\mathcal{A}^{(\mathbb{Z},\le)}$ spanned by functors $X\colon (\mathbb{Z},\le) \to \mathcal{A}$ sending $n \to n+2$ to a zero morphism for every n. Furthermore, $C(\mathcal{A})$ is an additive category. We have $(X\oplus Y)^n = X^n \oplus Y^n$ and the zero complex 0 with $0^n = 0$ is a zero object of $C(\mathcal{A})$. If $\mathcal{A}$ is an abelian category, then $C(\mathcal{A})$ is an abelian category as well, with $\ker(f)^n = \ker(f^n)$ and $\operatorname{coker}(f)^n = \operatorname{coker}(f^n)$.

Definition 2.1.3. We say that a complex X is *bounded below* (resp. *bounded above*) if $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$). We say that X is *bounded* if it is bounded below and bounded above. For an interval $I \subseteq \mathbb{Z}$, we say that X is concentrated in degrees in I if $X^n = 0$ for $n \notin I$. We let $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^I(\mathcal{A})$ denote the full subcategories of $C(\mathcal{A})$ consisting of complexes bounded below, bounded above, bounded, concentrated in I, respectively. These are additive subcategories.

The functor $C^{[0,0]}(\mathcal{A}) \to \mathcal{A}$ carrying X to X^0 is an equivalence of categories. A quasi-inverse is denoted by $A \mapsto A[0]$, or simply $A \mapsto A$. We will often use this equivalence to identify $\mathcal{A}$ with $C^{[0,0]}(\mathcal{A})$ and regard an object A of $\mathcal{A}$ as a complex concentrated in degree 0.

Remark 2.1.4. The inclusion functor $C^{\le n}(\mathcal{A}) \subseteq C(\mathcal{A})$ admits a left adjoint

$$\sigma^{\le n}\colon C(\mathcal{A}) \to C^{\le n}(\mathcal{A})$$

with $(\sigma^{\le n}X)^m = X^m$ for $m \le n$ and $(\sigma^{\le n}X)^m = 0$ for $m > n$. Similarly, the inclusion functor $C^{\ge n}(\mathcal{A}) \subseteq C(\mathcal{A})$ admits a right adjoint

$$\sigma^{\ge n}\colon C(\mathcal{A}) \to C^{\ge n}(\mathcal{A})$$

with $(\sigma^{\ge n}X)^m = X^m$ for $m \ge n$ and $(\sigma^{\ge n}X)^m = 0$ for $m < n$. These functors are called *naive truncation* functors, as opposed to the truncation functors introduced later. If $\mathcal{A}$ is an abelian category, the naive truncation functors are exact.

Definition 2.1.5. Let X be a complex and let k be an integer. We define a complex $X[k]$ by $X[k]^n = X^{n+k}$ and $d^n_{X[k]} = (-1)^k d^{n+k}_X$. For a morphism of complexes $f\colon X \to Y$, we define $f[k]\colon X[k] \to Y[k]$ by $f[k]^n = f^{n+k}$. The functor $[k]\colon C(\mathcal{A}) \to C(\mathcal{A})$ is called the *translation* (or shift) functor of degree k.

The sign in the definition of $X[k]$ will be explained later, after the definition of mapping cone (Definition 2.1.20). Note that if X is concentrated in degrees $[a,b]$, then $X[k]$ is concentrated in degrees $[a-k, b-k]$.

Remark 2.1.6. We define an isomorphism of categories $F\colon C(\mathcal{A})^{\mathrm{op}} \to C(\mathcal{A}^{\mathrm{op}})$ as follows. For X in $C(\mathcal{A})$, we define $Y = FX$ in $C(\mathcal{A}^{\mathrm{op}})$ by $Y^n = X^{-n}$, $d^n_Y = (-1)^n d_X^{-n-1}$. We have a natural isomorphism $F(X[1]) \simeq (FX)[-1]$ with $F(X[1])^n \simeq (FX)[-1]^n$ given by $(-1)^{n-1}\mathrm{id}_{X^{1-n}}$.

There are several other constructions for complexes in an additive category. We will return to them later. Now we proceed to define cohomology.

Cohomology

Let $\mathcal{A}$ be an abelian category. Consider a sequence

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

with $gf = 0$. We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{im}(f) \arrow[rr, hook, "\phi"] \arrow[dr, hook] & & \ker(g) \arrow[dl, hook] & \\
X' \arrow[ur, two heads] \arrow[rr, "f"] & & X \arrow[rr, "g"] \arrow[dl, two heads] \arrow[dr, two heads] & & X'' \\
 & \mathrm{coker}(f) \arrow[rr, two heads, "\psi"] & & \mathrm{im}(g) \arrow[ur, hook] &
\end{tikzcd}$$

We have isomorphisms

$$\mathrm{im}(\ker(g) \to \mathrm{coker}(f)) \simeq \mathrm{coker}(\phi) \simeq \ker(\psi).$$

These objects measure the failure of the exactness at X.

Definition 2.1.7. Let X be a complex in $\mathcal{A}$. We define

$$\begin{aligned}
Z^n X &= \ker(d_X^n \colon X^n \to X^{n+1}), \\
B^n X &= \mathrm{im}(d_X^{n-1} \colon X^{n-1} \to X^n), \\
H^n X &= \mathrm{coker}(B^n X \hookrightarrow Z^n X),
\end{aligned}$$

and call them the *cocycle*, *coboundary*, *cohomology* objects, of degree n.

The letter Z stands for German *Zyklus*, which means cycle. We get additive functors

$$Z^n, B^n, H^n \colon C(\mathcal{A}) \to \mathcal{A},$$

with Z^n left exact. Note that $H^n(X) = 0$ if and only if X is exact at X^n.

Example 2.1.8. Let M be a smooth manifold of dimension n. The de Rham complex $\Omega^\bullet(M)$ of M is a complex of $\mathbb{R}$-vector spaces:

$$\cdots \to 0 \to \Omega^0(M) \to \cdots \to \Omega^n(M) \to 0 \cdots,$$

where $\Omega^i(M)$ denotes the space of smooth differential i-forms on M. The i-th de Rham cohomology of M, $H^i_{\mathrm{dR}}(M)$, is by definition the cohomology of $\Omega^\bullet(M)$ of degree i.

Definition 2.1.9. A complex X is said to be *acyclic* if $H^n X = 0$ for all n. A morphism of complexes $X \to Y$ is called a *quasi-isomorphism* if $H^n f \colon H^n X \to H^n Y$ is an isomorphism for all n.

Later we will define the derived category $D(\mathcal{A})$ of $\mathcal{A}$. Roughly speaking, $D(\mathcal{A})$ is $C(\mathcal{A})$ modulo quasi-isomorphisms.

We have $H^n(X[k]) \simeq H^{n+k}X$.

Note that the morphisms $H^nX \to H^n\sigma^{\leq n}X$, $H^n\sigma^{\geq n}X \to H^nX$ are not isomorphisms in general. Moreover, if $f\colon X \to Y$ is a quasi-isomorphism, $\sigma^{\leq n}f\colon \sigma^{\leq n}X \to \sigma^{\leq n}Y$ and $\sigma^{\geq n}f\colon \sigma^{\geq n}X \to \sigma^{\geq n}Y$ are not quasi-isomorphisms in general. To remedy this problem, we introduce the following truncation functors.

Definition 2.1.10. Let X be a complex. We define

$$\tau^{\leq n}X = (\cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} Z^nX \to 0 \to \cdots),$$
$$\tau^{\geq n}X = (\cdots \to 0 \to X^n/B^nX \xrightarrow{d_X^n} X^{n+1} \to \cdots).$$

Here X^n/B^nX denotes $\operatorname{coker}(d_X^{n-1})$.

We obtain functors
$$\tau^{\leq n}, \tau^{\geq n}\colon C(\mathcal{A}) \to C(\mathcal{A}),$$
with $\tau^{\leq n}$ left exact and $\tau^{\geq n}$ right exact.

Remark 2.1.11. The morphism $\tau^{\leq n}X \to X$ induces an isomorphism $H^m\tau^{\leq n}X \to H^mX$ for $m \leq n$ and $H^m\tau^{\leq n}X = 0$ for $m > n$. The morphism $X \to \tau^{\geq n}X$ induces an isomorphism $H^mX \to H^m\tau^{\geq n}X$ for $m \geq n$ and $H^m\tau^{\geq n}X = 0$ for $m < n$. The functors $\tau^{\leq n}$ and $\tau^{\geq n}$ preserve quasi-isomorphisms.

Remark 2.1.12. For $a \leq b$, we have $\tau^{\leq a}\tau^{\geq b}X \simeq \tau^{\geq b}\tau^{\leq a}X$ and we write $\tau^{[a,b]}X$ for either of them. We have $\tau^{[n,n]}X \simeq (H^nX)[-n]$.

The functor H^n is neither left exact nor right exact in general. However, it has the following important property.

Proposition 2.1.13. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of complexes. Then we have a* long exact sequence

$$\cdots \to H^nL \xrightarrow{H^nf} H^nM \xrightarrow{H^ng} H^nN \xrightarrow{\delta} H^{n+1}L \xrightarrow{H^{n+1}f} H^{n+1}M \xrightarrow{H^{n+1}g} H^{n+1}N \to \cdots,$$

which is functorial with respect to the short exact sequence.

This generalizes the case of the snake lemma where the exact rows are short exact. The morphism δ is called the *connecting* morphism.

Proof. The sequence $\tau^{[n,n+1]}L \to \tau^{[n,n+1]}M \to \tau^{[n,n+1]}N$ provides a commutative diagram

$$\begin{tikzcd}
 & L^n/B^nL \arrow[r] \arrow[d] & M^n/B^nM \arrow[r] \arrow[d] & N^n/B^nN \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}L \arrow[r] & Z^{n+1}M \arrow[r] & Z^{n+1}N &
\end{tikzcd}$$

with exact rows. Applying the snake lemma, we obtain the desired exact sequence. □

Corollary 2.1.14. *Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "u"'] & M \arrow[r] \arrow[d, "v"] & N \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of complexes with exact rows. If two of the three morphisms u, v, w are quasi-isomorphisms, then the third one is a quasi-isomorphism too.

Proof. We prove the case where u and w are quasi-isomorphisms. The other cases are similar. By the proposition, we have a commutative diagram

$$\begin{tikzcd} H^{n-1}N \arrow[r] \arrow[d, "H^{n-1}w"'] & H^nL \arrow[r] \arrow[d, "H^nu"] & H^nM \arrow[r] \arrow[d, "H^nv"] & H^nN \arrow[r] \arrow[d, "H^nw"] & H^{n+1}L \arrow[d, "H^{n+1}u"] \\ H^{n-1}N' \arrow[r] & H^nL' \arrow[r] & H^nM' \arrow[r] & H^nN' \arrow[r] & H^{n+1}L' \end{tikzcd}$$

with exact rows. By assumption, $H^{n-1}w$, H^nw, H^nu, $H^{n+1}u$ are isomorphisms. It follows by the five lemma that H^nv is an isomorphism. □

For any morphism of complexes $L \to M$, not necessarily monomorphic, the induced morphisms $H^nL \to H^nM$ also extends naturally to a long exact sequence, by Proposition 2.1.23 below.

Resolutions

Let M be an R-module. Choosing a set of generators, we get an exact sequence $F^0 \xrightarrow{f} M \to 0$, where F^0 is a free R-module. Choosing a generating set of relations among the generators, namely a set of generators for $\ker(f)$, we get an exact sequence $F^{-1} \to F^0 \to M \to 0$, where F^{-1} is a free R-module. Further choosing relations among the relations, we get, by induction, an exact sequence

$$\cdots \to F^{-n} \to \cdots \to F^0 \to M \to 0,$$

where each F^i is a free R-module. Such an exact sequence induces a quasi-isomorphism $F^\bullet \to M$, where $F^\bullet \in C^{\leq 0}(R\text{-}\mathbf{Mod})$, and is called a *free resolution* of M.

Definition 2.1.15. Let X be an object of $\mathcal{A}$. A *left resolution* of X is an exact sequence

$$\cdots \to P^{-n} \to \cdots \to P^0 \to X \to 0$$

in $\mathcal{A}$, or equivalently a quasi-isomorphism $P^\bullet \to X$ with $P^\bullet \in C^{\leq 0}(\mathcal{A})$. It is called a *projective resolution* if each P^i is projective. Dually, a *right resolution* of X is an exact sequence

$$0 \to X \to I^0 \to \cdots \to I^n \to \cdots$$

in $\mathcal{A}$, or equivalently a quasi-isomorphism $X \to I^\bullet$ such that $I^\bullet \in C^{\geq 0}(\mathcal{A})$. It is called an *injective resolution* if each I^i is injective.

Definition 2.1.16. We say that $\mathcal{A}$ admits *enough injectives* if for every object X of $\mathcal{A}$, there exists a monomorphism $X \to I$ with I injective. We say that $\mathcal{A}$ admits *enough projectives* if for every object X of $\mathcal{A}$, there exists an epimorphism $P \to X$ with P projective.

If $\mathcal{A}$ admits enough injectives, then every object of X admits an injective resolution. If $\mathcal{A}$ admits enough projectives, then every object of X admits a projective resolution.

Example 2.1.17. Let R be a small ring. The abelian category R-**Mod** admits enough injectives and enough projectives.

Example 2.1.18. Let X be a small topological space. The category $\mathrm{Shv}(X)$ is an abelian category with enough injectives, but not enough projectives in general.

To show that $\mathrm{Shv}(X)$ admits enough injectives, consider, for every point $x \in X$, the stalk functor $i_x^*\colon \mathrm{Shv}(X) \to \mathbf{Ab}$ defined by $i_x^*\mathcal{F} = \mathrm{colim}_{U\in \mathrm{Nbhd}(x)^{\mathrm{op}}} \mathcal{F}(U)$, where $\mathrm{Nbhd}(x)$ is the partially ordered set of open neighborhoods of x. This functor admits a right adjoint $i_{x*}\colon \mathbf{Ab} \to \mathrm{Shv}(X)$ defined by $(i_{x*}A)(U) = A$ if $x \in U$ and $(i_{x*}A)(U) = 0$ if $x \notin U$. Since $\mathrm{Nbhd}(x)^{\mathrm{op}}$ is filtered, the functor i_x^* is exact. Moreover, i_{x*} is clearly exact. Let $\mathcal{F}$ be a sheaf of abelian groups on X. We choose $i_x^*\mathcal{F} \hookrightarrow I_x$ with I_x injective for each $x \in X$. We have

$$\mathcal{F} \hookrightarrow \prod_x i_{x*}i_x^*\mathcal{F} \hookrightarrow \prod_x i_{x*}I_x = I.$$

Since i_x^* is exact, the right adjoint i_{x*} preserves injectives, so that I is injective.

One can show that if X is a locally connected topological space, then $\mathrm{Shv}(X)$ admits enough projectives if and only if X is an Alexandrov space (namely, if any (infinite) intersection of open subsets is open).

A glimpse of derived functors

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. We define R^iFX by $H^iFI^\bullet$, where $X \to I^\bullet$ is an injective resolution. One can check that this is a well-defined functor. We have $R^0FX \simeq FX$. For any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, applying Proposition 2.1.13 to the short exact sequence of complexes given by the horseshoe lemma below, we obtain a long exact sequence

$$0 \to FX \to FY \to FZ \to R^1FX \to R^1FY \to R^1FZ \to \cdots.$$

Lemma 2.1.19 (Horseshoe lemma)**.** *Let $\mathcal{A}$ be a category with enough injectives. Any diagram*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] & X'' \arrow[r] \arrow[d] & 0 \\ & I'^\bullet & & I''^\bullet & \end{tikzcd}$$

with a short exact row in $\mathcal{A}$ such that the columns are injective resolutions in $\mathcal{A}$ can be completed into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I'^{\bullet} \arrow[r, "i^{\bullet}"] & I^{\bullet} \arrow[r, "p^{\bullet}"] & I''^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

with exact rows such that the columns are injective resolutions, $I^n = I'^n \oplus I''^n$ and i^n and p^n are given by the canonical morphisms for all $n \geq 0$.

As we will later prove the long exact sequence without using the lemma, we leave the proof of the lemma as an exercise.

Dually, let $G\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. We define L_iFX by $H^{-i}GP^{\bullet}$, where $P^{\bullet} \to X$ is a projective resolution. In particular, $L_0FX \simeq FX$. For any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, we have a long exact sequence

$$\cdots \to L_1GX \to L_1GY \to L_1GZ \to GX \to GY \to GZ \to 0.$$

Mapping cones

Let $\mathcal{A}$ be an additive category.

Definition 2.1.20. Let $f\colon X \to Y$ be a morphism of complexes in $\mathcal{A}$. We define the *mapping cone* of f to be the complex $\mathrm{Cone}(f)^n = X[1]^n \oplus Y^n = X^{n+1} \oplus Y^n$ with differential

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f[1]^n & d^n_Y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}.$$

Intuitively, for $\begin{pmatrix} x \\ y \end{pmatrix} \in X^{n+1} \oplus Y^n$, $d^n_{\mathrm{Cone}(f)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X x \\ f^{n+1}x + d^n_Y y \end{pmatrix}$.

Note that the sign in the definition of the differential of $X[1]$ makes $\mathrm{Cone}(f)$ a complex:

$$d^n_{\mathrm{Cone}(f)} d^{n-1}_{\mathrm{Cone}(f)} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \begin{pmatrix} -d^n_X & 0 \\ f^n & d^{n-1}_Y \end{pmatrix} = \begin{pmatrix} d^{n+1}_X d^n_X & 0 \\ d^n_Y f^n - f^{n+1} d^n_X & d^n_Y d^{n-1}_Y \end{pmatrix} = 0.$$

Example 2.1.21. If X and Y are concentrated in degree 0, then $\mathrm{Cone}(f)$ can be identified with the complex $X^0 \xrightarrow{f^0} Y^0$ concentrated on degrees -1 and 0.

Remark 2.1.22. Let X and Y be CW complexes and let $f\colon X \to Y$ be a cellular map. The (topological) mapping cone $\mathrm{Cone}(f)$ of f is obtained by gluing the base of the cone $\mathrm{Cone}(X)$ to Y via f. If we let c denote the cone point, then $\mathrm{Cone}(C_{\bullet}(f))$ can be identified with $C_{\bullet}(\mathrm{Cone}(f))/C_{\bullet}(c)$. Here $C_{\bullet}(f)$ denotes the cellular chain complex.[1] As in Remark 2.1.2, a chain complex $(X_{\bullet}, d_{\bullet})$ is regarded as a cochain complex by $X^{-n} = X_n$, $d^{-n} = d_n$.

[1]We invite readers unfamiliar with CW complexes to replace “CW” and “cellular” by “simplicial”.

Assume that $\mathcal{A}$ is an abelian category. We have a short exact sequence of complexes
$$0 \to Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0,$$
where $i\colon Y \to \mathrm{Cone}(f)$ is the inclusion and $p\colon \mathrm{Cone}(f) \to X[1]$ is the projection. The short exact sequence induces a long exact sequence
$$\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta} H^nY \xrightarrow{H^ni} H^n(\mathrm{Cone}(f)) \xrightarrow{H^np} H^n(X[1]) \to \cdots.$$

Proposition 2.1.23. *Via the isomorphism $H^{n-1}(X[1]) \simeq H^nX$, the connecting morphism can be identified with H^nf.*

The long exact sequence thus has the form
$$\cdots \to H^nX \xrightarrow{H^nf} H^nY \xrightarrow{H^ni} H^n(\mathrm{Cone}(f)) \xrightarrow{H^np} H^{n+1}X \to \cdots.$$

Proof. The connecting morphism is constructed using the snake lemma applied to the commutative diagram
$$\begin{tikzcd}
& Y^{n-1}/B^{n-1}Y \arrow[r] \arrow[d] & C^{n-1}/B^{n-1}C \arrow[r] \arrow[d] & X^n/B^nX \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^nY \arrow[r] & Z^nC \arrow[r] & Z^{n+1}X, &
\end{tikzcd}$$
where $C = \mathrm{Cone}(f)$. We reduce by the Freyd-Mitchell Theorem to the case of modules. Let $x \in Z^nX$. Then $\begin{pmatrix} x \\ 0 \end{pmatrix} + B^{n-1}C$ is a lifting of $x + B^nX$. We conclude by $d_C^{n-1}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f^n(x) \end{pmatrix}$. □

Proposition 2.1.24. *A morphism of complexes $f\colon X \to Y$ is a quasi-isomorphism if and only if its cone* $\mathrm{Cone}(f)$ *is acyclic.*

Proof. Indeed, by the long exact sequence, H^nf is an isomorphism for all n if and only if $H^n(\mathrm{Cone}(f)) = 0$ for all n. □

Proposition 2.1.25. *Consider a short exact sequence of complexes $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. Then the map $\phi\colon \mathrm{Cone}(f) \to Z$ defined by $\phi^n = (0, g^n)$ is a quasi-isomorphism.*

Proof. We have a short exact sequence
$$0 \to \mathrm{Cone}(\mathrm{id}_X) \xrightarrow{\psi} \mathrm{Cone}(f) \xrightarrow{\phi} Z \to 0,$$
where ψ is associated to the commutative square
$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$
Since $\mathrm{Cone}(\mathrm{id}_X)$ is acyclic (in fact homotopy equivalent to zero), the long exact sequence implies that ϕ is a quasi-isomorphism. □

Remark 2.1.26. We have a commutative diagram of long exact sequences

$$\begin{tikzcd} H^nX \arrow[r, "H^nf"] \arrow[d, equal] & H^nY \arrow[r, "H^ni"] \arrow[d, equal] & H^n(\mathrm{Cone}(f)) \arrow[r, "H^np"] \arrow[d, "H^n\phi"] \arrow[dr, phantom, "(*)"] & H^{n+1}X \arrow[r, "H^{n+1}f"] \arrow[d, equal] & H^{n+1}Y \arrow[d, equal] \\ H^nX \arrow[r, "H^nf"] & H^nY \arrow[r, "H^ng"] & H^nZ \arrow[r, "-\delta"] & H^{n+1}X \arrow[r, "H^{n+1}f"] & H^{n+1}Y. \end{tikzcd}$$

Indeed, for the commutativity of the square $(*)$ we reduce by the Freyd-Mitchell Theorem to the case of modules, and it suffices to note that for $\begin{pmatrix} x \\ y \end{pmatrix} \in Z^n\mathrm{Cone}(f)$, we have $f^n(x)+d^ny=0$. By the five lemma, this gives another proof of Proposition 2.1.25.

2.2 Homotopy category of complexes

Let $\mathcal{A}$ be an additive category. Let X and Y be complexes in $\mathcal{A}$. We let

$$\mathrm{Ht}(X,Y)=\prod_n \mathrm{Hom}_{\mathcal{A}}(X^n,Y^{n-1})$$

denote the abelian group of families of morphisms $h=(h^n\colon X^n\to Y^{n-1})_{n\in\mathbb{Z}}$. Given h, consider $f^n=d_Y^{n-1}h^n+h^{n+1}d_X^n\colon X^n\to Y^n$. We have

$$d_Y^nf^n=d_Y^{n-1}d_Y^nh^n+d_Y^nh^{n+1}d_X^n=d_Y^nh^{n+1}d_X^n=d_Y^nh^{n+1}d_X^n+h^{n+2}d_X^{n+1}d_X^n=f^{n+1}d_X^n.$$

Thus we get a morphism of complexes $f\colon X\to Y$. We get a homomorphism of abelian groups

$$\mathrm{Ht}(X,Y)\to\mathrm{Hom}_{C(\mathcal{A})}(X,Y). \tag{2.2.1}$$

Definition 2.2.1. We say that a morphism of complexes $f\colon X\to Y$ is *null-homotopic* if there exists $h\in\mathrm{Ht}(X,Y)$ such that $f^n=d_Y^{n-1}h^n+h^{n+1}d_X^n$. We say that two morphisms of complexes $f,g\colon X\to Y$ are *homotopic* if $f-g$ is null-homotopic.

Lemma 2.2.2. *Let $f\colon X\to Y$, $g\colon Y\to Z$ be morphisms of complexes in $\mathcal{A}$. If f or g is null-homotopic, then gf is null-homotopic.*

Proof. If $f=dh+hd$ for $h\in\mathrm{Ht}(X,Y)$, then $gf=gdh+ghd=d(gh)+(gh)d$, where $gh\in\mathrm{Ht}(X,Z)$. The other case is similar. □

Definition 2.2.3. We define the *homotopy category of complexes in* $\mathcal{A}$, $K(\mathcal{A})$, as follows. The objects of $K(\mathcal{A})$ are objects of $C(\mathcal{A})$, that is, complexes in $\mathcal{A}$. For complexes X and Y, we put

$$\mathrm{Hom}_{K(\mathcal{A})}(X,Y)=\mathrm{coker}(\mathrm{Ht}(X,Y)\xrightarrow{(2.2.1)}\mathrm{Hom}_{C(\mathcal{A})}(X,Y)).$$

In other words, morphisms in $K(\mathcal{A})$ are homotopy classes of morphisms of complexes.

An isomorphism in $K(\mathcal{A})$ is called a *homotopy equivalence*. By definition, a morphism of complexes $f\colon X \to Y$ is a homotopy equivalence if there exists a morphism of complexes $g\colon Y \to X$ such that fg is homotopic to id_Y and gf is homotopic to id_X. We say that two complexes are *homotopy equivalent* if they are isomorphic in $K(\mathcal{A})$. A complex X is homotopy equivalent to zero if and only if id_X is null-homotopic.

Remark 2.2.4. The category $K(\mathcal{A})$ is an additive category and the functor $C(\mathcal{A}) \to K(\mathcal{A})$ carrying a complex to itself and a morphism of complexes to its homotopy class is an additive functor.

Notation 2.2.5. Let $I \subseteq \mathbb{Z}$ be an interval. We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$, $K^I(\mathcal{A})$ denote the full subcategories of $K(\mathcal{A})$ consisting of complexes in $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^I(\mathcal{A})$, respectively. These are additive subcategories. The functor $C^{[0,0]}(\mathcal{A}) \to K^{[0,0]}(\mathcal{A})$ is an isomorphism of categories. We thus obtain an equivalence of categories between $\mathcal{A}$ and $K^{[0,0]}(\mathcal{A})$.

Remark 2.2.6. Let $\mathcal{A}$ be an abelian category. Note that $dh + hd$ induces zero morphisms on cohomology. Thus if $f, g\colon X \to Y$ are homotopic, then $H^n f = H^n g\colon H^n X \to H^n Y$. The additive functor $H^n\colon C(\mathcal{A}) \to \mathcal{A}$ factorizes through an additive functor

$$H^n\colon K(\mathcal{A}) \to \mathcal{A}.$$

In particular, a homotopy equivalence is a quasi-isomorphism. The converse does *not* hold in general. Indeed, an acyclic complex is *not* homotopic to zero in general, as shown by the following lemma.

Similarly, the additive functors $\tau^{\leq n}, \tau^{\geq n}\colon C(\mathcal{A}) \to C(\mathcal{A})$ induce additive functors

$$\tau^{\leq n}, \tau^{\geq n}\colon K(\mathcal{A}) \to K(\mathcal{A}).$$

Lemma 2.2.7. *Let $\mathcal{A}$ be an abelian category. Then a complex X in $\mathcal{A}$ is homotopy equivalent to zero if and only if X is acyclic and the short exact sequences*

$$0 \to Z^n X \to X^n \to Z^{n+1} X \to 0$$

are split.

Thus a complex in $\mathcal{A}$ is homotopy equivalent to zero if and only if it is isomorphic to $(Z^n \oplus Z^{n+1})$, with $d^n\colon Z^n \oplus Z^{n+1} \to Z^{n+1} \to Z^{n+1} \oplus Z^{n+2}$. This holds in fact more generally for idempotent-complete[2] additive categories.

Proof. If the sequences are split short exact sequences, so that X^n can be identified with $Z^n \oplus Z^{n+1}$, then $h^n\colon Z^n \oplus Z^{n+1} \to Z^n \to Z^{n-1} \oplus Z^n$ satisfies $hd + dh = \mathrm{id}_X$. Conversely, if $hd + dh = \mathrm{id}_X$, then h^{n+1} restricted to $Z^{n+1}X$ provides a splitting of the short exact sequence. □

[2]In a category $\mathcal{C}$, a morphism $e\colon X \to X$ such that $e^2 = e$ is called an *idempotent*. A typical example for $\mathcal{C}$ additive is the composition $e = gf\colon A \oplus B \xrightarrow{f} A \xrightarrow{g} A \oplus B$. We have $fg = \mathrm{id}_A$. We say that an idempotent $e\colon X \to X$ *splits* if there exist morphisms $f\colon X \to Y$, $g\colon Y \to X$ such that $e = gf$ with $fg = \mathrm{id}_Y$. We say that $\mathcal{C}$ is *idempotent-complete* if every idempotent in $\mathcal{C}$ splits. Every abelian category is idempotent-complete.

Remark 2.2.8. The homotopy category brings us one step closer to the derived category of an abelian category $\mathcal{A}$. We will see that under the condition that $\mathcal{A}$ admits enough injectives, $D^+(\mathcal{A})$ is equivalent to $K^+(\mathcal{I})$, where $\mathcal{I}$ is the full subcategory of $\mathcal{A}$ spanned by injective objects.

Remark 2.2.9. Even for $\mathcal{A}$ abelian, $K(\mathcal{A})$ is not an abelian category in general. In fact, one can show that for $\mathcal{A}$ abelian, $K(\mathcal{A})$ admits kernels if and only if every short exact sequence in $\mathcal{A}$ splits (in this case $K(\mathcal{A})$ is equivalent to $\prod_{\mathbb{Z}} \mathcal{A}$ and is an abelian category) [V2, Propositions II.1.2.9, II.1.3.6].

2.3 Triangulated categories

Given a category $\mathcal{D}$ equipped with a functor $X \mapsto X[1]$, diagrams of the form $X \to Y \to Z \to X[1]$ are called *triangles*. It is sometimes useful to visualize such diagrams as

$$\begin{tikzcd} & Z \arrow[dl, "+1"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

Such a morphism is an isomorphism if and only if f, g, h are isomorphisms.

Definition 2.3.1 (Verdier)**.** A *triangulated category* consists of the following data:

(1) An additive category $\mathcal{D}$.
(2) A *translation functor* $\mathcal{D} \to \mathcal{D}$ which is an equivalence of categories. We denote the functor by $X \mapsto X[1]$.
(3) A collection of *distinguished triangles* $X \to Y \to Z \to X[1]$.

These data are subject to the following axioms:

(TR1)
(a) The collection of distinguished triangles is stable under isomorphism.
(b) For every object X of $\mathcal{D}$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle.
(c) Every morphism $f\colon X \to Y$ in $\mathcal{D}$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

(T2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then the (clockwise) rotated diagram $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(TR4) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} Y[1],$$
$$X \xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} X[1],$$

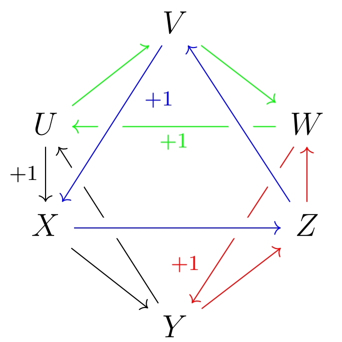
with $h = gf$, there exists a distinguished triangle

$$U \xrightarrow{i} V \xrightarrow{i'} W \xrightarrow{i''} U[1]$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"] & & Z \arrow[rr, "g'"] \arrow[dr, "h'"] & & W \arrow[rr, "i''"] \arrow[dr, "g''"] & & U[1] \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"] & & V \arrow[ur, "i'"] \arrow[dr, "h''"] & & Y[1] \arrow[ur, "f'[1]"] & \\
& & U \arrow[ur, "i"] \arrow[rr, "f''"] & & X[1] \arrow[ur, "f[1]"] & &
\end{tikzcd}$$

This notion was introduced by Verdier in his 1963 notes [V1] and 1967 thesis of *doctorat d'État* [V2]. Some authors call the translation functor the suspension functor and denote it by Σ. (TR4) is sometimes known as the octahedron axiom, as the four distinguished triangles and the four commutative triangles can be visualized as the faces of an octahedron, as shown below:



Remark 2.3.2. The octahedron axiom can be compared with the following form of the Third Isomorphism Theorem: Given three short exact sequences in an abelian category $\mathcal{A}$

$$0 \to X \xrightarrow{f} Y \xrightarrow{f'} Y/X \to 0$$

$$0 \to Y \xrightarrow{g} Z \xrightarrow{g'} Z/Y \to 0$$

$$0 \to X \xrightarrow{h} Z \xrightarrow{h'} Z/X \to 0$$

with $h = gf$, there exists a unique short exact sequence

$$0 \to Y/X \xrightarrow{i} Z/X \xrightarrow{i'} Z/Y \to 0$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"] & & Z \arrow[rr, "g'"] \arrow[dr, "h'"] & & Z/Y \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"] & & Z/X \arrow[ur, "i'"] & \\
& & Y/X \arrow[ur, "i"] & &
\end{tikzcd}$$

Remark 2.3.3. The original definition included an axiom (TR3) and the following stronger form of (T2) instead of (T2):

(TR2) A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if the (clockwise) rotated diagram $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

May [M2, Section 2] observed that (TR3) can be deduced from (TR1) and (TR4), as we shall see in Proposition 2.3.9. He also observed that (TR2) follows from (TR1), (T2), and (TR3), as we shall see in Remark 2.3.15.

In a category equipped with a translation functor, we write $[n]$ for $[1]$ composed n times for $n \geq 0$. We often fix a quasi-inverse $[-1]$ of $[1]$ and we write $[-n]$ for $[-1]$ composed n times.

Remark 2.3.4. (TR2) is equivalent to (T2) and the following dual of (T2):

(T2′) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then the counterclockwise rotated triangle $Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$ is distinguished.

Rotating a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ thrice, we get $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$. Thus (TR2) is equivalent to (T2) and the following condition: if $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$ is a distinguished triangle, then the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished.

Remark 2.3.5. We have an isomorphism of triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "-\mathrm{id}_Y"] & Z \arrow[r, "h"] \arrow[d, equal] & X[1] \arrow[d, equal] \\ X \arrow[r, "-f"] & Y \arrow[r, "-g"] & Z \arrow[r, "h"] & X[1]. \end{tikzcd}$$

Similarly, modifying exactly two signs in a triangle T produces a triangle isomorphic to T.

Theorem 2.3.6. *Let $\mathcal{A}$ be an additive category. We equip $K(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.5. We say that a triangle in $K(\mathcal{A})$ is* distinguished *if it is isomorphic to a standard triangle, namely a triangle of the form $X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1]$, where f is a morphism of complexes and i and p are the canonical morphisms. Then $K(\mathcal{A})$ is a triangulated category.*

The proof will be given later in this section.

Example 2.3.7. The stable homotopy category of spectra hSp equipped with the suspension functor Σ and the collection of (triangles isomorphic to) mapping cone triangles is a triangulated category.

Example 2.3.8. The definition of the ∞-categorical analogue of triangulated category is much simpler [L3, Section 1.1]: An ∞-category $\mathcal{C}$ is said to be *stable* if it satisfies the following conditions:

(1) $\mathcal{C}$ admits a zero object, pullbacks, and pushouts;

(2) A square in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a pullback if and only if it is a pushout.

The homotopy category of a stable ∞-category is a triangulated category, with distinguished triangles given by pullback/pushout squares as above. The triangulated categories $K(\mathcal{A})$ and hSp are both homotopy categories of stable ∞-categories.

Proposition 2.3.9. *(TR1) and (TR4) imply the following property.*

(TR3) Given a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "j"] \arrow[d, "g"] & Z \arrow[r, "k"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "j'"] & Z' \arrow[r, "k'"] & X'[1] \end{tikzcd}$$

in which both rows are distinguished triangles and the square on the left is commutative, there exists a dotted arrow rendering the entire diagram commutative.

Remark 2.3.10. The dotted arrow is not unique in general. This is the source of many troubles.

It seems that there is no known example of a category (equipped with a translation functor and a collection of distinguished triangles) satisfying (TR1), (TR2), and (TR3), but not (TR4) [N, Remark 1.3.15].

Proof of Proposition 2.3.9. By (TR1c), we may extend $gi = i'f$ to a distinguished triangle

$$X \xrightarrow{gi} Y' \xrightarrow{j''} Z'' \xrightarrow{k''} X[1].$$

Applying (TR1c) to g and (TR4) to the distinguished triangles with bases g, i, and gi, we get a morphism $Z \xrightarrow{h'} Z''$ such that $h'j = j''g$ and $k = k''h'$. Similarly, applying (TR1c) to f and (TR4) to the distinguished triangles with bases f, i', and gi, we get $Z'' \xrightarrow{h''} Z'$ such that $j' = h''j''$ and $f[1]k'' = k'h''$. It suffices to take $h = h''h'$. ☐

Corollary 2.3.11. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle. Then $gf = 0$.*

Proof. By (TR1b), $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle. By (TR3), there exists a morphism $0 \to Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "\mathrm{id}_{X[1]}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd} \tag{2.3.1}$$

commutes. The commutativity of the square in the middle implies $gf = 0$. ☐

Proposition 2.3.12. *Let $\mathcal{D}$ be a triangulated category. Let W be an object of $\mathcal{D}$ and let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle. Then the sequences*

$$\mathrm{Hom}_{\mathcal{D}}(W,X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{D}}(W,Y) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{D}}(W,Z),$$
$$\mathrm{Hom}_{\mathcal{D}}(Z,W) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{D}}(Y,W) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathcal{D}}(X,W)$$

are exact.

Proof. Since $gf = 0$, the compositions are zero. For the exactness of the first sequence, it suffices to show that for $j\colon W \to Y$ satisfying $gj = 0$, there exists $i\colon W \to X$ such that $j = fi$. Applying (TR1b), (T2), (TR3), we get the following commutative diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "j"'] & 0 \arrow[r] \arrow[d] & W[1] \arrow[r, "-\mathrm{id}_{W[1]}"] \arrow[d, dashed, "i[1]"] & W[1] \arrow[d, "j[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r] & X[1] \arrow[r, "-f[1]"] & Y[1].
\end{tikzcd}$$

For the exactness of the second sequence, it suffices to show that for $k\colon Y \to W$ satisfying $kf = 0$, there exists $l\colon Z \to W$ such that $k = lg$. Applying (TR1b), (T2) (twice), (TR3), we get the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r] \arrow[d, dashed, "l"] & X[1] \arrow[d] \\
0 \arrow[r] & W \arrow[r, "\mathrm{id}_W"] & W \arrow[r] & 0[1].
\end{tikzcd}$$

□

Corollary 2.3.13. *Let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

be a morphism of distinguished triangles. If two of the three morphisms f, g, h are isomorphisms, then so is the third one.

Proof. By (T2), we may assume that f and g are isomorphisms. Let W be any object of the triangulated category. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(W,X) \arrow[r] \arrow[d, "\mathrm{Hom}(W,f)"] & \mathrm{Hom}(W,Y) \arrow[r] \arrow[d, "\mathrm{Hom}(W,g)"] & \mathrm{Hom}(W,Z) \arrow[r] \arrow[d, "\mathrm{Hom}(W,h)"] & \mathrm{Hom}(W,X[1]) \arrow[r] \arrow[d, "\mathrm{Hom}(W,f[1])"] & \mathrm{Hom}(W,Y[1]) \arrow[d, "\mathrm{Hom}(W,g[1])"] \\
\mathrm{Hom}(W,X') \arrow[r] & \mathrm{Hom}(W,Y') \arrow[r] & \mathrm{Hom}(W,Z') \arrow[r] & \mathrm{Hom}(W,X'[1]) \arrow[r] & \mathrm{Hom}(W,Y'[1])
\end{tikzcd}$$

By Proposition 2.3.12 and (T2), the two rows are exact. By assumption, $\mathrm{Hom}(W,f)$, $\mathrm{Hom}(W,g)$, $\mathrm{Hom}(W,f[1])$, $\mathrm{Hom}(W,g[1])$ are isomorphisms. By the five lemma, it follows that $\mathrm{Hom}(W,h)$ is an isomorphism. Therefore h is an isomorphism by Yoneda's lemma. □

Corollary 2.3.14. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} X[1]$$

be distinguished triangles. Then there exists an isomorphism $i\colon Z \xrightarrow{\sim} Z'$ rendering the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, dashed, "i"] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X[1] \end{tikzcd}$$

Thus triangles extending a morphism $X \to Y$ are unique up to non-unique isomorphisms.

Proof. By (TR3), there exists a morphism i rendering the above diagram commutative. By Corollary 2.3.13, i is an isomorphism. □

Remark 2.3.15. We can now show that (TR1), (T2), and (TR3) imply (TR2) (see Remark 2.3.3). Indeed, Corollary 2.3.13 holds under these axioms. Let $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ be a distinguished triangle. By (TR1c), there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} X[1]$. By (T2), $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$ and $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g'[1]} Z'[1] \xrightarrow{-h'[1]} X[2]$ are distinguished triangles. By Corollary 2.3.14, there exists an isomorphism $i\colon Z[1] \to Z'[1]$, rendering the diagram

$$\begin{tikzcd} X[1] \arrow[r, "-f[1]"] \arrow[d, equal] & Y[1] \arrow[r, "-g[1]"] \arrow[d, equal] & Z[1] \arrow[r, "-h[1]"] \arrow[d, dashed, "i"] & X[2] \arrow[d, equal] \\ X[1] \arrow[r, "-f[1]"] & Y[1] \arrow[r, "-g'[1]"] & Z'[1] \arrow[r, "-h'[1]"] & X[2] \end{tikzcd}$$

commutative. It follows that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is isomorphic to the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} X[1]$, and hence is a distinguished triangle by (TR1a).

Corollary 2.3.16. *In a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, f is an isomorphism if and only if Z is a zero object.*

Proof. Applying Corollary 2.3.13 to the diagram (2.3.1), we see that f is an isomorphism if and only if h is an isomorphism. □

Example 2.3.17. A morphism of complexes $f\colon X \to Y$ is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is homotopy equivalent to zero.

Proof of Theorem 2.3.6. (TR1a) and (TR1c) are clear from the definition of distinguished triangles. For any complex X, $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0[1]$ is a distinguished triangle as X can be identified with the cone of $0 \to X$. (TR1b) follows thus from (T2).

For (T2), it suffices to show that, for every morphism $f\colon X \to Y$ of complexes, there exists a homotopy equivalence $g\colon X[1] \to \mathrm{Cone}(i)$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r, "p"] \arrow[d, equal] & X[1] \arrow[r, "-f[1]"] \arrow[d, "g"] & Y[1] \arrow[d, equal] \\ Y \arrow[r, "i"] & \mathrm{Cone}(f) \arrow[r, "i'"] & \mathrm{Cone}(i) \arrow[r, "p'"] & Y[1] \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Here i, p, i', p' denote the canonical morphisms of complexes. We have

$$\begin{aligned} \mathrm{Cone}(i)^n &= Y^{n+1} \oplus \mathrm{Cone}(f)^n \simeq Y^{n+1} \oplus X^{n+1} \oplus Y^n, \\ d^n_{\mathrm{Cone}(i)} &= \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ \mathrm{id}_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix}. \end{aligned}$$

We define $g^n\colon X[1]^n \to \mathrm{Cone}(i)^n$ and $g'^n\colon \mathrm{Cone}(i)^n \to X[1]^n$ by

$$g^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}, \quad g'^n = (0, \mathrm{id}_{X^{n+1}}, 0).$$

It is clear that g, g' are morphism of complexes, and $g'g = \mathrm{id}_{X[1]}$, $g'i' = p$, $p'g = -f[1]$. Moreover,

$$\mathrm{id}_{\mathrm{Cone}(i)^n} - g^n g'^n = \begin{pmatrix} \mathrm{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{Y^n} \end{pmatrix} = h^{n+1} d^n_{\mathrm{Cone}(i)} + d^{n-1}_{\mathrm{Cone}(i)} h^n,$$

where $h^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Next we prove (TR3), which will be used in the proof of (TR4). We may assume that the two distinguished triangles in (TR3) are standard. In other words, it suffices to show that given a square in $C(\mathcal{A})$

(2.3.2)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "j'"] & Y' \end{tikzcd}$$

that commutes in $K(\mathcal{A})$, there exists a morphism of complexes $h\colon \mathrm{Cone}(j) \to \mathrm{Cone}(j')$ such that the diagram

(2.3.3)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & Y \arrow[r, "i"] \arrow[d, "g"] & \mathrm{Cone}(j) \arrow[r, "p"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "j'"] & Y' \arrow[r, "i'"] & \mathrm{Cone}(j') \arrow[r, "p'"] & X'[1] \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Here i, p, i', p' denote the canonical morphisms. The commutativity of the square (2.3.2) in $K(\mathcal{A})$ means that there exists $k \in \mathrm{Ht}(X, Y)$ such that $g^n j^n - j'^n f^n = k^{n+1} d_X^n + d_{Y'}^{n-1} k^n$. We define $h\colon \mathrm{Cone}(j) \to \mathrm{Cone}(j')$ by

$$h^n = \begin{pmatrix} f^{n+1} & 0 \\ k^{n+1} & g^n \end{pmatrix}.$$

It is clear that h is a morphism of complexes and that the squares in the middle and on the right of (2.3.3) commute in $C(\mathcal{A})$.

For (TR4), by Lemma 2.3.18 below (which depends on (TR3)), we may assume that the three distinguished triangles are standard. We choose representatives of f and g in $C(\mathcal{A})$, which we still denote by f and g and we let $h = gf$. We have

$$U^n = X^{n+1} \oplus Y^n, \quad V^n = X^{n+1} \oplus Z^n, \quad W^n = Y^{n+1} \oplus Z^n.$$

Let

$$i^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix}, \quad i'^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}_{Z^n} \end{pmatrix}, \quad i''^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y^{n+1}} & 0 \end{pmatrix}.$$

Then the diagram in (TR4) commutes in $C(\mathcal{A})$. It remains to find a homotopy equivalence $k\colon W \to \mathrm{Cone}(i)$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, equal] & V \arrow[r, "i'"] \arrow[d, equal] & W \arrow[r, "i''"] \arrow[d, "k"] & U[1] \arrow[d, equal] \\ U \arrow[r, "i"] & V \arrow[r, "j'"] & \mathrm{Cone}(i) \arrow[r, "j''"] & U[1] \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Here j' and j'' are the canonical morphisms. We have

$$\mathrm{Cone}(i)^n = U^{n+1} \oplus V^n \simeq X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n,$$

$$d^n_{\mathrm{Cone}(i)} = \begin{pmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -f^{n+2} & -d_Y^{n+1} & 0 & 0 \\ \mathrm{id}_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & g^{n+1} & h^{n+1} & d_Z^n \end{pmatrix}.$$

We define $k\colon W \to \mathrm{Cone}(i)$ and $k'\colon \mathrm{Cone}(i) \to W$ by

$$k^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{Z^n} \end{pmatrix}, \quad k'^n = \begin{pmatrix} 0 & \mathrm{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z^n} \end{pmatrix}.$$

It is clear that k and k' are morphism of complexes and $k'k = \mathrm{id}_W$, $j''k = i''$, $k'j' = i'$. Moreover,

$$\mathrm{id}_{\mathrm{Cone}(i)^n} - k^n k'^n = \begin{pmatrix} \mathrm{id}_{X^{n+2}} & 0 & 0 & 0 \\ 0 & 0 & -f^{n+1} & 0 \\ 0 & 0 & \mathrm{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = l^{n+1} d^n_{\mathrm{Cone}(i)} + d^{n-1}_{\mathrm{Cone}(i)} l^n,$$

where $l^n = \begin{pmatrix} 0 & 0 & -\mathrm{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. □

Lemma 2.3.18. *Let $X \xrightarrow{f} Y$ be a morphism in $C(\mathcal{A})$. A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $K(\mathcal{A})$ is distinguished if and only if there exists a homotopy equivalence $j\colon Z \to \mathrm{Cone}(f)$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "j"] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "i"] & \mathrm{Cone}(f) \arrow[r, "p"] & X[1] \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Here i and p are the canonical morphisms.

Proof. The "if" part is trivial. The "only if" part follows from Corollary 2.3.14 (which depends on (TR3)). □

Remark 2.3.19. By Remark 2.3.4 and the isomorphism $\mathrm{Cone}(f) \simeq \mathrm{Cone}(-f[1])[-1]$, (T2) and (T2′) for $K(\mathcal{A})$ are equivalent to each other. One can also prove (T2′) for $K(\mathcal{A})$ directly as follows. It suffices to show, for every morphism $f\colon X \to Y$ of complexes, that there exists a homotopy equivalence $g\colon Y \to \mathrm{Cyl}(f)$ such that the diagram

$$\begin{tikzcd} \mathrm{Cone}(f)[-1] \arrow[r, "-p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "g"] & \mathrm{Cone}(f) \arrow[d, equal] \\ \mathrm{Cone}(f)[-1] \arrow[r, "-p"] & X \arrow[r, "i'"] & \mathrm{Cyl}(f) \arrow[r, "p'"] & \mathrm{Cone}(f) \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Here $\mathrm{Cyl}(f)$ denotes the mapping cone of $-p$, and is called the *mapping cylinder* of f. We have

$$\mathrm{Cyl}(f)^n = \mathrm{Cone}(f)[-1]^{n+1} \oplus X^n \simeq X^{n+1} \oplus Y^n \oplus X^n,$$
$$d^n_{\mathrm{Cyl}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 & 0 \\ f^{n+1} & d_Y^n & 0 \\ -\mathrm{id}_{X^{n+1}} & 0 & d_X^n \end{pmatrix}.$$

We define $g\colon Y \to \mathrm{Cyl}(f)$ and $g'\colon \mathrm{Cyl}(f) \to Y$ by

$$g^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \\ 0 \end{pmatrix}, \quad g'^n = (0, \mathrm{id}_{Y^n}, f^n).$$

It is clear that g, g' are morphism of complexes, and $g'g = \mathrm{id}_Y$, $p'g = i$, $g'i' = f$. Moreover,

$$\mathrm{id}_{\mathrm{Cyl}(Y)} - g^n g'^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 & 0 \\ 0 & 0 & -f^n \\ 0 & 0 & \mathrm{id}_{X^n} \end{pmatrix} = h^{n+1} d^n_{\mathrm{Cyl}(Y)} + d^{n-1}_{\mathrm{Cyl}(Y)} h^n,$$

where $h^n = \begin{pmatrix} 0 & 0 & -\mathrm{id}_{X^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Remark 2.3.20. Let $f\colon X \to Y$ be a cellular map of CW complexes. The (topological) mapping cylinder $\mathrm{Cyl}(f)$ of f is obtained by gluing the base of the cylinder $\mathrm{Cyl}(X)$ of X to Y via f. The mapping cylinder $\mathrm{Cyl}(C_\bullet(f))$ can be identified with $C_\bullet(\mathrm{Cyl}(f))$. The homotopy equivalence in the proof of (T2′) mirrors the fact that Y is a deformation retract of $\mathrm{Cyl}(f)$.

Definition 2.3.21. Let $\mathcal{D}$ be a triangulated category and let $\mathcal{A}$ be an abelian category. An additive functor $H\colon \mathcal{D} \to \mathcal{A}$ is called a *cohomological functor* if for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$, the sequence $HX \xrightarrow{Hf} HY \xrightarrow{Hg} HZ$ is exact.

For a cohomological functor H, we write H^nX for $H(X[n])$. Applying (TR2), we get a long exact sequence

$$\cdots \to H^nX \to H^nY \to H^nZ \to H^{n+1}X \to \cdots.$$

Example 2.3.22. For any abelian category $\mathcal{A}$, $H^n\colon K(\mathcal{A}) \to \mathcal{A}$ is a homological functor by Proposition 2.1.23.

Example 2.3.23. If $\mathcal{D}$ has small Hom sets, then Proposition 2.3.12 means that the functors

$$\mathrm{Hom}_{\mathcal{D}}(W,-)\colon \mathcal{D} \to \mathbf{Ab}, \quad \mathrm{Hom}_{\mathcal{D}}(-,W)\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Ab}$$

are cohomological functors. See Remark 2.3.24 below for the triangulated structure of $\mathcal{D}^{\mathrm{op}}$.

Remark 2.3.24. Let $\mathcal{D}$ be a triangulated category. We endow $\mathcal{D}^{\mathrm{op}}$ with the translation functor $[-1]$. We say that a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1]$ in $\mathcal{D}^{\mathrm{op}}$ is *distinguished* if $Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{h[1]} Z[1]$ is a distinguished triangle in $\mathcal{D}$. Then $\mathcal{D}^{\mathrm{op}}$ is a triangulated category.

The isomorphisms in Remark 2.1.6 induce an isomorphism of triangulated categories $K(\mathcal{A})^{\mathrm{op}} \simeq K(\mathcal{A}^{\mathrm{op}})$.

Remark 2.3.25. Let $\mathcal{D}$ be a triangulated category. We define a triangulated category $\mathcal{D}^{\mathrm{anti}}$ with the same underlying additive category $\mathcal{D}$ and translation functor $[1]$ as follows: A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished in $\mathcal{D}^{\mathrm{anti}}$ if and only if $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} X[1]$ is distinguished in $\mathcal{D}$. We have an isomorphism of triangulated categories $\mathcal{D} \xrightarrow{\sim} \mathcal{D}^{\mathrm{anti}}$ carrying X to X and f to $-f$.

Triangulated functors

Definition 2.3.26. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories. A *triangulated functor* consists of the following data:

(1) An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$.

(2) A natural isomorphism $\phi_X\colon F(X[1]) \simeq (FX)[1]$ of functors $\mathcal{D} \to \mathcal{D}'$.

These data are subject to the condition that F carries distinguished triangles in $\mathcal{D}$ to distinguished triangles in $\mathcal{D}'$. That is, for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}$, $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\phi_Z \circ Fh} (FX)[1]$ is a distinguished triangle in $\mathcal{D}'$.

Let $(F,\phi), (F',\phi')\colon \mathcal{D} \to \mathcal{D}'$ be triangulated functors. A *natural transformation* of triangulated functors is a natural transformation $\alpha\colon F \to F'$ such that the following diagram commutes for all X:

$$\begin{tikzcd} F(X[1]) \arrow[r, "\phi_X"] \arrow[d, "\alpha(X[1])"'] & (FX)[1] \arrow[d, "\alpha(X)[1]"] \\ F'(X[1]) \arrow[r, "\phi'_X"] & (F'X)[1]. \end{tikzcd}$$

Triangulated functors from $\mathcal{D}$ to $\mathcal{D}'$ form an additive category $\mathrm{TrFun}(\mathcal{D},\mathcal{D}')$. Isomorphisms in $\mathrm{TrFun}(\mathcal{D},\mathcal{D}')$ are called *natural isomorphisms* of triangulated functors.

Example 2.3.27. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F extends to an additive functor $C(F)\colon C(\mathcal{A}) \to C(\mathcal{B})$ with $(C(F)X)^n = F(X^n)$ and $d^n_{C(F)X} = F(d^n_X)$. The functor $C(F)$ induces an additive functor $K(F)\colon K(\mathcal{A}) \to K(\mathcal{B})$, which is the unique functor such that the diagram

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r] \arrow[d, "C(F)"'] & K(\mathcal{A}) \arrow[d, "K(F)"] \\ C(\mathcal{B}) \arrow[r] & K(\mathcal{B}) \end{tikzcd}$$

commutes. Here the horizontal arrows are the canonical functors. We have $C(F)(X[1]) = (C(F)X)[1]$ and $K(F)(X[1]) = (K(F)X)[1]$. Equipped with the latter, $K(F)$ is a triangulated functor, because $C(F)$ preserves cones. We abbreviate $C(F)$ and $K(F)$ to F when no confusion arises.

Remark 2.3.28. A composition of triangulated functors is a triangulated functor. If $F\colon \mathcal{D} \to \mathcal{D}'$ is a triangulated functor and $H\colon \mathcal{D}' \to \mathcal{A}$ is a cohomological functor, then $HF\colon \mathcal{D} \to \mathcal{A}$ is a cohomological functor.

A triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called an *isomorphism of triangulated categories* if there exists a triangulated functor $G\colon \mathcal{D}' \to \mathcal{D}$ such that $GF = \mathrm{id}_{\mathcal{D}}$ and $FG = \mathrm{id}_{\mathcal{D}'}$ as triangulated functors. A triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called an *equivalence of triangulated categories* if there exist a triangulated functor $G\colon \mathcal{D}' \to \mathcal{D}$ and natural isomorphisms of triangulated categories $FG \simeq \mathrm{id}_{\mathcal{D}'}$ and $\mathrm{id}_{\mathcal{D}} \simeq GF$.

Definition 2.3.29. Let $\mathcal{D}$ be a triangulated category. A *triangulated subcategory* of $\mathcal{D}$ consists of a subcategory $\mathcal{D}'$ of $\mathcal{D}$, stable under $[1]$, and a class of distinguished triangles such that $\mathcal{D}'$ is a triangulated category and the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is a triangulated functor.

Remark 2.3.30. Let $\mathcal{D}'$ be a full triangulated subcategory of $\mathcal{D}$. Then a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{D}'$ is a distinguished triangle in $\mathcal{D}'$ if and only if it is a distinguished triangle in $\mathcal{D}$. Indeed, the "only if" part is trivial and for the "if" part, note that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z' \to X[1]$ in $\mathcal{D}'$, and by Corollary 2.3.14 applied to $\mathcal{D}$, there exists an isomorphism $i\colon Z \to Z'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, dashed, "i"] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & X[1] \end{tikzcd}$$

commutes. By (TR1a) applied to $\mathcal{D}'$, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle in $\mathcal{D}'$.

It follows that a full triangulated subcategory is determined by its set of objects. A nonempty set $\mathcal{D}'$ of objects of $\mathcal{D}$, stable under [1], spans a full triangulated subcategory if and only if for every morphism $g\colon Y \to Z$ with Y and Z in $\mathcal{D}'$, there exists a distinguished triangle $X \to Y \to Z \to X[1]$ with X in $\mathcal{D}'$. The only nontrivial point here is that in the "if" part, $\mathcal{D}'$ is stable under $[-1]$ and finite direct sums up to isomorphisms in $\mathcal{D}$. For this we use the fact that $Y \to 0 \to Y[1] \xrightarrow{-\mathrm{id}} Y[1]$ and $X \oplus Y \to X \to Y[1] \xrightarrow{-\mathrm{id}} Y[1]$ are distinguished triangles for X and Y in $\mathcal{D}$ (Exercise).

Example 2.3.31. Let $\mathcal{A}$ be an additive category. Then $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$. On the other hand, for a nonempty interval $I \subsetneq \mathbb{Z}$, $K^I(\mathcal{A})$ is *not* stable under translation unless $\mathcal{A}$ has no nonzero objects.

2.4 Localization of categories

Proposition 2.4.1. *Let $\mathcal{C}$ be a category and let S be a collection of morphisms. Then there exists a category $\mathcal{C}[S^{-1}]$ and a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that*

(1) For any $s \in S$, $Q(s)$ is an isomorphism.

(2) For any functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is invertible for all $s \in S$, there exists a unique functor $G\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = GQ$.

Note that in (2) we require an equality of functors, not just natural isomorphism. The pair $(\mathcal{C}[S^{-1}], Q)$ is clearly unique up to unique isomorphism (not just equivalence). We call $\mathcal{C}[S^{-1}]$ the localization of $\mathcal{C}$ with respect to S.

Proof. Let $\mathrm{Ob}(\mathcal{C}[S^{-1}]) = \mathrm{Ob}(\mathcal{C})$. Consider diagrams in $\mathcal{C}$ of the form

$$\to \cdots \to \leftarrow \cdots \leftarrow \to \cdots \to \cdots \leftarrow \cdots \leftarrow,$$

where each $\leftarrow$ represents an element of S. More formally, such a diagram is a finite sequence $(f_i) = f_n \cdots f_0$, $f_i \in T = \mathrm{Mor}(\mathcal{C}) \coprod S$, with $\mathrm{source}(f_{i+1}) = \mathrm{target}(f_i)$. Here

$$\mathrm{source}(\alpha(f)) = \mathrm{source}(f), \quad \mathrm{target}(\alpha(f)) = \mathrm{target}(f),$$
$$\mathrm{source}(\beta(s)) = \mathrm{target}(s), \quad \mathrm{target}(\beta(s)) = \mathrm{source}(s),$$

where $\alpha\colon \mathrm{Mor}(\mathcal{C}) \to T$, $\beta\colon S \to T$ are the inclusions. We adopt the convention that a sequence of length zero is uniquely determined by an object X and we write i_X for the sequence. Consider the equivalence relation on the set of such diagrams that is stable under concatenation and generated by the following relations:

- $\alpha(fg) \sim \alpha(f)\alpha(g)$, $\alpha(\mathrm{id}_X) \sim i_X$.
- For $s\colon X \to Y$ in S, $\beta(s)\alpha(s) \sim \mathrm{id}_X$ and $\alpha(s)\beta(s) \sim \mathrm{id}_Y$.

We define morphisms of $\mathcal{C}[S^{-1}]$ to be equivalence classes of such diagrams. We define $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ by $QX = X$ and $Qf = [\alpha(f)]$. Then (1) is clearly satisfied. For F as in (2), the unique functor G is given by $GX = FX$, $G([\alpha(f)]) = Ff$, and $G([\beta(s)]) = (Fs)^{-1}$. □

Remark 2.4.2. Property (2) above implies that the functor $-Q\colon \mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ given by composition with Q is injective on objects and describes its image. This functor is fully faithful. Indeed, given functors G and G' from $\mathcal{C}[S^{-1}]$ to $\mathcal{D}$ natural transformations $G \to G'$ and $QG \to QG'$ amount to the same data.

Remark 2.4.3. If $\mathcal{C}$ is a small category, then $\mathcal{C}[S^{-1}]$ is a small category. If the Hom sets of $\mathcal{C}$ are small, then the Hom sets of $\mathcal{C}[S^{-1}]$ is *not* small in general. However, as we shall see, in most of our applications, the Hom sets of $\mathcal{C}[S^{-1}]$ are small.

We now give conditions guaranteeing simpler descriptions of $\mathcal{C}[S^{-1}]$.

Definition 2.4.4. We say that S is a *right multiplicative system* if the following conditions hold:

(M1) If $s\colon X \to Y$, $t\colon Y \to Z$ belong to S, then ts belongs to S. For any X, id_X belongs to S.

(M2) Given morphisms $f\colon X \to Y$ and $s \in X \to X'$ with $s \in S$, there exist $t\colon Y \to Y'$ and $g\colon X' \to Y'$ with $t \in S$ such that $gs = tf$ as shown by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ X' \arrow[r, dashed, "g"] & Y'. \end{tikzcd}$$

(M3) Let $f, g\colon X \rightrightarrows Y$ be morphisms such that there exists $s\colon W \to X$ in S satisfying $fs = gs$. Then there exists $t\colon Y \to Z$ in S such that $tf = tg$.

We say that S is a *left multiplicative system* if it is a right multiplicative system in $\mathcal{C}^{\mathrm{op}}$). That is, if it satisfies (M1), (M2′) and (M3′), where (M2′) and (M3′) are (M2) and (M3) with all arrows reversed. We say that S is a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system.

For any collection S of morphisms in a category $\mathcal{C}$, and for any object Y of $\mathcal{C}$, we let $S_{Y/}$ denote the full subcategory of $\mathcal{C}_{Y/}$ consisting of $(Y, s\colon Y \to Y')$ with s in S. A morphism from (Y', s) to (Y'', t) is a morphism $f\colon Y' \to Y''$ in $\mathcal{C}$ such that $t = fs$. Dually, for any object X of $\mathcal{C}$, we let $S_{/X}$ denote the full subcategory of $\mathcal{C}_{/X}$ consisting of $(X', s\colon X' \to X)$ with s in S.

Proposition 2.4.5. *Let S be a right multiplicative system in a category $\mathcal{C}$.*

(1) The category $S_{Y/}$ is filtered for any object Y of $\mathcal{C}$.

(2) Moreover, for objects X and Y of $\mathcal{C}$, the map

$$\operatorname*{colim}_{(Y',s)\in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X,Y') \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$$

carrying a map $f\colon X\to Y'$ indexed by $s\colon Y\to Y'$ to $Q(s)^{-1}Q(f)$ is a bijection.

This property is sometimes summarized as follows: $\mathcal{C}[S^{-1}]$ admits a *calculus of left fractions* [GZ, Section I.2]. By the description of filtered colimits of sets (Remark 1.6.25), a morphism from X to Y in $\mathcal{C}[S^{-1}]$ is an equivalence class of diagrams of the form

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

in $\mathcal{C}$ with $s\in S$, sometimes called "right roofs" [GM, Remark III.2.9] ("left roofs" for certain authors). Two such diagrams (f,Z,s), (f',Z',s') are said to be equivalent if there exists a third diagram (f'',Z'',s'') and a commutative diagram

$$\begin{tikzcd} & Z \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & Z'' & Y \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\ & Z'. \arrow[u] & \end{tikzcd}$$

Proof. (1) Let $s\colon Y\to Y'$, $s'\colon Y\to Y''$ be in S, defining objects (Y',s), (Y'',s') of $S_{Y/}$. Applying (M2), we get $t\colon Y'\to Z$, $g\colon Y''\to Z$ in $\mathcal{C}$ with t in S such that $ts=gs'$. By (M1), ts is in S. Then t and g define morphisms $(Y',s)\to(Z,ts)$ and $(Y'',s')\to(Z,ts)$ in $S_{Y/}$, respectively. Now let $u,v\colon (Y',s)\rightrightarrows(Y'',s')$ be morphisms in $S_{Y/}$. Then $s'=us=vs$. Applying (M3), we get $w\colon Y''\to W$ in S with $wu=wv$. By (M1), ws' is in S, so that w defines a morphism $(Y'',s')\to(W,ws')$ in $S_{Y/}$. Therefore, $S_{Y/}$ is filtered.

(2) We define a category $\mathcal{D}$ by $\mathrm{Ob}(\mathcal{D})=\mathrm{Ob}(\mathcal{C})$ and

$$\mathrm{Hom}_{\mathcal{D}}(X,Y)=\operatorname*{colim}_{Y'\in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X,Y').$$

Given $(f,Y',s)\colon X\to Y$, we let $[f,Y',s]$ denote its equivalence class in the colimit. Composition is defined as follows. Given $(f,Y',s)\colon X\to Y$ and $(g,Z',t)\colon Y\to Z$, we apply (M2) to g and s to get the commutative diagram

$$\begin{tikzcd} & & Z'' & & \\ & Y' \arrow[ur, "g'"] & & Z' \arrow[ul, "s'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \end{tikzcd}$$

with s' in S and we set $[g,Z',t][f,Y',s]=[g'f,Z'',s't]$, where $s't$ is in S by (M1). It is easy to check that this does not depend on the choices of (g',Z'',s'). Indeed, if (g'',Z''',s'') is another choice, then by (M2) applied to s' and s'', we get $is'=i's''$

with $i\colon Z'' \to V$, $i'\colon Z''' \to V$, i in S. Since $ig's = is'g = is''g = ig''s$, applying (M3) to $(ig', i'g'')$, we get $v\colon V \to V'$ in S such that $vig' = vi'g''$. Thus we get $(g', Z'', s') \sim (vig', V', vis') = (vi'g'', V', vi's'') \sim (g'', Z''', s'')$, so that $(g'f, Z'', s't) \sim (g''f, Z''', s''t)$. The identity $X \to X$ is given by $(\mathrm{id}_X, X, \mathrm{id}_X)$. To check the associativity of the composition, we apply (M2) to get the commutative diagram

$$\begin{tikzcd}
 & & & W''' & & & \\
 & & Z'' \arrow[ur, "h''"] & & W'' \arrow[ul, "s''"'] & & \\
 & Y' \arrow[ur, "g'"] & & Z' \arrow[ul, "s'"'] \arrow[ur, "h'"] & & W' \arrow[ul, "t'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \arrow[ur, "h"] & & W. \arrow[ul, "u"']
\end{tikzcd}$$

Consider the functor $F\colon \mathcal{C} \to \mathcal{D}$ carrying X to X and $f\colon X \to Y$ to $[f, Y, \mathrm{id}_Y]$.

It remains to show that the pair $(\mathcal{D}, F)$ solves the same universal problem for $(\mathcal{C}[S^{-1}], Q)$. For $s\colon X \to Y$ in S, $F(s) = [s, Y, \mathrm{id}_Y]$ has an inverse given by $[\mathrm{id}_Y, Y, s]$. For any functor $F'\colon \mathcal{C} \to \mathcal{D}'$ such that $F'(s)$ is invertible for all s in S, we define $G\colon \mathcal{D} \to \mathcal{D}'$ by $G([f, Y', s]) = F'(s)^{-1}F'(f)$. Note that for a morphism $(f, Y', s) \to (f', Y'', s')$, we have $F'(s)^{-1}F'(f) = F'(s')^{-1}F'(f')$. Thus the definition of G does not depend on the choice of (f, Y', s). Moreover, $F' = GF$. The uniqueness of G is clear. $\square$

Remark 2.4.6. Dually, if S is a left multiplicative system, then $(S_{/X})^{\mathrm{op}}$ is a filtered category for any object X and the map

$$\operatorname*{colim}_{(X',s) \in (S_{/X})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$$

carrying a map $f\colon X' \to Y$ indexed by $s\colon X' \to X$ to $Q(f)Q(s)^{-1}$ is a bijection for objects X and Y in $\mathcal{C}$. If S is a multiplicative system, then the map

$$\operatorname*{colim}_{(X',Y') \in (S_{/X})^{\mathrm{op}} \times S_{Y'/}} \mathrm{Hom}_{\mathcal{C}}(X', Y') \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$$

carrying a map $f\colon X' \to Y'$ indexed by $s\colon X' \to X$ and $t\colon Y \to Y'$ to $Q(t)^{-1}Q(f)Q(s)^{-1}$ is a bijection for objects X and Y in $\mathcal{C}$.

Remark 2.4.7. If S is a right multiplicative system and $\mathcal{C}$ admits finite coproducts, then $\mathcal{C}[S^{-1}]$ admits finite coproducts and the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ preserves finite coproducts by the proposition. Indeed, if $X = \coprod_i X_i$ is a finite coproduct in $\mathcal{C}$, then we have a commutative square of bijections

$$\begin{tikzcd}
\operatorname{colim}_{(Y',s) \in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X, Y') \arrow[rr, "\sim"] \arrow[d, "\simeq"'] & & \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) \arrow[d] \\
\operatorname{colim}_{(Y',s) \in S_{Y/}} \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y') \arrow[r, "\sim"] & \prod_i \operatorname{colim}_{(Y',s) \in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y') \arrow[r, "\sim"] & \prod_i \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X_i, Y)
\end{tikzcd}$$

Dually, if S is a left multiplicative system and $\mathcal{C}$ admits finite products, then $\mathcal{C}[S^{-1}]$ admits finite products and the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ preserves finite products by the proposition.

If S is a multiplicative system of an additive category $\mathcal{A}$, then $\mathcal{A}[S^{-1}]$ is an additive category and the localization functor $Q\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$ is an additive functor.

Localization of triangulated categories

Let $\mathcal{D}$ be a triangulated category and let $\mathcal{N}$ be a full triangulated subcategory. We let $S_\mathcal{N} = S_\mathcal{N}^\mathcal{D}$ denote the collection of morphisms $f\colon X \to Y$ in $\mathcal{D}$ such that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}$ with Z in $\mathcal{N}$.

Proposition 2.4.8. *$S_\mathcal{N}$ is a multiplicative system.*

Proof. We check the axioms of a right multiplicative system.

(M1) Since there exists a zero object in $\mathcal{N}$, id_X is in $S_\mathcal{N}$ by (TR1b). Let $f\colon X \to Y$, $g\colon Y \to Z$ be in $S_\mathcal{N}$. There exist distinguished triangles $X \xrightarrow{f} Y \to U \to X[1]$, $Y \xrightarrow{g} Z \to W \to Y[1]$ with U and W in $\mathcal{N}$. By (TR1c), there exists a distinguished triangle $X \xrightarrow{gf} Z \to V \to X[1]$. By (TR4), there exists a distinguished triangle $U \to V \to W \to U[1]$. Thus V is isomorphic to an object of $\mathcal{N}$. By (TR2), it follows that gf is in $S_\mathcal{N}$.

(M2) Let $f\colon X \to Y$, $s\colon X \to X'$ with s in $S_\mathcal{N}$. By (TR2), there exists a distinguished triangle $Z \xrightarrow{g} X \xrightarrow{s} X' \to Z[1]$ with Z in $\mathcal{N}$. Applying (TR1c) and (TR3), we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "\mathrm{id}_Z"'] & X \arrow[r, "s"] \arrow[d, "f"] & X' \arrow[r] \arrow[d] & Z[1] \arrow[d, "\mathrm{id}_{Z[1]}"] \\
Z \arrow[r, "fg"] & Y \arrow[r, "t"] & Y' \arrow[r] & Z[1].
\end{tikzcd}$$

By (TR2), t is in $S_\mathcal{N}$.

(M3) It suffices to show that for $f\colon X \to Y$ such that there exists $s\colon W \to X$ in $S_\mathcal{N}$ with $fs = 0$, there exists $t\colon Y \to Z$ in $S_\mathcal{N}$ such that $tf = 0$. We have a distinguished triangle $W \xrightarrow{s} X \xrightarrow{g} X' \to W[1]$ with X' in $\mathcal{N}$. By the long exact sequence of $\mathrm{Hom}_\mathcal{D}$, there exists $f'\colon X' \to Y$ such that $f = f'g$. By (TR1c), we get a distinguished triangle $X' \xrightarrow{f'} Y \xrightarrow{t} Z \to X'[1]$. By (TR2), t is in $S_\mathcal{N}$. Moreover, $tf' = 0$ so that $tf = 0$.

Similarly one shows that $S_\mathcal{N}$ is a left multiplicative system. □

We write $\mathcal{D}/\mathcal{N} = \mathcal{D}[S_\mathcal{N}^{-1}]$. Let $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the localization functor. The composition $\mathcal{D} \xrightarrow{[1]} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{N}$ carries $S_\mathcal{N}$ to isomorphisms and thus factors uniquely through a functor $[1]\colon \mathcal{D}/\mathcal{N} \to \mathcal{D}/\mathcal{N}$. We say that a triangle in $\mathcal{D}/\mathcal{N}$ is distinguished if it is isomorphic to the image of a distinguished triangle in $\mathcal{D}$ under Q.

Proposition 2.4.9. *(1) $\mathcal{D}/\mathcal{N}$ is a triangulated category and $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is a triangulated functor.*

(2) *For any object X of $\mathcal{N}$, QX is a zero object. Moreover, for any triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ such that $FX \simeq 0$ for every object X of $\mathcal{N}$, there exists a unique triangulated functor $G\colon \mathcal{D}/\mathcal{N} \to \mathcal{D}'$ such that $F = GQ$.*
(3) *For any cohomological functor $H\colon \mathcal{D} \to \mathcal{A}$ such that $HX \simeq 0$ for every object X of $\mathcal{N}$, there exists a unique cohomological functor $I\colon \mathcal{D}/\mathcal{N} \to \mathcal{A}$ such that $H = IQ$.*

We call $\mathcal{D}/\mathcal{N}$ the *quotient category* of $\mathcal{D}$ by $\mathcal{N}$.

Proof. (1) The axioms for $\mathcal{D}/\mathcal{N}$ follow from the axioms for $\mathcal{D}$. This is clear for (TR1), (TR2), and (TR3). For (TR4), we note that there exists a commutative diagram in $\mathcal{D}/\mathcal{N}$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X' & \xrightarrow{Q(\tilde{f})} & Y' & \xrightarrow{Q(\tilde{g})} & Z' \end{array}$$

where the vertical arrows are isomorphisms. Indeed, for $f = Q(\tilde{f})Q(s)^{-1}$ and $g = Q(t)^{-1}Q(\tilde{g})$, where $s, t \in S_{\mathcal{N}}$, we can take $u = Q(s)^{-1}$, $v = \mathrm{id}_Y$, and $w = Q(t)$. Using the claim and Corollary 2.3.14 (which depends on (TR3)), we reduce (TR4) to the case where the triangles are images of distinguished triangles in $\mathcal{D}$, which follows from (TR4) for $\mathcal{D}$. It is clear that Q is a triangulated functor.

(2) This follows from the universal property for the localization and the fact that, for any triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$, $FX \simeq 0$ for every object X of $\mathcal{N}$ if and only if Ff is an isomorphism for every morphism f in $S_{\mathcal{N}}$.

(3) For any distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}$ with Z in $\mathcal{N}$, it follows from the exactness of the sequence

$$0 = H^{-1}Z \to HX \xrightarrow{Hf} HY \to HZ = 0$$

that Hf is an isomorphism. We conclude by the universal property for the localization. $\square$

Definition 2.4.10. A full triangulated subcategory $\mathcal{N}$ of a triangulated category $\mathcal{D}$ is said to be *thick* ("saturated" in the terminology of Verdier's thesis [V2, II.2.1.6]) if it is stable under direct summands in $\mathcal{D}$.

Remark 2.4.11. Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor. We let $\ker(F)$ denote the *kernel* of F, namely the full subcategory of $\mathcal{D}$ spanned by objects X such that $FX \simeq 0$. Then $\ker(F)$ is a thick subcategory of $\mathcal{D}$. Moreover F can be decomposed as $\mathcal{D} \xrightarrow{Q} \mathcal{D}/\ker(F) \xrightarrow{G} \mathcal{D}'$, where $\ker(G)$ is spanned by zero objects.

Remark 2.4.12. Let $\mathcal{N}$ be a full triangulated subcategory of $\mathcal{D}$. It was shown in Verdier's thesis [V2, Corollaire II.2.2.11] (and independently by Rickard) that the following conditions are equivalent:
(1) $\mathcal{N}$ is thick.
(2) $\mathcal{N} = \ker(Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N})$ (namely, an object X of $\mathcal{D}/\mathcal{N}$ is zero if and only if X is in $\mathcal{N}$).

It follows that in general $\ker(Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N})$ is the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{N}$.

2.5 Derived categories

Let $\mathcal{A}$ be an abelian category. We let $N(\mathcal{A})$ denote the full subcategory of $K(\mathcal{A})$ consisting of acyclic complexes. Then $N(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$.

Definition 2.5.1. We call $D(\mathcal{A}) = K(\mathcal{A})/N(\mathcal{A})$ the *derived category* of $\mathcal{A}$.

By definition, $D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$, where $S = S_{N(\mathcal{A})}$. For a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $K(\mathcal{A})$, Z is acyclic if and only if f is a quasi-isomorphism, by the long exact sequence. Thus S is the collection of quasi-isomorphisms in $K(\mathcal{A})$. Objects of $D(\mathcal{A})$ are complexes in $\mathcal{A}$ and we have

$$\mathrm{Hom}_{D(\mathcal{A})}(X,Y) \simeq \operatorname*{colim}_{(Y',s)\in S_{Y/}} \mathrm{Hom}_{\mathcal{C}}(X,Y') \simeq \operatorname*{colim}_{(X',s)\in (S_{/X})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(X',Y).$$

In general, $D(\mathcal{A})$ does not have small Hom sets, even if $\mathcal{A}$ has small Hom sets. See however Remark 2.5.25 below.

Moreover, $D(\mathcal{A})$ is a triangulated category and the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is triangulated. A triangle in $D(\mathcal{A})$ is distinguished if and only if it is isomorphic to a standard triangle $X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1]$, where f is a morphism of complexes.

Remark 2.5.2. Let $\tilde{S}$ denote the collection of quasi-isomorphisms in $C(\mathcal{A})$. There exists a unique functor $F\colon C(\mathcal{A})[\tilde{S}^{-1}] \to D(\mathcal{A})$ rending the diagram

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r] \arrow[d] & K(\mathcal{A}) \arrow[d] \\ C(\mathcal{A})[\tilde{S}^{-1}] \arrow[r, "F"] & D(\mathcal{A}). \end{tikzcd}$$

commutative. Here the unnamed arrows are the canonical functors. One can show that F is an isomorphism of categories. See [M3, Theorem 3.2.1].

The cohomological functors $H^n\colon K(\mathcal{A}) \to \mathcal{A}$ carry acyclic complexes to zero, and hence induce cohomological functors

$$H^n\colon D(\mathcal{A}) \to \mathcal{A}.$$

For any distinguished triangle $X \to Y \to Z \to X[1]$ in $D(\mathcal{A})$, we have a long exact sequence

$$\cdots \to H^nX \to H^nY \to H^nZ \to H^{n+1}X \to \cdots . \tag{2.5.1}$$

The functors $\tau^{\leq n}, \tau^{\geq n}\colon K(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$ induce additive functors

$$\tau^{\leq n}, \tau^{\geq n}\colon D(\mathcal{A}) \to D(\mathcal{A}).$$

These are not triangulated functors unless all objects of $\mathcal{A}$ are zero objects.

Proposition 2.5.3. *(1) A morphism $f\colon X \to Y$ in $D(\mathcal{A})$ is an isomorphism if and only if H^nf is an isomorphism for all n.*

(2) An object Z of $D(\mathcal{A})$ is a zero object if and only if H^nZ is zero for all n.

Proof. By Corollary 2.3.16, given a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $D(\mathcal{A})$, f is an isomorphism and only if Z is zero. By the long exact sequence (2.5.1), H^nf is an isomorphism if and only if H^nZ is zero for all n. Thus (1) and (2) and equivalent. The "only if" parts of (1) and (2) are trivial. The "if" part of (2) is also trivial. □

Example 2.5.4. For any short exact sequence of complexes $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, we have an isomorphism of triangles in $D(\mathcal{A})$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r, "p"] \arrow[d, "\phi"] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

where i and p are the canonical morphisms, ϕ is the quasi-isomorphism defined by $\phi^n = (0, g^n)$ (Proposition 2.1.25), and $h = p\phi^{-1}$. Thus

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in $D(\mathcal{A})$. By Remark 2.1.26, the long exact sequence associated to it is

$$H^nX \xrightarrow{H^nf} H^nY \xrightarrow{H^ng} H^nZ \xrightarrow{-\delta} H^{n+1}X,$$

where δ is the connecting morphism in the long exact sequence associated to the short exact sequence.

Remark 2.5.5. For a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, identified with $C^{[0,0]}(\mathcal{A})$, we get a morphism $h\colon Z \to X[1]$ in $D(\mathcal{A})$. We have $H^nh = 0$ for every n, but h is *not* the zero morphism unless the short exact sequence splits.

Example 2.5.6. Let X be a complex and let $a \le b < c$, where $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z} \cup \{+\infty\}$. The short exact sequence of complexes

$$0 \to \sigma^{[b+1,c]}X \to \sigma^{[a,b]}X \to \sigma^{[a,c]}X \to 0$$

induces a distinguished triangle in $D(\mathcal{A})$

$$\sigma^{[b+1,c]}X \to \sigma^{[a,b]}X \to \sigma^{[a,c]}X \to (\sigma^{[b+1,c]}X)[1]$$

Example 2.5.7. Let X be a complex and let b be an integer. We have a short exact sequence of complexes

$$0 \to \tau^{\le b}X \to X \to X/\tau^{\le b}X \to 0.$$

Although $X/\tau^{\le b}X$ is in $C^{\ge b}(\mathcal{A})$ but not in $C^{\ge b+1}(\mathcal{A})$ in general, the morphism of complexes $X/\tau^{\le b}X \to \tau^{\ge b+1}X$ is a quasi-isomorphism. Thus we get a distinguished triangle in $D(\mathcal{A})$

$$\tau^{\le b}X \to X \to \tau^{\ge b+1}X \to (\tau^{\le b}X)[1].$$

For $a \le b < c$ (a could be $-\infty$ and c could be $+\infty$), applying the above to $\tau^{[a,c]}X$, we get a distinguished triangle in $D(\mathcal{A})$

$$\tau^{[a,b]}X \to \tau^{[a,c]}X \to \tau^{[b+1,c]}X \to (\tau^{[a,b]}X)[1].$$

Notation 2.5.8. Let $I \subseteq \mathbb{Z}$ be an interval. We let $D^I(\mathcal{A})$ denote the full subcategory of $D(\mathcal{A})$ consisting of complexes X such that $H^n X = 0$ for $n \notin I$. We let $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, resp. $D^b(\mathcal{A})$) denote the full subcategory of $D(\mathcal{A})$ consisting of complexes X such that $H^n X = 0$ for $n \ll 0$ (resp. $n \gg 0$, resp. $|n| \gg 0$).

The full subcategories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are triangulated subcategories of $D(\mathcal{A})$.

Proposition 2.5.9. *The functor $H^0\colon D^{[0,0]}(\mathcal{A}) \to \mathcal{A}$ is an equivalence of categories.*

Proof. Consider the functor $F\colon \mathcal{A} \to D^{[0,0]}(\mathcal{A})$ carrying A to a complex X concentrated in degree 0 with $X^0 = A$. We have $H^0 FA \simeq A$. For any complex X in $D^{[0,0]}(\mathcal{A})$, we have $X \simeq \tau^{[0,0]} X \simeq FH^0 X$. □

To give descriptions of $D^*(\mathcal{A})$ in terms of $K^*(\mathcal{A})$, we need a general result on colimits.

Definition 2.5.10. We say that a category $\mathcal{C}$ is *connected* if for every pair of objects X and Y, there exists a sequence of objects $X = X_0, \dots, X_n = Y$ such that for each $0 \le i \le n-1$, there exists either a morphism $X_i \to X_{i+1}$ or a morphism $X_{i+1} \to X_i$. We say that a functor $\phi\colon J \to I$ is *cofinal* if $(i \downarrow \phi)$ is nonempty and connected for every object i of I. We say that a subcategory is cofinal if the inclusion functor is cofinal.

Proposition 2.5.11. *Let $\phi\colon J \to I$ be a functor of categories. The following conditions are equivalent:*

(1) ϕ is cofinal.

(2) For every functor $F\colon I \to \mathcal{C}$, the functor $\mathcal{C}_{F/} \to \mathcal{C}_{F\phi/}$ carrying $(X, (f_i\colon Fi \to X))$ to $(X, (f_{\phi(j)}\colon F\phi(j) \to X))$ is an isomorphism of categories.

(3) For every functor $F\colon I \to \mathcal{C}$ and every colimit diagram $(X, (f_i\colon Fi \to X))$ of F, $(X, (f_{\phi(j)}\colon F\phi(j) \to X))$ is a colimit diagram of $F\phi$.

Proof. (1) $\Longrightarrow$ (2). We construct an inverse $\mathcal{C}_{F\phi/} \to \mathcal{C}_{F/}$ as follows. Let

$$(X, (g_j\colon F\phi(j) \to X))$$

be an object of $\mathcal{C}_{F\phi/}$. For each object i of I, and each object $x = (j, \alpha\colon i \to \phi(j))$ of $(i \downarrow \phi)$, we put $f_{i,x} = g_j F(\alpha)\colon F(i) \to X$. For a morphism $\beta\colon x \to x' = (j', \alpha')$,

$$f_{i,x} = g_j F(\alpha) = g_{j'} F\phi(\beta) F(\alpha) = g_{j'} F(\alpha') = f_{i,x'}.$$

Since $(i \downarrow \phi)$ is nonempty and connected, $f_{i,x}$ is independent of x and we let f_i denotes the common value. For a morphism $a\colon i \to i'$ in I, and for $x = (j, \alpha')$ in $(i' \downarrow \phi)$,

$$f_{i'} F(a) = g_j F(\alpha') F(a) = g_j F(\alpha' a) = f_i$$

since $(j, \alpha' a)$ is in $(i \downarrow \phi)$. We have thus constructed an object $(X, (f_i\colon F(i) \to X))$ of $\mathcal{C}_{F/}$. This construction is clearly functorial and provides the inverse as claimed.

(2) $\Longrightarrow$ (3). This follows from the definition of colimit: a colimit diagram of F is by definition an initial object of $\mathcal{C}_{F/}$ and similarly for $F\phi$.

(3) $\Longrightarrow$ (1). We take $\mathcal{C}$ to be the category of sets in a universe for which the Hom sets of I are small. Let i be an object of I and let $F = \mathrm{Hom}_I(i, -)\colon I \to \mathcal{C}$. The set $\operatorname{colim} F = \operatorname{colim}_{i' \in I} \mathrm{Hom}_I(i, i')$ can be identified with the set of connected components of the category $\mathcal{C}_{i/}$, which is a singleton because $\mathcal{C}_{i/}$ admits the initial object (i, id_i). By (3), $\operatorname{colim} F\phi = \operatorname{colim}_{j\in J} \mathrm{Hom}_I(i, \phi(j))$, which can be identified with the set of connected components of $(i \downarrow \phi)$, is a singleton. In other words, $(i \downarrow \phi)$ is connected. □

Example 2.5.12. If I admits a final object i, then the functor $\{*\} \to I$ carrying $*$ to i is cofinal. In this case, for any functor $F\colon I \to \mathcal{C}$, $F(i)$ is a colimit of F.

Proposition 2.5.13. *Let I be a filtered category and let J be a full subcategory. Let $\iota\colon J \to I$ be the inclusion functor. Then J is a cofinal subcategory of I if and only if for every object i of I, there exist an object j in J and a morphism $i \to j$. In this case, J is a filtered category.*

The first assertion means the inclusion functor ι is cofinal if and only if $(i \downarrow \iota)$ is nonempty for every object i of I.

Proof. The "only if" part of the first assertion is trivial. To show the "if" part of the first assertion, let $(i, a\colon i \to j)$ and $(i, a'\colon i \to j')$ be objects of $(i \downarrow \iota)$. Since I is filtered, there exist an object k of I and morphisms $b\colon j \to k$, $b'\colon j' \to k$. Furthermore, since I is filtered, we may assume $ba = b'a'$. By assumption, we may assume that k is in J. Then we get morphisms $(j, a) \xrightarrow{b} (k, ba) \xleftarrow{b'} (j', a')$ in $(i \downarrow \iota)$.

Now let J be a cofinal full subcategory of I. For j and j' in J, there exist an object k in I and morphisms $j \to k$, $j' \to k$. By the cofinality of J, we may assume that k is in J. For $j \rightrightarrows j'$ in J, there exists $j' \to k$ in I equalizing the arrows. By the cofinality of J, we may assume that k is in J. □

For $Y \in D^*(\mathcal{A})$, where $*$ is $+$ or $\geq n$, we let $S^*_{Y/}$ denote the full subcategory of $S_{Y/}$ consisting of pairs $(Y', s\colon Y \to Y')$ such that $Y' \in K^*(\mathcal{A})$. For $X \in D^*(\mathcal{A})$, where $*$ is $-$ or $\leq n$, we let $S^*_{/X}$ denote the full subcategory of $S_{/X}$ consisting of pairs $(X', s\colon X' \to X)$ such that $X' \in K^*(\mathcal{A})$.

Proposition 2.5.14. *(1) Let $*$ be $+$ or $\geq n$. Let Y be a complex in $D^*(\mathcal{A})$. Then $S^*_{Y/}$ is a cofinal full subcategory of $S_{Y/}$.*
(2) Let $*$ be $-$ or $\leq n$. Let X be a complex in $D^*(\mathcal{A})$. Then $(S^*_{/X})^{\mathrm{op}}$ is a cofinal full subcategory of $(S_{/X})^{\mathrm{op}}$.

Proof. We treat the case where $*$ is $\geq n$, the other cases being similar. Let Y be in $D^{\geq n}(\mathcal{A})$ and let $(Y', a\colon Y \to Y')$ be a quasi-isomorphism. Then $f\colon Y' \to \tau^{\geq n} Y' = Y''$ provides a morphism $(Y', a) \to (Y'', fa)$ in $S_{Y/}$ with (Y'', fa) in $S^{\geq n}_{Y/}$. □

Corollary 2.5.15. *Let $*$ be $+$, $-$, or b. The functor $K^*(\mathcal{A}) \to D^*(\mathcal{A})$ induces an equivalence of triangulated categories $K^*(\mathcal{A})/N^*(\mathcal{A}) \to D^*(\mathcal{A})$, where $N^*(\mathcal{A}) = N(\mathcal{A}) \cap K^*(\mathcal{A})$.*

Proof. The functor is clearly essentially surjective, because every object of $D^*(\mathcal{A})$ is isomorphic to a truncation in the image of $K^*(\mathcal{A})$. It remains to show that the functor is fully faithful. We first treat the case $* = +$. For X, Y in $K^+(\mathcal{A})$, by the cofinality of $S^+_{Y/}$ in $S_{Y/}$, we have

$$\begin{aligned}\mathrm{Hom}_{K^+(\mathcal{A})/N^+(\mathcal{A})}(X,Y) &\simeq \operatorname*{colim}_{Y'\in S^+_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X,Y') \\ &\simeq \operatorname*{colim}_{Y'\in S_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X,Y') \simeq \mathrm{Hom}_{D^+(\mathcal{A})}(X,Y).\end{aligned}$$

The case $* = -$ is similar. (See Lemma 2.5.20 for a formalization of the argument here.)

Finally let $* = b$. Let $X, Y \in K^{[m,n]}(\mathcal{A})$. We construct an inverse of the map

$$\begin{aligned}&\mathrm{Hom}_{K^b(\mathcal{A})/N^b(\mathcal{A})}(X,Y) \simeq \operatorname*{colim}_{(X',Y')\in (S^b_{/X})^{\mathrm{op}}\times S^b_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y') \\ &\to \operatorname*{colim}_{(X',Y')\in (S_{/X})^{\mathrm{op}}\times S_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y') \simeq \operatorname*{colim}_{(X',Y')\in (S^{\le n}_{/X})^{\mathrm{op}}\times S^{\ge m}_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y') \\ &\simeq \mathrm{Hom}_{D^b(\mathcal{A})}(X,Y)\end{aligned}$$

For $X' \in K^{\le n}(\mathcal{A})$ and $Y' \in K^{\ge m}(\mathcal{A})$, any morphism of complexes $f\colon X' \to Y'$ factorizes as

$$X' \to \tau^{\ge m-1}X' \xrightarrow{g} \tau^{\le n+1}Y' \to Y'.$$[3]

It is easy to check that $f \mapsto g$ provides the inverse as claimed. □

Corollary 2.5.16. *Let X be an object of $D^{\le n}(\mathcal{A})$ and let Y be an object of $D^{\ge n+1}(\mathcal{A})$. Then* $\mathrm{Hom}_{D(\mathcal{A})}(X,Y) = 0$.

Proof. Indeed, $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y) \simeq \mathrm{colim}_{(X',Y')\in (S^{\le n}_{/X})^{\mathrm{op}}\times S^{\ge n+1}_{Y/}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y') = 0$. □

Warning 2.5.17. Under the assumptions of Corollary 2.5.16, we do not have $\mathrm{Hom}_{D(\mathcal{A})}(Y,X) = 0$ in general. See Remark 2.5.5 for a counterexample.

Corollary 2.5.18. *For X in $D^{\le n}(\mathcal{A})$, Y in $D(\mathcal{A})$, Z in $D^{\ge n}(\mathcal{A})$, the maps*

$$\mathrm{Hom}_{D(\mathcal{A})}(X,\tau^{\le n}Y) \to \mathrm{Hom}_{D(\mathcal{A})}(X,Y), \quad \mathrm{Hom}_{D(\mathcal{A})}(\tau^{\ge n}Y,Z) \to \mathrm{Hom}_{D(\mathcal{A})}(Y,Z)$$

are isomorphisms.

Thus the functor $\tau^{\le n}\colon D(\mathcal{A}) \to D^{\le n}(\mathcal{A})$ is a right adjoint of the inclusion functor $D^{\le n}(\mathcal{A}) \to D(\mathcal{A})$; the functor $\tau^{\ge n}\colon D(\mathcal{A}) \to D^{\ge n}(\mathcal{A})$ is a left adjoint of the inclusion functor $D^{\ge n}(\mathcal{A}) \to D(\mathcal{A})$.

Proof. The distinguished triangle $\tau^{\le n}Y \to Y \to \tau^{\ge n+1}Y \to (\tau^{\le n}Y)[1]$ induces a long exact sequence

$$\mathrm{Hom}(X,(\tau^{\ge n+1}Y)[-1]) \to \mathrm{Hom}(X,\tau^{\le n}Y) \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,\tau^{\ge n+1}Y).$$

Since $\tau^{\ge n+1}Y$ and $(\tau^{\ge n+1}Y)[-1]$ are in $D^{\ge n+1}(\mathcal{A})$, the first and fourth terms are zero. The first assertion follows. The proof of the second assertion is similar. □

[3] In fact, f even factorizes as $X' \to \tau^{\ge m}X' \to \tau^{\le n}Y' \to Y'$.

Resolutions

Part (1) of the following theorem generalizes the construction of injective resolutions.

Theorem 2.5.19. *Let $\mathcal{J} \subseteq \mathcal{A}$ be a full additive subcategory. Assume that for every object X of $\mathcal{A}$, there exists a monomorphism $X \to Y$ with Y in $\mathcal{J}$.*

(1) For every $K \in C^{\geq n}(\mathcal{A})$, there exist $L \in C^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $f\colon K \to L$ such that $\tau^{\geq m} f$ is a monomorphism of complexes for each m.

(2) The functor $K^+(\mathcal{J}) \to D^+(\mathcal{A})$ induces an equivalence of triangulated categories

$$K^+(\mathcal{J})/N^+(\mathcal{J}) \to D^+(\mathcal{A}),$$

where $N^+(\mathcal{J}) = N(\mathcal{A}) \cap K^+(\mathcal{J})$.

In (1) the condition on $\tau^{\geq m} f$ means that f is a monomorphism of complexes and the morphism $K^i/B^iK \to L^i/B^iL$ induced by f is a monomorphism for each i. In (2) $N^+(\mathcal{J})$ is a thick triangulated subcategory of $K^+(\mathcal{J})$.

Proof. (1) It suffices to construct $L_m = (\cdots \to L^m \to 0 \to \cdots) \in C^{[n,m]}(\mathcal{J})$ and a morphism $f_m\colon K \to L_m$ of complexes for each m such that f_m^i and $K^i/B^iK \to L^i/B^iL$ are monomorphisms for each $i \leq m$, $H^i f_m$ is an isomorphism for each $i < m$, $L_m = \sigma^{\leq m} L_{m+1}$ and f_m equals the composite $K \xrightarrow{f_{m+1}} L_{m+1} \to L_m$. We proceed by induction on m. For $m < n$, we take $L_m = 0$. Given L_m, we construct L_{m+1} as follows. Form the pushout square

$$\begin{tikzcd}
K^m/B^mK \arrow[r] \arrow[d] & L^m/B^mL \arrow[d] \\
K^{m+1} \arrow[r] & X.
\end{tikzcd}$$

By induction hypothesis, the upper horizontal arrow is a monomorphism. It follows that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K^m/B^mK \arrow[r] \arrow[d] & L^m/B^mL \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K^{m+1} \arrow[r] & X \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By assumption, there exists a monomorphism $X \to L^{m+1}$ with L^{m+1} in $\mathcal{J}$. We define $f^{m+1}\colon K^{m+1} \to L^{m+1}$ and $d_L^m\colon L^m \to L^{m+1}$ by the obvious compositions. Then f_{m+1} is a morphism of complexes. It is clear that f^{m+1} is a monomorphism. Applying the snake lemma to the above diagram, we see that $K^{m+1}/B^{m+1}K \to L^{m+1}/B^{m+1}L$ is a monomorphism and $H^m f$ is an isomorphism.

(2) This follows from (1) and the following lemma. $\square$

Lemma 2.5.20. *Let $\mathcal{K}$ be a triangulated category and let $\mathcal{N}$, $\mathcal{J}$ be full triangulated subcategories of $\mathcal{K}$, with* $\mathrm{Ob}(\mathcal{N})$ *stable under isomorphisms. Assume that for each $Y \in \mathcal{K}$, there exists a morphism $Y \to Y'$ in $S_{\mathcal{N}}$ such that with $Y' \in \mathcal{J}$. Then the triangulated functor $F\colon \mathcal{J}/\mathcal{J} \cap \mathcal{N} \to \mathcal{K}/\mathcal{N}$ is an equivalence of triangulated categories.*

Proof. By assumption, F is essentially surjective. Let us show that F is fully faithful. For $Y \in \mathcal{J}$, the full subcategory $(S_{\mathcal{N}}^{\mathcal{J}})_{Y/} \subseteq (S_{\mathcal{N}})_{Y/}$ consisting of pairs $(Y, f\colon Y \to Y')$ with $Y' \in \mathcal{J}$ is cofinal. Moreover, if $Y \xrightarrow{f} Y' \to Y'' \to Y[1]$ is a distinguished triangle in $\mathcal{J}$ extending such an f, then Y'' is in $\mathcal{J} \cap \mathcal{N}$, since $\mathrm{Ob}(\mathcal{N})$ is stable under isomorphisms. Thus $(S_{\mathcal{N}}^{\mathcal{J}})_{Y/} = (S_{\mathcal{J}\cap\mathcal{N}}^{\mathcal{J}})_{Y/}$. For X in $\mathcal{J}$, F induces

$$\begin{aligned}\mathrm{Hom}_{\mathcal{J}/\mathcal{J}\cap\mathcal{N}}(X, Y) &\simeq \operatorname*{colim}_{Y' \in (S_{\mathcal{J}\cap\mathcal{N}}^{\mathcal{J}})_{Y/}} \mathrm{Hom}_{\mathcal{J}}(X, Y') \\ &\simeq \operatorname*{colim}_{Y' \in (S_{\mathcal{N}})_{Y/}} \mathrm{Hom}_{\mathcal{K}}(X, Y') \simeq \mathrm{Hom}_{\mathcal{K}/\mathcal{N}}(X, Y).\end{aligned}$$

□

Dually we have the following.

Theorem 2.5.21. *Let $\mathcal{J} \subseteq \mathcal{A}$ be a full additive subcategory. Assume that for every object X of $\mathcal{A}$, there exists an epimorphism $Y \to X$ with Y in $\mathcal{J}$.*

(1) *For every $K \in C^{\leq n}(\mathcal{A})$, there exist $L \in C^{\leq n}(\mathcal{J})$ and a quasi-isomorphism $f\colon L \to K$ such that $\tau^{\leq m} f$ is an epimorphism of complexes for each m.*
(2) *The functor $K^-(\mathcal{J}) \to D^-(\mathcal{A})$ induces an equivalence of triangulated categories*
$$K^-(\mathcal{J})/N^-(\mathcal{J}) \to D^-(\mathcal{A}),$$
where $N^-(\mathcal{J}) = N(\mathcal{A}) \cap K^-(\mathcal{J})$.

Corollary 2.5.22. *Let $\mathcal{A}$ be an abelian category with enough injectives. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects. Then the triangulated functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence of triangulated categories.*

It follows that for $X, I \in K^+(\mathcal{I})$, $\mathrm{Hom}_{K(\mathcal{A})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(X, I)$. This extends to $X \in K(\mathcal{A})$ and for this the assumption on $\mathcal{A}$ can be dropped by Propositions 2.5.26 and 2.5.28 below.

Proof. This follows from Theorem 2.5.19 and the lemma below. □

Lemma 2.5.23. *Let $\mathcal{A}$ be an abelian category. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects. Then $N^+(\mathcal{I})$ is equivalent to zero.*

Proof. Let $L \in K^+(\mathcal{I})$ be an acyclic complex. Then L breaks into short exact sequences
$$0 \to Z^n L \to L^n \to Z^{n+1} L \to 0.$$
One shows by induction on i that $Z^n L$ is injective and the sequence splits. Thus L is homotopy equivalent to 0. □

Dually we have the following.

Corollary 2.5.24. *Let $\mathcal{A}$ be an abelian category with enough projectives. We let $\mathcal{P}$ denote the full subcategory of $\mathcal{A}$ consisting of projective objects. Then $N^-(\mathcal{P})$ is equivalent to zero and the triangulated functor $K^-(\mathcal{P}) \to D^-(\mathcal{A})$ is an equivalence of triangulated categories. Moreover, for $L \in K(\mathcal{A})$, $P \in K^-(\mathcal{P})$, $\mathrm{Hom}_{K(\mathcal{A})}(P, L) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(P, L)$.*

Remark 2.5.25. By the corollaries, if $\mathcal{A}$ has small Hom sets and admits enough injectives (resp. projectives), then $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$) has small Hom sets.

Proposition 2.5.26. *Let $\mathcal{A}$ be an abelian category. For any complex I, the following conditions are equivalent:*

(1) $\mathrm{Hom}_{K(\mathcal{A})}(X, I) = 0$ *for all* $X \in N(\mathcal{A})$.
(2) $\mathrm{Hom}_{K(\mathcal{A})}(X, I) \to \mathrm{Hom}_{D(\mathcal{A})}(X, I)$ *is an isomorphism for all* $X \in K(\mathcal{A})$.

Proof. (2) $\Longrightarrow$ (1). Clear.

(1) $\Longrightarrow$ (2). We have $\mathrm{Hom}_{D(\mathcal{A})}(X, I) \simeq \mathrm{colim}_{(X',s)\in(S_{/X})^{\mathrm{op}}} \mathrm{Hom}_{K(\mathcal{A})}(X', I)$. Applying (1) to the cone of s, we see that $\mathrm{Hom}_{K(\mathcal{A})}(s, I) \colon \mathrm{Hom}_{K(\mathcal{A})}(X, I) \to \mathrm{Hom}_{K(\mathcal{A})}(X', I)$ is an isomorphism. □

Definition 2.5.27. A complex I is said to be *homotopically injective* if it satisfies the conditions of the above proposition.

We let $K_{\mathrm{hi}}(\mathcal{A}) \subseteq K(\mathcal{A})$ denote the full subcategory spanned by homotopically injective complexes. The functor $K_{\mathrm{hi}}(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful.

Proposition 2.5.28. *Let $\mathcal{I} \subseteq \mathcal{A}$ denote the full subcategory consisting of injective objects. Let $X \in N(\mathcal{A})$, $I \in K(\mathcal{I})$. Assume $X \in K^+(\mathcal{A})$ or $I \in K^+(\mathcal{A})$. Then* $\mathrm{Hom}_{K(\mathcal{A})}(X, I) = 0$. *In particular, we have $K^+(\mathcal{I}) \subseteq K_{\mathrm{hi}}(\mathcal{A})$.*

Proof. Let $f \colon X \to I$ be a morphism of complexes. We construct a homotopy h satisfying

$$(*^n) \qquad f^n = h^{n+1} d_X^n + d_I^{n-1} h^n$$

for all n as follows. Assume $X \in K^{\geq m}(\mathcal{A})$ or $I \in K^{\geq m}(\mathcal{A})$. For $n \leq m$, we take $h^n = 0$. Then $(*^n)$ holds for $n < m$. for For a general n, assume h^n constructed satisfying $(*^{n-1})$. Then

$$(f^n - d_I^{n-1} h^n) d_X^{n-1} = d_I^{n-1} f^{n-1} - d_I^{n-1}(f^{n-1} - d_I^{n-2} h^{n-1}) = 0.$$

Thus $f^n - d_I^{n-1} h^n$ factorizes via $d^n \colon X^n / B^n X \xrightarrow{\sim} Z^{n+1} X$ through $g^n \colon Z^{n+1} X \to I^n$. We take $h^{n+1} \colon X^{n+1} \to I^n$ to be an extension of g^n. Then $(*^n)$ holds. □

Dually one defines homotopically projective complexes.

Grothendieck categories

Definition 2.5.29. Let $\mathcal{C}$ be a category and let G be an object of $\mathcal{C}$. We say that G is a *generator* of $\mathcal{C}$ if every morphism $f \colon X \to Y$ such that $\mathrm{Hom}_{\mathcal{C}}(G, X) \to \mathrm{Hom}_{\mathcal{C}}(G, Y)$ is a bijection is an isomorphism.

Definition 2.5.30. A *Grothendieck category* is an abelian category $\mathcal{A}$ with small Hom sets admitting a generator and satisfying the following axiom:

(AB5) $\mathcal{A}$ admits small colimits and small filtered colimits are exact.

Example 2.5.31. Let R be a small ring and X a small topological space. The categories R-**Mod** and $\mathrm{Shv}(X)$ are Grothendieck categories. The R-module R is a generator of R-**Mod**. A generator of $\mathrm{Shv}(X)$ is $\bigoplus_U \mathbb{Z}_U$, where U runs through open subsets of X, and $\mathbb{Z}_U = a(\mathbb{Z}'_U)$, where $\mathbb{Z}'_U$ is the presheaf carrying $V \subseteq U$ to $\mathbb{Z}$ and other V to 0. Indeed, $\mathrm{Hom}_{\mathrm{Shv}(X)}(\mathbb{Z}_U, \mathcal{F}) \simeq \mathcal{F}(U)$.

For R nonzero, R-$\mathbf{Mod}^{\mathrm{op}}$ is not a Grothendieck category. For X nonempty, $\mathrm{Shv}(X)^{\mathrm{op}}$ is not a Grothendieck category. Indeed, sequential limit $\lim_{\mathbb{N}^{\mathrm{op}}}$ is not exact in R-**Mod** or $\mathrm{Shv}(X)$.

Theorem 2.5.32 (Grothendieck [G, Théorème 1.10.1])**.** *Grothendieck categories admit enough injectives.*

Theorem 2.5.33. *Let $\mathcal{A}$ be a Grothendieck category. There are enough homotopically injective complexes: for every complex I in $\mathcal{A}$, there exists a quasi-isomorphism $I \to I'$ with I' homotopically injective. In particular, the functor $K_{\mathrm{hi}}(\mathcal{A}) \to D(\mathcal{A})$ is an equivalence of categories.*

It follows from the theorem that $D(\mathcal{A})$ has small hom-sets when $\mathcal{A}$ does. We refer the reader to [KS2, Corollary 14.1.8] for a proof (of a generalization) of the theorem.

Proposition 2.5.34. *An abelian category $\mathcal{A}$ admitting a generator and small colimits (for example, a Grothendieck category) satisfies*

(AB3) $\mathcal{A}$ admits small limits.*

We refer the reader to [KS2, Proposition 5.2.8, Corollary 5.2.10] for a generalization of the proposition to categories that are not necessarily abelian categories.

Lemma 2.5.35. *Let $\mathcal{C}$ be a category with small* Hom*-sets and admitting a generator G and fiber products. Then for any object X, the set of subobjects of X is small.*

By a *subobject* of X, we mean an isomorphism class of objects (Y, i) of $\mathcal{C}_{/X}$ with i a monomorphism.

Proof. Consider the map ϕ from the set of subobjects of X to the set of subsets of $\mathrm{Hom}(G, X)$ carrying a subobject Y of X to $\mathrm{im}(\mathrm{Hom}(G, Y) \to \mathrm{Hom}(G, X))$. Since $\mathrm{Hom}(G, X)$ is a small set, it suffices to show that ϕ is an injection. If $\phi(Y) = \phi(Y')$, then $\mathrm{Hom}(G, p_Y)$ and $\mathrm{Hom}(G, p_{Y'})$ are bijections. Here $p_Y : Y \times_X Y' \to Y$ and $p_{Y'} : Y \times_X Y' \to Y'$ are the projections. Thus p_Y and $p_{Y'}$ are isomorphisms. It follows that Y equals Y' as subobjects of X. □

Lemma 2.5.36. *Let $\mathcal{C}$ be a category with small* Hom*-sets and admitting a generator G and equalizers. Let X be an object such that $G^{\amalg \mathrm{Hom}(G,X)}$ exists. Then the morphism $G^{\amalg \mathrm{Hom}(G,X)} \to X$ is an epimorphism.*

Proof. Indeed, $\mathrm{Hom}(G, -)$ is faithful and the map $\mathrm{Hom}(G, G^{\amalg \mathrm{Hom}(G,X)}) \to \mathrm{Hom}(G, X)$ is surjective. □

Proof of Proposition 2.5.34. Let $F\colon I \to \mathcal{A}$ be a small diagram. The category $\mathcal{A}_{/F}$ admits small colimits and the forgetful functor $\mathcal{A}_{/F} \to \mathcal{A}$ preserves such colimits. We need to show that it admits a final object. By Theorem 1.4.19 applied to $(\mathcal{A}_{/F})^{op}$, it suffices to find a small set of objects of $\mathcal{A}_{/F}$ that is weakly final. For any object X of $\mathcal{A}$, $\mathrm{Nat}(\Delta X, F)$ is a small set. Let $Z = G^{\amalg \mathrm{Nat}(\Delta G, F)}$. There is an obvious morphism $\Delta Z \to F$. For any cone $u\colon \Delta X \to F$, form the pushout

$$\begin{tikzcd} G^{\amalg \mathrm{Hom}(G,X)} \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{A}$. The morphism $X \to Y$ lifts to a morphism with source (X, u) in $\mathcal{A}_{/F}$. By Lemma 2.5.36, the vertical arrow on the left is an epimorphism. It follows that the same holds for the vertical arrow on the right. Thus the set of cones $v\colon \Delta Y \to F$ with Y a quotient of Z is weakly final and it suffices to apply Lemma 2.5.35. □

2.6 Extensions

Let $\mathcal{A}$ be an abelian category.

Notation 2.6.1. For $K, L \in D(\mathcal{A})$, the *hyper* Ext *groups* are defined to be

$$\mathrm{Ext}^n(K, L) = \mathrm{Hom}_{D(\mathcal{A})}(K, L[n]).$$

For $n = 0$, $\mathrm{Ext}^0(K, L) = \mathrm{Hom}_{D(\mathcal{A})}(K, L)$.

We are particularly interested in the case where $K = X$ and $L = Y$ are objects of $\mathcal{A}$, regarded as complexes concentrated in degree 0. In this case, we drop the word "hyper". For $n < 0$, $\mathrm{Ext}^n(X, Y) = 0$ by Corollary 2.5.16. We have $\mathrm{Ext}^0(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$.

For any short exact sequence $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{A}$, we have long exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}(X, Y') \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y'') & \\ \to \mathrm{Ext}^1(X, Y') \to \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X, Y'') \to \cdots, & \\ 0 \to \mathrm{Hom}(Y'', X) \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y', X) & \\ \to \mathrm{Ext}^1(Y'', X) \to \mathrm{Ext}^1(Y, X) \to \mathrm{Ext}^1(Y', X) \to \cdots. & \end{aligned}$$

Remark 2.6.2. We have $\mathrm{Ext}^n_{\mathcal{A}^{\mathrm{op}}}(X, Y) \simeq \mathrm{Ext}^n_{\mathcal{A}}(Y, X)$.

Yoneda extensions

For $n \geq 1$, we will now give an interpretation of $\mathrm{Ext}^n(X, Y)$, which is due to Yoneda for $n \geq 2$.

Definition 2.6.3. Let $n \geq 1$. An *n-extension* of X by Y is an exact sequence

$$0 \to Y \to K^{-n+1} \to \cdots \to K^0 \to X \to 0.$$

An *extension* of Y by X is a 1-extension of Y by X, namely, a short exact sequence

$$0 \to Y \to K^0 \to X \to 0.$$

A morphism of n-extensions of X by Y is a morphism of exact sequences inducing identities on X and on Y.

By the snake lemma, a morphism of 1-extensions of X by Y is necessarily an isomorphism. This fails for n-extensions, $n \geq 2$ in general. We let $E^n(X,Y)$ denote the set of equivalence classes of n-extensions of X by Y, the equivalence relation being generated by morphisms.

We have seen how to produce a morphism $X \to Y[1]$ in $D(\mathcal{A})$ from an extension of X by Y. More generally, given an n-extension of X by Y as above, we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & X \arrow[r] & \cdots \\
\cdots \arrow[r] & Y \arrow[r] \arrow[u] \arrow[d, "\mathrm{id}_Y"'] & K^{-n+1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & K^0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots ,
\end{tikzcd}$$

giving morphisms of complexes $X \xleftarrow{s} K \to Y[n]$. Note that s is a quasi-isomorphism, so that the morphisms induce a morphism $X \to Y[n]$ in $D(\mathcal{A})$. If $E \to E'$ is a morphism of n-extensions, then E and E' induce the same morphism $X \to Y[n]$. We thus obtain a map $\phi\colon E^n(X,Y) \to \operatorname{Ext}^n(X,Y)$.

Proposition 2.6.4. *The map $\phi\colon E^n(X,Y) \to \operatorname{Ext}^n(X,Y)$ is a bijection.*

Proof. We construct an inverse $\psi\colon \operatorname{Ext}^n(X,Y) \to E^n(X,Y)$ as follows. Let $X \xleftarrow{s} K \xrightarrow{f} Y[n]$ represent a morphism $X \to Y[n]$ in $D(\mathcal{A})$. For any quasi-isomorphism $t\colon K' \to K$, we may replace (s,K,f) by (st,K',ft), without changing the class of (s,K,f). Conversely, if for a quasi-isomorphism $t\colon K \to K'$, we have $s = s't$ and $f = f't$, then we may replace (s,K,f) by (s',K',f'). Thus, by truncation, we may assume that $K \in C^{[-n,0]}(\mathcal{A})$. Then we have an exact sequence

$$0 \to K^{-n} \xrightarrow{d_K^{-n}} K^{-n+1} \cdots \to K^0 \xrightarrow{s^0} X \to 0.$$

Taking the pushout of d_K^{-n} by f^{-n}, we get a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & K^{-n} \arrow[r, "d_K^{-n}"] \arrow[d, "f^{-n}"] & K^{-n+1} \arrow[r] \arrow[d] & K^{-n+2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & K'^{-n+1} \arrow[r] & K^{-n+2} \arrow[r] & \cdots \arrow[r] & X \arrow[r] & 0
\end{tikzcd} \tag{2.6.1}$$

where the second row is also exact. This corresponds to a commutative diagram in $C(\mathcal{A})$

$$\begin{tikzcd}
 & K \arrow[dl, "s"'] \arrow[d, "t"] \arrow[dr, "f"] & \\
X & K' \arrow[l, "s'"'] \arrow[r, "f'"] & Y[n],
\end{tikzcd}$$

where s' and t are quasi-isomorphisms. We define $\psi([s,K,f]) \in E^n(X,Y)$ to be the class of the second row of (2.6.1). Then $\phi\psi([s,K,f]) = [s',K',f'] = [s,K,f]$. Moreover, it is clear that $\psi\phi(E) = E$. $\square$

The zero element of $\mathrm{Ext}^n(X,Y)$ corresponds to the class of the n-extension given by the direct sum of $X \xrightarrow{\mathrm{id}} X$ (put in degrees 0 and 1) and $Y \xrightarrow{\mathrm{id}} Y$ (put in degrees $-n$ and $-n+1$). In particular, $0 \in \mathrm{Ext}^1(X,Y)$ corresponds to the class of split short exact sequences.

Remark 2.6.5. (1) Let $f\colon X' \to X$ and $g\colon Y \to Y'$ be morphisms. The map $E^n(X,Y) \to E^n(X',X)$ corresponding to $\mathrm{Ext}^n(f,Y)$ is given by taking pullback along f. The map $E^n(X,Y) \to E^n(X,X')$ corresponding to $\mathrm{Ext}^n(X,g)$ is given by taking pushout along g.

(2) The group structure of $\mathrm{Ext}^n(X,Y)$ can be described in terms of n-extensions. Given $E, E' \in E(X,Y)$, $E + E'$ is obtained from the direct sum of the two n-extensions by taking pullback along the diagonal $(\mathrm{id}_X, \mathrm{id}_X)\colon X \to X \oplus X$ and pushout along the sum morphism $(\mathrm{id}_Y, \mathrm{id}_Y)\colon Y \oplus Y \to Y$. (The order of the two operations does not matter.)

Remark 2.6.6. Let X, Y, Z be objects of $\mathcal{A}$. For $E \in E^n(X,Y)$, $E' \in E^m(Y,Z)$, represented by

$$0 \to Y \xrightarrow{d_K^{-n}} K^{-n+1} \to \cdots \to K^0 \to X \to 0,$$
$$0 \to Z \to K'^{-m+1} \to \cdots \to K'^0 \xrightarrow{s'^0} Y \to 0.$$

We define $E'' = E' \circ E \in E^{n+m}(X,Y)$ to be the class of the spliced exact sequence

$$0 \to Z \xrightarrow{(-1)^n d_{K'}^{-m}} K'^{-m+1} \to \cdots \xrightarrow{(-1)^n d_{K'}^{-1}} K'^0 \xrightarrow{d_K^{-n} s'^0} K^{-n+1} \to \cdots \to K^0 \to X \to 0.$$

Then $\phi(E' \circ E)$ is the composite $X \xrightarrow{\phi(E)} Y[n] \xrightarrow{\phi(E')[n]} Z[n+m]$. Indeed, we have a commutative diagram

$$\begin{tikzcd}
 & & K'' \arrow[dl, "t"'] \arrow[dr, "g"] & & \\
 & K \arrow[dl, "s"'] \arrow[dr, "f"] & & K'[n] \arrow[dl, "{s'[n]}"'] \arrow[dr, "{f'[n]}"] & \\
X & & Y[n] & & Z[n+m]
\end{tikzcd}$$

with $s'' = st$, $f'' = f'[n]g$, where $g^i = \mathrm{id}$ for $-m-n \le i \le -n$, $t^i = \mathrm{id}$ for $-n+1 \le i \le 0$ and $t^{-n} = s'^0$.

We could define the composition $E' \circ E$ without adding signs. To make it compatible with ϕ, we need to modify ϕ by a factor of $(-1)^{n(n+1)}$ (or $(-1)^{n(n-1)}$).

Corollary 2.6.7. *Let X and Y be objects of $\mathcal{A}$ and let $m, n \ge 0$ be integers. Every element $e'' \in \mathrm{Ext}^{n+m}(X,Y)$ has the form $e'' = e'[n] \circ e$ for some object Z of $\mathcal{A}$ and some $e \in \mathrm{Ext}^n(X,Z)$, $e' \in \mathrm{Ext}^m(Z,Y)$.*

Proof. We may assume $m, n \geq 1$. An $(n+m)$-extension of X by Y

$$0 \to Y \to K^{-n-m+1} \to \cdots \to K^{-n} \xrightarrow{d^{-n}} K^{-n+1} \to \cdots \to X \to 0$$

can be decomposed into exact sequences

$$0 \to Y \to K^{-n-m+1} \to \cdots \to K^{-n} \to \mathrm{im}(d^{-n}) \to 0,$$
$$0 \to \mathrm{im}(d^{-n}) \to K^{-n+1} \to \cdots \to X \to 0.$$

□

Homological dimension

Proposition 2.6.8. *Let X be an object of $\mathcal{A}$ and let $m \geq 0$ be an integer. The following conditions are equivalent:*
(1) $\mathrm{Ext}^m(X, Y) = 0$ *for every object Y of $\mathcal{A}$.*
(2) $\mathrm{Ext}^n(X, Y) = 0$ *for every object Y of $\mathcal{A}$ and every $n \geq m$.*

Dually, for an object Y of $\mathcal{A}$, the following conditions are equivalent:
(1) $\mathrm{Ext}^m(X, Y) = 0$ for every object X of $\mathcal{A}$.
(2) $\mathrm{Ext}^n(X, Y) = 0$ for every object X of $\mathcal{A}$ and every $n \geq m$.

Proof. That (2) implies (1) is trivial. That (1) implies (2) follows from Corollary 2.6.7. □

Definition 2.6.9. Let X and Y be objects of $\mathcal{A}$. The *projective dimension of X* and *injective dimension of Y* are defined to be

$$\mathrm{proj.dim}(X) = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^n(X, Y) \neq 0 \text{ for some } Y\},$$
$$\mathrm{inj.dim}(Y) = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^n(X, Y) \neq 0 \text{ for some } X\}.$$

The *homological dimension* of $\mathcal{A}$ is defined to be

$$\mathrm{hom.dim}(\mathcal{A}) = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^n(X, Y) \neq 0 \text{ for some } X, Y\}.$$

We adopt the convention $\sup \emptyset = -\infty$. The above dimensions take values in $\mathbb{Z}_{\geq 0} \cup \{\pm\infty\}$. Proposition 2.6.8 gives equivalent conditions for $\mathrm{proj.dim}(X) < m$. By definition,

$$\mathrm{hom.dim}(\mathcal{A}) = \sup_{X \in \mathcal{A}} \mathrm{proj.dim}(X) = \sup_{Y \in \mathcal{A}} \mathrm{inj.dim}(Y).$$

Remark 2.6.10. The following conditions are equivalent:
- $X = 0$;
- $\mathrm{proj.dim}(X) = -\infty$;
- $\mathrm{inj.dim}(X) = -\infty$.

Proposition 2.6.11. *The following conditions are equivalent:*
(1) $\mathrm{proj.dim}(X) \leq 0$*;*
(2) $\mathrm{Ext}^1(X, Y) = 0$ *for all Y;*

(3) X is projective.

Proof. (1) $\iff$ (2). This follows from Proposition 2.6.8.

(2) $\implies$ (3). It follows from the long exact sequence that $\mathrm{Hom}(X,-)$ is exact.

(3) $\implies$ (2). Every short exact sequence $0 \to Y \to E \to X \to 0$ is split. Thus, by Proposition 2.6.4, $\mathrm{Ext}^1(X,Y) = 0$. □

Dually, the following conditions are equivalent:

(1) $\mathrm{inj.dim}(Y) \le 0$;
(2) $\mathrm{Ext}^1(X,Y) = 0$ for all X;
(3) Y is injective.

Corollary 1.7.5 admits the following extension.

Corollary 2.6.12. *Let $\mathcal{A}$ be an abelian category. The following conditions are equivalent:*

(1) $\mathrm{hom.dim}(\mathcal{A}) \le 0$.
(2) Every object of $\mathcal{A}$ is projective.
(3) Every object of $\mathcal{A}$ is injective.
(4) Every short exact sequence in $\mathcal{A}$ is split.

Proposition 2.6.13 (Dimension shifting)**.** *Let $0 \to X' \to P^{-k+1} \to \cdots \to P^0 \to X \to 0$ be an exact sequence with P^i projective. Then $\mathrm{Ext}^n(X',Y) \simeq \mathrm{Ext}^{n+k}(X,Y)$ for all Y and all $n \ge 1$. In particular,*

$$\max\{\mathrm{proj.dim}(X'), 0\} = \max\{\mathrm{proj.dim}(X) - k, 0\}. \tag{2.6.2}$$

Moreover, if $\mathrm{proj.dim}(X) \ge k$*, then we have*

$$\mathrm{proj.dim}(X') = \mathrm{proj.dim}(X) - k. \tag{2.6.3}$$

Proof. For the first assertion, decomposing the exact sequence into short exact sequence, we reduce by induction to the case $k = 1$. In this case, the assertion follows from the long exact sequence

$$0 = \mathrm{Ext}^n(P^0,Y) \to \mathrm{Ext}^n(X',Y) \to \mathrm{Ext}^{n+1}(X,Y) \to \mathrm{Ext}^{n+1}(P^0,Y) = 0.$$

The first assertion implies that $\mathrm{proj.dim}(X') < n$ if and only if $\mathrm{proj.dim}(X) < n+k$ for all $n \ge 1$, and hence (2.6.2). If $\mathrm{proj.dim}(X) \ge k$, then X' is nonzero (by the first part of Corollary 2.6.14 below), and (2.6.3) follows. □

Corollary 2.6.14. *If X has a projective resolution concentrated in $[-n, 0]$ (namely, there exists an exact sequence $0 \to P^{-n} \to \cdots \to P^0 \to X \to 0$ with P^i projective), then* $\mathrm{proj.dim}(X) \le n$*. Conversely, if* $\mathrm{proj.dim}(X) \le n$ *and $\mathcal{A}$ admits enough projectives, then X has a projective resolution concentrated in $[-n, 0]$.*

Proof. The first assertion follows from (2.6.2):

$$\mathrm{proj.dim}(X) \le \max\{\mathrm{proj.dim}(P^{-n}), 0\} + n = n.$$

For the second assertion, by assumption there exists an exact sequence $0 \to P^{-n} \to \cdots \to P^0 \to X \to 0$ with P^i projective for $i \ge -n+1$. It the follows from (2.6.2) that

$$\mathrm{proj.dim}(P^{-n}) \le \max\{\mathrm{proj.dim}(X) - n, 0\} = 0,$$

namely P^{-n} is projective. □

Corollary 2.6.15. *Consider the following conditions.*
(1) $\mathrm{hom.dim}(\mathcal{A}) \leq 1$*.*
(2) Subobjects of projectives are projective.
(3) Quotients of injectives are injective.
Then (1) $\Longrightarrow$ *(2) and (1)* $\Longrightarrow$ *(3). Moreover, if* $\mathcal{A}$ *admits enough projectives, then (1)* $\Longleftrightarrow$ *(2). Dually, if* $\mathcal{A}$ *admits enough injectives, then (1)* $\Longleftrightarrow$ *(3).*

Definition 2.6.16. The *left global dimension* of a ring R is defined to be

$$\mathrm{l.gl.dim}(R) = \mathrm{hom.dim}(R\text{-}\mathbf{Mod}).$$

Dually, the *right global dimension* of a ring R is defined to be

$$\mathrm{r.gl.dim}(R) = \mathrm{hom.dim}(\mathbf{Mod}\text{-}R).$$

For a commutative ring R, we drop the words “left” and “right” and we denote the global dimension by $\mathrm{gl.dim}(R)$.

Example 2.6.17. The following conditions are equivalent:
(1) R is a semisimple ring;
(2) $\mathrm{l.gl.dim}(R) \leq 0$;
(3) $\mathrm{r.gl.dim}(R) \leq 0$.

Example 2.6.18. Recall that a ring R is left hereditary (Corollary 1.8.8) if and only if submodules of projective R-modules are projective. Thus, taking $\mathcal{A} = R\text{-}\mathbf{Mod}$ in the above corollary, we get that $\mathrm{l.gl.dim}(R) \leq 1$ if and only if R is left hereditary. Moreover, we obtain Proposition 1.8.38.

Dually $\mathrm{r.gl.dim}(R) \leq 1$ if and only if R is right hereditary. Since there are left hereditary rings that are not right hereditary, $\mathrm{l.gl.dim}(R) \neq \mathrm{r.gl.dim}(R)$ in general.

Proposition 2.6.19. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories, and let* $F\colon \mathcal{A} \to \mathcal{B}$ *be an exact functor preserving projectives. Assume that* $\mathcal{A}$ *admits enough projectives. Then* $\mathrm{proj.dim}(FX) \leq \mathrm{proj.dim}(X)$*.*

Proof. This follows from Corollary 2.6.14. □

Example 2.6.20. Let R and S be rings and let ${}_RM_S$ be a bimodule such that ${}_RM$ is projective and M_S is flat. Then ${}_RM_S \otimes_S -\colon S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is an exact functor carrying projectives to projectives and $\mathrm{Hom}_R({}_RM_S, -)\colon R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ is an exact functor carrying injectives to injectives by Lemma 1.9.17. Thus, for every S-module N and every R-module L,

$$\mathrm{proj.dim}_R(M \otimes_S N) \leq \mathrm{proj.dim}_S(N), \quad \mathrm{inj.dim}_S(\mathrm{Hom}_R(M, L)) \leq \mathrm{inj.dim}_R(L).$$

Here are two special cases.
(1) Let $R \to S$ be a ring homomorphism such that ${}_RS$ is projective. Then $\mathrm{proj.dim}_R(N) \leq \mathrm{proj.dim}_S(N)$ for every S-module N.
(2) Let $S \to R$ be a ring homomorphism such that R_S is flat. Then $\mathrm{inj.dim}_S(L) \leq \mathrm{inj.dim}_R(L)$ for every R-module L.

Theorem 2.6.21. *Let R be a ring. For any $R[t]$-module N, we have*

$$\operatorname{proj.dim}_{R[t]}(N) \leq \operatorname{proj.dim}_R(N) + 1.$$

In particular, $\operatorname{l.gl.dim}(R[t]) \leq \operatorname{l.gl.dim}(R) + 1$.

Note that $R[t]$ is a free left R-module, so that $\operatorname{proj.dim}_R(N) \leq \operatorname{proj.dim}_{R[t]}(N)$.

Proof. We may assume that $d = \operatorname{proj.dim}_R(N) < \infty$. For $d \geq 1$, choose a short exact sequence of $R[t]$-modules $0 \to N' \to P \to N \to 0$ such that ${}_{R[t]}P$ is projective. Then $N' \neq 0$ and ${}_RP$ is projective, so that we have identities $\operatorname{proj.dim}_{R[t]}(N') = \operatorname{proj.dim}_{R[t]}(N) - 1$ and $\operatorname{proj.dim}_R(N') = d - 1$. We are thus reduced to the case of $d-1$. Thus, by induction, we may assume $d = 0$, namely that ${}_RN$ is projective. Then ${}_{R[t]}(R[t] \otimes_R N)$ is projective. To prove $\operatorname{proj.dim}_{R[t]}(N) \leq 1$, it suffices to check that the sequence of $R[t]$-modules

$$0 \to R[t] \otimes_R N \xrightarrow{f} R[t] \otimes_R N \xrightarrow{g} N \to 0 \tag{2.6.4}$$

is exact, and hence projective resolution of ${}_{R[t]}N$. Here $g(1 \otimes n) = n$ and $f(1 \otimes n) = t \otimes n - 1 \otimes tn$. We will show that (2.6.4) is a split short exact sequence of R-modules. We define R-module homomorphisms $s\colon N \to R[t] \otimes_R N$ and $r\colon R[t] \otimes_R N \to R[t] \otimes_R N$ by $s(n) = 1 \otimes n$ and

$$r(t^i \otimes n) = \sum_{\substack{j+k+1=i \\ j,k \geq 0}} t^j \otimes t^k n.$$

It is easy to check that s is a section of g, r is a retraction of f, and $\mathrm{id}_{R[t]\otimes_R N} = fr + sg$. We conclude by Remark 1.5.5. □

Remark 2.6.22. In fact, we have $\operatorname{l.gl.dim}(R[t]) = \operatorname{l.gl.dim}(R) + 1$ (Exercise). We refer the reader to [GM, Theorem III.5.16] for a categorical statement.

Corollary 2.6.23. *For any semisimple ring R, we have* $\operatorname{l.gl.dim}(R[x_1, \dots, x_n]) \leq n$.

Combining this with the theorem of Quillen and Suslin, we get the following.

Corollary 2.6.24 (Hilbert's syzygy theorem)**.** *Let k be a field and let $R = k[x_1, \dots, x_n]$. Then any R-module N admits a free resolution concentrated in $[-n, 0]$. Moreover, for any exact sequence*

$$F^{-n+1} \xrightarrow{d^{-n+1}} F^{-n+2} \to \cdots \to F^0 \to N \to 0$$

of R-modules with F^i free, $F^{-n} = \ker(d^{-n+1})$ is free.

Proof. By dimension shifting, $\operatorname{proj.dim}_R(F^{-n}) \leq 0$. In other words, F^{-n} is projective. We conclude by Theorem 1.8.29. □

Here F^{-n} is sometimes called the n-th (or $(n-1)$-th depending on the convention) syzygy of the (partial) free resolution of N. The word "syzygy" comes from astronomy, in which it describes the alignment of three celestial bodies. The formulation of the theorem in Hilbert's 1890 paper [H] is for finitely generated graded R-modules. It follows from Nakayama's lemma that finitely generated graded R-modules whose underlying R-modules are projective are graded free.

Proposition 2.6.25. *Let R be a ring, let M be an R-module, and let $n \geq 0$ be an integer. Then* $\mathrm{inj.dim}_R(M) \leq n$ *if and only if* $\mathrm{Ext}^{n+1}_R(R/I, M) = 0$ *for every left ideal I of R.*

Proof. The "only if" part is clear. For the "if" part, take an exact sequence $0 \to M \to I^0 \to \cdots \to I^{n-1} \to N \to 0$ with I^i injective. Then $\mathrm{Ext}^1_R(R/I, N) \simeq \mathrm{Ext}^{n+1}_R(R/I, M) = 0$. Thus the restriction map $\mathrm{Hom}_R(R, N) \to \mathrm{Hom}_R(I, N)$ is surjective. It follows from Baer's test that N is injective. □

Remark 2.6.26. There is no obvious analogue of the proposition for projective dimensions. Whitehead asks whether for every abelian group A, $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$ (which implies $\mathrm{Ext}^1_{\mathbb{Z}}(A, B) = 0$ for every finitely generated abelian group B) implies that A is free (or, equivalently, projective). Shelah proved that Whitehead's problem is undecidable within ZFC even assuming the Continuum Hypothesis.

Corollary 2.6.27. *For any ring R, we have* $\mathrm{l.gl.dim}(R) = \sup_I \mathrm{proj.dim}_R(R/I)$, *where I runs through left ideals of R.*

Hyper Ext groups

For $K, L \in D(\mathcal{A})$, we have long exact sequences

(2.6.5)
$$\cdots \to \mathrm{Ext}^n(K, \tau^{\leq m}L) \to \mathrm{Ext}^n(K, L) \to \mathrm{Ext}^n(K, \tau^{\geq m+1}L) \to \mathrm{Ext}^{n+1}(K, \tau^{\leq m}L) \to \cdots,$$
(2.6.6)
$$\cdots \to \mathrm{Ext}^n(\tau^{\geq m+1}K, L) \to \mathrm{Ext}^n(K, L) \to \mathrm{Ext}^n(\tau^{\leq m}K, L) \to \mathrm{Ext}^{n+1}(\tau^{\geq m+1}K, L) \to \cdots.$$

It follows by induction that for $K \in D^-(\mathcal{A})$ and $L \in D^+(\mathcal{A})$, $\mathrm{Ext}^n(K, L)$ is obtained from the groups $\mathrm{Ext}^n(H^kK[-k], H^lL[-l]) \simeq \mathrm{Ext}^{n+k-l}(H^kK, H^lL)$ (zero for all but finitely many pairs (k, l)) by successively taking subquotients and extensions. (A subquotient is a subobject of a quotient.) Here we used the fact that $\mathrm{Ext}^n(\tau^{\leq k}K, \tau^{\geq l}L) = 0$ for $n + k - l > 0$.

Lemma 2.6.28. *Assume* $d = \mathrm{hom.dim}(\mathcal{A}) < \infty$. *Let $K, L \in D^b(\mathcal{A})$ with $K \in D^{\geq k}$ and $L \in D^{\leq l}$. For $k - l + n > d$, we have* $\mathrm{Ext}^n(K, L) = 0$. *For $k - l + n = d$, the map* $\mathrm{Ext}^n(K, L) \to \mathrm{Ext}^n((H^kK)[-k], (H^lL)[-l]) \simeq \mathrm{Ext}^d(H^kK, H^lL)$ *is a bijection.*

Proof. The first assertion follows from the above description of $\mathrm{Ext}^n(K, L)$. More explicitly, assume that $K \in D^{[k',k'']}$ and $L \in D^{[l',l'']}$. We proceed by induction on $k'' - k'$ and $l'' - l'$. In the case where $k'' = k'$ and $l'' - l'$, we have $K = A[-k']$ and $L = B[-l']$ for $A, B \in \mathcal{A}$, $k' \geq k$, and $l' \leq l$. Then $\mathrm{Ext}^n(K, L) = \mathrm{Ext}^{k'-l'+n}(A, B) = 0$, since $k' - l' + n \geq k - l + n > d$. For $k'' > k'$ or $l'' > l$, it suffices to apply (2.6.5) for $m = l'$, (2.6.5) for $m = l''$, and the induction hypothesis.

The map in the second assertion is the composite of the maps

$$\mathrm{Ext}^n(K, L) \to \mathrm{Ext}^n((H^kK)[-k], L) \to \mathrm{Ext}^n((H^kK)[-k], (H^lL)[-l]),$$

which are bijections since $\mathrm{Ext}^i(\tau^{\geq k+1}K, L) = 0$ and $\mathrm{Ext}^i((H^kK)[-k], \tau^{\leq l-1}L) = 0$ for $i = n, n+1$ by the first assertion. □

Proposition 2.6.29. *If* hom.dim$(\mathcal{A}) \leq 1$*, then for every* $K \in D^b(\mathcal{A})$*, and every integer* n*, we have we have* $K \simeq \bigoplus_n (H^n K)[-n]$.

The decomposition is not canonical. We note that $D^b(\mathcal{A})$ is *not* the direct sum of $\mathcal{A}[-n]$ unless hom.dim$(\mathcal{A}) \leq 0$.

Proof. Assume that $K \in D^{[n,m]}$. We proceed by induction on $m-n$. If $m = n$, then $K \simeq (H^n K)[-n]$. Assume $m > n$. Consider the distinguished triangle

$$\tau^{\leq n} K \to K \to \tau^{\geq n+1} K \xrightarrow{h} (\tau^{\leq n} K)[1]$$

By the lemma, $\mathrm{Ext}^1(\tau^{\geq n+1}K, \tau^{\leq n}K) = 0$, so that $h = 0$. It follows that $K \simeq \tau^{\leq n} K \oplus \tau^{\geq n+1} K$ (Exercise). We conclude by the induction hypothesis. □

Corollary 2.6.30 (Künneth formula for hyper Ext)**.** *Assume* hom.dim$(\mathcal{A}) \leq 1$*. For every* $K \in D^-(\mathcal{A})$ *and every* $L \in D^+(\mathcal{A})$*, we have a split short exact sequence*

$$0 \to \bigoplus_{l-k=n-1} \mathrm{Ext}^1(H^k K, H^l L) \xrightarrow{f} \mathrm{Ext}^n(K, L) \xrightarrow{g} \bigoplus_{l-k=n} \mathrm{Hom}(H^k K, H^l L) \to 0.$$

In particular, for $K \in D^-(\mathcal{A})$ *and* $Y \in \mathcal{A}$*, we have a split short exact sequence*

$$0 \to \mathrm{Ext}^1(H^{1-n}K, Y) \to \mathrm{Ext}^n(K, Y) \to \mathrm{Hom}(H^{-n}K, Y) \to 0.$$

The exact sequences are canonical, with g carrying $a\colon K \to L[n]$ to the family of $H^k a\colon H^k K \to H^{k+n} L$ (zero for all but finitely many k), and f given by the maps

$$\mathrm{Ext}^1(H^k K, H^l L) \simeq \mathrm{Ext}^n(\tau^{\geq k} K, \tau^{\leq l} L) \to \mathrm{Ext}^n(K, L).$$

Here we used Lemma 2.6.28. The splittings are not canonical.

Proof. We have $L \in D^{\geq k}$ for some k. Then $\mathrm{Ext}^n(K, L) \to \mathrm{Ext}^n(\tau^{\geq k-n} K, L)$ is an isomorphism. Thus we may assume $K \in D^b(\mathcal{A})$. Similarly, we may assume $L \in D^b(\mathcal{A})$. Then, by the proposition,

$$\mathrm{Ext}^n(K, L) \simeq \bigoplus_{k,l} \mathrm{Ext}^{n+k-l}(H^k K, H^l L).$$

□

2.7 Derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. We have remarked that F extends to an additive functor $C(F)\colon C(\mathcal{A}) \to C(\mathcal{B})$, which induces a triangulated $K(F)\colon K(\mathcal{A}) \to K(\mathcal{B})$. The composite

$$K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B}) \to D(\mathcal{B})$$

factorizes through a functor $D(F)\colon D(\mathcal{A}) \to D(\mathcal{B})$ if and only if F is exact. Indeed, F is exact if and only if it carries $N(\mathcal{A})$ into $N(\mathcal{B})$, as acyclic complexes break into short exact sequences. Note that $D(F)$ is equipped with the structure of a triangulated functor. Even when F is not exact, it is possible to define a localization of $K(F)$ in many cases.

Definition 2.7.1. Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a triangulated functor and let $\mathcal{N} \subseteq \mathcal{K}$, $\mathcal{N}' \subseteq \mathcal{K}'$ be full triangulated subcategories. Let $Q\colon \mathcal{K} \to \mathcal{K}/\mathcal{N}$ and $Q'\colon \mathcal{K}' \to \mathcal{K}'/\mathcal{N}'$ be the localization functors. A *right derived functor* of F with respect to $\mathcal{N}$ and $\mathcal{N}'$ is an initial object of $(Q'F \downarrow -Q)$, where $-Q\colon \mathrm{TrFun}(\mathcal{K}/\mathcal{N}, \mathcal{K}'/\mathcal{N}') \to \mathrm{TrFun}(\mathcal{K}, \mathcal{K}'/\mathcal{N}')$ is the functor induced by composition with Q. In other words, a right derived functor is a pair (RF, ϵ), where $RF\colon \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}'$ is a triangulated functor, and $\epsilon\colon Q'F \to (RF)Q$ is a natural transformation of triangulated functors, as shown in the diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "Q"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\epsilon}{\Rightarrow}"] & \mathcal{K}/\mathcal{N} \arrow[d, "RF"] \\
\mathcal{K}' \arrow[r, "Q'"] & \mathcal{K}'/\mathcal{N}',
\end{tikzcd}$$

such that for every pair (G, η), where $G\colon \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}'$ is a triangulated functor and $\eta\colon Q'F \to GQ$ is a natural transformation of triangulated functors, there exists a unique natural transformation of triangulated functors $\alpha\colon RF \to G$ such that $\eta = (\alpha Q)\epsilon$.

Dually, a *left derived functor* of F with respect to $\mathcal{N}$ and $\mathcal{N}'$ is a final object of $(-Q \downarrow Q'F)$.

Right (resp. left) derived functors of F with respect to $\mathcal{N}$ and $\mathcal{N}'$ are unique up to unique natural isomorphism.

Remark 2.7.2.
- The above definition depends on the category $\mathcal{K}'$ only via its localization $\mathcal{K}'/\mathcal{N}'$.
- A similar notion of derived functors can be defined for (non-triangulated) localizations of categories $\mathcal{C} \to \mathcal{C}[S^{-1}]$.

Proposition 2.7.3. *Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a triangulated functor and let $\mathcal{N} \subseteq \mathcal{K}$, $\mathcal{N}' \subseteq \mathcal{K}'$ be full triangulated subcategories such that* $\mathrm{Ob}(\mathcal{N})$ *stable under isomorphisms. Let $\mathcal{J} \subseteq \mathcal{K}$ be a full triangulated subcategory satisfying the following conditions:*

(1) For every $X \in \mathcal{K}$, there exists $X \to Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{J}$.

(2) For every $Y \in \mathcal{J} \cap \mathcal{N}$, $FY \in \mathcal{N}'$.

Then the right derived functor $(RF\colon \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}', \epsilon)$ exists and the restriction of ϵ to $\mathcal{J}$ is a natural isomorphism.

We refer to [KS2, Proposition 7.3.2] for a non-triangulated version of the proposition.

Proof. By (1) and Lemma 2.5.20, the inclusion $\iota\colon \mathcal{J} \to \mathcal{K}$ induces an equivalence of triangulated categories $\phi\colon \mathcal{J}/\mathcal{J} \cap \mathcal{N} \to \mathcal{K}/\mathcal{N}$. By (2), $F\iota$ induces a triangulated functor $\bar{F}\colon \mathcal{J}/\mathcal{J} \cap \mathcal{N} \to \mathcal{K}'/\mathcal{N}'$.

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "Q_{\mathcal{J}}"] \arrow[d, "\iota"'] & \mathcal{J}/\mathcal{J} \cap \mathcal{N} \arrow[d, "\phi"] \arrow[ddr, "\bar{F}"] & \\
\mathcal{K} \arrow[r, "Q"] \arrow[d, "F"'] & \mathcal{K}/\mathcal{N} & \\
\mathcal{K}' \arrow[rr, "Q'"] & & \mathcal{K}'/\mathcal{N}'
\end{tikzcd}$$

For each $X \in \mathcal{K}$, we choose $f_X \colon X \to Y_X$ as in (1). By Proposition 1.2.19, there exists a quasi-inverse ψ of ϕ such that $\psi X = Y_X$ and Qf_X defines a natural isomorphism $\mathrm{id}_{\mathcal{K}/\mathcal{N}} \to \phi\psi$. In particular, for every morphism $g \colon X \to X'$ in $\mathcal{K}$, the square

$$\begin{array}{ccc} X & \xrightarrow{Qf_X} & Y_X \\ {\scriptstyle Qg}\downarrow & & \downarrow{\scriptstyle \phi\psi Qg} \\ X' & \xrightarrow{Qf_{X'}} & Y_{X'} \end{array}$$

commutes. We have $\psi g = Q(h)Q(s)^{-1}$ for $h \colon Y_X \to Y'$ in $\mathcal{K}$ and $s \colon Y' \to Y_{X'}$ in $S^{\mathcal{J}}_{\mathcal{J}\cap\mathcal{N}}$. By the commutativity of the above square, there exists $t \colon Y' \to Y''$ in $S_{\mathcal{N}}$ such that

$$thf_X = tsf_{X'}g. \tag{2.7.1}$$

By (1), we may assume $Y'' \in \mathcal{J}$, so that $t \in S^{\mathcal{J}}_{\mathcal{J}\cap\mathcal{N}}$.

We define $RF = \bar{F}\psi$, which is a triangulated functor. In particular, $(RF)X = FY_X$. We define $\epsilon \colon Q'F \to (RF)Q$ by $\epsilon_X = Q'Ff_X$. By $Q'F$ applied to (2.7.1), ϵ is a natural transformation. For $X \in \mathcal{J}$, $f_X \in S^{\mathcal{J}}_{\mathcal{J}\cap\mathcal{N}}$ and ϵ_X is an isomorphism. Next we prove a non-triangulated universal property for (RF, ϵ): for any functor, $G \colon \mathcal{K}/\mathcal{N} \to \mathcal{K}'/\mathcal{N}'$, the composition c of the first column of the following commutative diagram is a bijection:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K}/\mathcal{N},\mathcal{K}'/\mathcal{N}')}(RF, G) & \xrightarrow[\sim]{-\phi} & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}/\mathcal{J}\cap\mathcal{N},\mathcal{K}'/\mathcal{N}')}(RF\phi, G\phi) \\ {\scriptstyle\simeq}\downarrow{\scriptstyle -Q} & & {\scriptstyle\simeq}\downarrow{\scriptstyle -Q_{\mathcal{J}}} \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{K}'/\mathcal{N}')}(RGQ, GQ) & \xrightarrow{-\iota} & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{K}'/\mathcal{N}')}(RFQ\iota, GQ\iota) \\ \downarrow{\scriptstyle -\circ\epsilon} & & {\scriptstyle\simeq}\downarrow{\scriptstyle -\circ\epsilon\iota} \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{K}'/\mathcal{N}')}(Q'F, GQ) & \xrightarrow{-\iota} & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{K}'/\mathcal{N}')}(Q'F\iota, GQ\iota) \end{array}$$

where the vertical arrows of the upper square are bijections by Remark 2.4.2. It follows from the commutativity of the diagram that c is an injection. To prove the surjectivity, let $\eta \colon Q'F \to GQ$ be a natural transformation. For any $X \in \mathcal{K}$, we have a commutative square

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle Q'Ff_X}\downarrow & & \downarrow{\scriptstyle GQf_X} \\ FY_X & \xrightarrow{\eta_{Y_X}} & GY_X. \end{array}$$

We define $\alpha \colon RF \to G$ by taking α_X to be the composite $FY_X \xrightarrow{\eta_{Y_X}} GY_X \xrightarrow{G(Qf_X)^{-1}} GX$. Then $c(\alpha) = \eta$.

Finally, the non-triangulated universal property implies that ϵ is a natural transformation of triangulated functors and the desired triangulated universal property holds. □

The proof shows that $\epsilon_X \colon FX \to RFX$ can be computed by choosing $f \colon X \to Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{J}$ and taking $\epsilon_X = Q'Ff \colon FX \to FY$.

Remark 2.7.4. One important case is $\mathcal{K} = K^+(\mathcal{A})$, $\mathcal{N} = N^+(\mathcal{A})$, and $\mathcal{J} = K^+(\mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{A}$ denotes the full subcategory spanned by injectives. Since in this case $\mathcal{J} \cap \mathcal{N}$ is equivalent to zero, the conditions of Proposition 2.7.3 are satisfied whenever $\mathcal{A}$ admits enough injectives.

Another important case is $\mathcal{K} = K(\mathcal{A})$, $\mathcal{N} = N(\mathcal{A})$, and $\mathcal{J} = K_{\mathrm{hi}}(\mathcal{A})$. Since in this case $\mathcal{J} \cap \mathcal{N}$ is equivalent to zero, the conditions of Proposition 2.7.3 are satisfied whenever there are enough homotopically injective complexes.

Definition 2.7.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. By a *right derived functor* of F, we mean a right derived functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ of $K^+(F)\colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ with respect to $N^+(\mathcal{A})$ and $N^+(\mathcal{B})$. If RF exists, we put $R^nFK = H^nRFK \in \mathcal{B}$ for $K \in D^+(\mathcal{A})$ (sometimes called the *hypercohomology* of K with respect to RF). The functor $R^nF\colon \mathcal{A} \to \mathcal{B}$ is called the n-th right derived functor of F.

By a *left derived functor* of F, we mean a left derived functor of $K^-(F)\colon K^-(\mathcal{A}) \to K^-(\mathcal{B})$ with respect to $N^-(\mathcal{A})$ and $N^-(\mathcal{B})$. If LF exists, we put $L_nFX = H^{-n}LFX \in \mathcal{B}$ for $X \in \mathcal{A}$. The functor $L_nF\colon \mathcal{A} \to \mathcal{B}$ is called the n-th left derived functor of F.

Remark 2.7.6. Consider a short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$. If RF exists, then we have a distinguished triangle

$$RFX' \to RFX \to RFX'' \to RFX'[1], \tag{2.7.2}$$

which induces a long exact sequence

$$\cdots \to R^nFX' \to R^nFX \to R^nFX'' \to R^{n+1}FX' \to \cdots . \tag{2.7.3}$$

Similarly, if LF exists, then we have a long exact sequence

$$\cdots \to L_nFX' \to L_nFX' \to L_nFX \to L_nFX'' \to L_{n-1}FX' \to \cdots .$$

Definition 2.7.7. Let $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. A full additive subcategory $\mathcal{J} \subseteq \mathcal{A}$ is said to be *F-injective* if it satisfies the following conditions:

(a) For every $X \in \mathcal{A}$, there exists a monomorphism $X \to Y$ with $Y \in \mathcal{J}$.

(b) For every $L \in N^+(\mathcal{J})$, FL is acyclic.

A full additive subcategory $\mathcal{J} \subseteq \mathcal{A}$ is said to be *F-projective* if $\mathcal{J}^{\mathrm{op}} \subseteq \mathcal{A}^{\mathrm{op}}$ is F-injective.

Note that the conditions (a) and (b) are equivalent to the conditions of Proposition 2.7.3 applied to the functor $K^+(F)\colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$, the subcategories $N^+(\mathcal{A})$, $N^+(\mathcal{B})$, and the subcategory $K^+(\mathcal{J})$. Indeed, (b) is the same as (2). By Theorem 2.5.19, (a) implies (1). Conversely, for $X \in \mathcal{A}$, by (1) there exists a quasi-isomorphism $X \to L$ with $L \in N^+(\mathcal{J})$. Since $X \to Z^0L \to H^0L$ is an isomorphism, $X \to Z^0L$ is a monomorphism. It follows that $X \to Z^0L \to L^0$ is a monomorphism.

By Proposition 2.7.3, if there exists an F-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$, then the right derived functor $(RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B}), \epsilon)$ exists and the restriction of ϵ to $K^+(\mathcal{J})$ is a natural isomorphism.

The terminology is not completely standard. Our definition here follows [KS2, Definitions 10.3.2, 13.3.4]. The same authors gave a more restrictive definition of F-injective categories in their previous book [KS1, Definition 1.8.2] ((a), (b′) below, and $\mathcal{J}$ stable under isomorphisms in $\mathcal{A}$).

Proposition 2.7.8. *Condition (b′) below implies (b).*

(b′) *Every monomorphism $X' \to X$ in $\mathcal{A}$ with $X', X \in \mathcal{J}$ can be completed into a short exact sequence*
$$0 \to X' \to X \to X'' \to 0$$
in $\mathcal{A}$ with $X'' \in \mathcal{J}$ such that the sequence
$$0 \to FX' \to FX \to FX'' \to 0$$
is exact.

Proof. Let $L \in K^+(\mathcal{J})$ be an acyclic complex. Then L breaks into short exact sequences
$$0 \to Z^nL \to L^n \to Z^{n+1}L \to 0.$$
By (b′), one shows by induction on n that Z^nL is isomorphic to an object in $\mathcal{J}$ and we have short exact sequences
$$0 \to F(Z^nL) \to F(L^n) \to F(Z^{n+1}L) \to 0,$$
so that $K^+(F)(L)$ is acyclic. □

Remark 2.7.9. If $\mathcal{A}$ admits enough injectives, then the full subcategory $\mathcal{I} \subseteq \mathcal{A}$ consisting of injective objects satisfies conditions (a) and (b′) for every F.

Proposition 2.7.10. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and let $\mathcal{J} \subseteq \mathcal{A}$ be an F-injective subcategory. The functor RF carries $D^{\geq n}(\mathcal{A})$ into $D^{\geq n}(\mathcal{B})$. In particular, R^0F is left exact and $R^nFX = 0$ for $X \in \mathcal{A}$ and $n < 0$. Moreover, F is left exact if and only if the morphism $FX \to R^0FX$ is an isomorphism for all $X \in \mathcal{A}$.*

Proof. The first assertion is that $RFK \in D^{\geq n}(\mathcal{B})$ for $K \in D^{\leq n}(\mathcal{A})$. For this, up to replacing K by $\tau^{\geq n}K$, we may assume $K \in K^{\geq n}(\mathcal{B})$. By Theorem 2.5.19 (1), there exists a quasi-isomorphism $K \to K'$ with $K' \in K^{\geq n}(\mathcal{J})$. By Proposition 2.7.3, $RFK \simeq FK' \in D^{\geq n}(\mathcal{B})$. The second assertion follows from the first one and the long exact sequence (2.7.3). For the third assertion, the "if" part is then trivial. Finally, assume that F is left exact and let $X \in \mathcal{A}$. Choose a quasi-isomorphism $X \to L$ with $L \in K^{\geq 0}(\mathcal{J})$, corresponding to an exact sequence
$$0 \to X \to L^0 \to L^1 \to \cdots .$$
Applying F, we obtain an exact sequence
$$0 \to FX \to FL^0 \to FL^1.$$
Thus $R^0FX \simeq H^0FL \simeq FX$. □

Remark 2.7.11. If $\mathcal{J}$ satisfies (a) and (b′) for F, then the same holds for R^0F. Indeed, the second part of condition (b′) for R^0F follows from the long exact sequence (2.7.3) and $RFX \simeq FX$ for $X \in \mathcal{J}$. Moreover, the natural transformation $F \to R^0F$ induces a natural isomorphism $RF \xrightarrow{\sim} R(R^0F)$.

Proposition 2.7.12. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories admitting an F-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$. Then the full subcategory $\mathcal{I}$ of $\mathcal{A}$ spanned by objects X such that $R^nFX = 0$ for all $n \geq 1$ satisfies (a) and (b′) for F. In particular, $\mathcal{I}$ is F-injective. Moreover, for any short exact sequence*

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$, the following holds

(1) If $X' \in \mathcal{I}$, then the induced sequence $0 \to FX' \to FX \to FX'' \to 0$ is exact.
(2) If $X', X \in \mathcal{I}$, then $X'' \in \mathcal{I}$.
(3) If $X', X'' \in \mathcal{I}$, then $X \in \mathcal{I}$.

Clearly $\mathcal{I}$ is the largest F-injective subcategory of $\mathcal{A}$. Objects of $\mathcal{I}$ are sometimes said to be *F-acyclic*.

Proof. It is clear that $\mathcal{J} \subseteq \mathcal{I}$. Condition (a) for $\mathcal{I}$ then follows from condition (a) for $\mathcal{J}$. (1), (2), (3) follow easily from the long exact sequence for (2.7.2). (b) follows from (1) and (2). □

Proposition 2.7.13. *Let $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories. Let $\mathcal{I} \subseteq \mathcal{A}$ be an F-injective subcategory and let $\mathcal{J} \subseteq \mathcal{B}$ be a G-injective subcategory. Assume that F carries $\mathcal{I}$ into $\mathcal{J}$. Then $\mathcal{I}$ is a GF-injective subcategory and the natural transformation $\eta\colon R(GF) \to (RG)(RF)$ given by the universal property of right derived functors is a natural isomorphism.*

This applies in particular to the case where $\mathcal{I} \subseteq \mathcal{A}$ and $\mathcal{J} \subseteq \mathcal{B}$ are the full subcategories spanned by injectives if $\mathcal{A}$ and $\mathcal{B}$ admit enough injectives and F preserves injectives.

Proof. It is clear that $\mathcal{I}$ is GF-injective. For the second assertion, let ϵ, ϵ', ϵ'' be the natural transformations underlying RF, RG, $R(GF)$. Then we have a commutative diagram

$$\begin{tikzcd} GFL \arrow[r, "\epsilon''"] \arrow[dr, "\epsilon'\epsilon"'] & R(GF)L \arrow[d, "\eta"] \\ & (RG)(RF)L \end{tikzcd}$$

For $L \in K^+(\mathcal{I})$, the horizontal and oblique arrows are isomorphisms. Hence so is η_L. □

We leave it to the reader to give dual statements of the above.

Example 2.7.14. Let G be a group. By a (left) *G-module*, we mean an abelian group equipped with a (left) G-action, or equivalently, a (left) $\mathbb{Z}G$-module, where $\mathbb{Z}G = \mathbb{Z}[G]$ is the group ring. The functor $\mathbf{Ab} \to \mathbb{Z}G$-$\mathbf{Mod}$ carrying an abelian group A to A equipped with trivial G-action admits a right adjoint $(-)^G\colon \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ and a left adjoint $(-)_G\colon \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$, which can be described as follows. For a G-module M, M^G is the maximal G-invariant subgroup of M, which is the group of G-invariants of M. Since $\mathbb{Z}G\text{-}\mathbf{Mod}$ admits enough injectives and enough projectives, these functors admit left and right derived functors. Moreover, M_G is the maximal G-invariant quotient group of M, called the group of G-coinvariants of M. We define $H^n(G,-)$ to be the n-th right derived functor of $(-)^G$, and $H_n(G,-)$ to be the n-th left derived functor of $(-)_G$. For a G-module M, we call $H^n(G,M)$ the *n-th cohomology group* of G with coefficients in M, and $H_n(G,M)$ the *n-th homology group* of G with coefficients in M. Thus $H^0(G,M) = M^G$ and $H_0(G,M) = M_G$. For a short exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules, we have long exact sequences

$$0 \to M'^G \to M^G \to M''^G \to H^1(G,M') \to \cdots,$$
$$\cdots \to H_1(G,M'') \to M'_G \to M_G \to M''_G \to 0.$$

$H^n(G,M)$ can be computed as the n-th cohomology of I^G, where $M \to I$ is an injective resolution. Dually, $H_n(G,M)$ can be computed as the $-n$-th cohomology of P_G, where $P \to M$ is a projective resolution. We will give better recipes for the computation later.

Example 2.7.15. Let $\mathcal{A}$ be an abelian category and let $\mathcal{B} = \mathcal{A}^{\bullet\to\bullet}$ be the category of morphisms in $\mathcal{A}$. The functor $\ker\colon \mathcal{B} \to \mathcal{A}$ is left exact. The full subcategory $\mathcal{J}$ of $\mathcal{B}$ spanned by epimorphisms in $\mathcal{A}$ satisfies (a) and (b′) for ker. On $\mathcal{B}$, we have $R^1\ker \simeq \operatorname{coker}$ and $R^n\ker = 0$ for $n > 1$. The long exact sequence associated to a short exact sequence in $\mathcal{B}$ recovers the snake lemma. More generally, for $K \in C(\mathcal{B})$, corresponding to a morphism f in $C(\mathcal{A})$, we have $R\ker K \simeq \mathrm{Cone}(f)[-1]$.

Dually, $\operatorname{coker}\colon \mathcal{B} \to \mathcal{A}$ is right exact. On $\mathcal{B}$, $L_1\operatorname{coker} \simeq \ker$ and $L_n\operatorname{coker} = 0$ for $n > 1$. More generally, for $K \in C(\mathcal{B})$, corresponding to a morphism f in $C(\mathcal{A})$, we have $L\operatorname{coker} K \simeq \mathrm{Cone}(f)$. (Exercise. See also Remark 2.7.18.)

Example 2.7.16. Let X be a topological space. The global section functor

$$\Gamma(X,-)\colon \mathrm{Shv}(X) \to \mathbf{Ab}$$

is left exact. Since $\mathrm{Shv}(X)$ admits enough injectives, $\Gamma(X,-)$ admits a right derived functor $R\Gamma(X,-)\colon D^+(\mathrm{Shv}(X)) \to D^+(\mathbf{Ab})$. We write $H^n(X,-)$ for the n-th right derived functor of $R\Gamma(X,-)$. For $\mathcal{F} \in \mathrm{Shv}(X)$, $H^n(X,\mathcal{F})$ is called the *n-th cohomology group of X* with coefficients in $\mathcal{F}$. We have $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$. By extension, for $K \in D^+(\mathrm{Shv}(X))$, we write $H^n(X,K)$ for $H^nR\Gamma(X,K)$, which is sometimes called the *n-th hypercohomology of X* with coefficients in K. A sheaf $\mathcal{F}$ on X is called *flabby* (*flasque* in French) if the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective for every inclusion $U \subseteq V$ of open subsets of X. Any sheaf can be canonically embedded into a flabby sheaf: $\mathcal{F} \hookrightarrow \prod_{x\in X} i_{x*}i_x^*\mathcal{F}$. Using Zorn's lemma, one can show that the full subcategory of $\mathrm{Shv}(X)$ spanned by flabby sheaves satisfies (a) and (b′) for $\Gamma(X,-)$.

The functor $\Gamma(X,-)$ admits an exact left adjoint, carrying an abelian group M to the *constant sheaf* M_X on X of value M. The sheaf M_X is the sheafification of the constant presheaf $U \mapsto M$. For X locally contractible, we have $H^n(X, M_X) \simeq H^n_{\mathrm{sing}}(X, M)$, where $H^n_{\mathrm{sing}}(X, M)$ is the cohomology of the singular cochain complex $C^\bullet(X, M)$. To see this, consider the sequence

$$0 \to M_X \to \mathcal{C}^0(X, M) \to \cdots \to \mathcal{C}^n(X, M) \to \cdots,$$

where $\mathcal{C}^n(X, M)$ is the sheafification of $U \mapsto C^n(U, M)$. The sequence is exact. Indeed, the cohomology of the sequence at $\mathcal{C}^n(X, M)$ is the sheafification of $U \mapsto \tilde{H}^n_{\mathrm{sing}}(U, M)$, which is zero by local contractibility. Moreover, $\mathcal{C}^n(X, M)$ is flabby. Thus, $H^n(X, M) \simeq H^n(\Gamma(X, \mathcal{C}^\bullet(X, M)))$. A subdivision argument shows that the morphism of complexes $C^\bullet(X, M) \to \Gamma(X, \mathcal{C}^\bullet(X, M))$, surjective in each degree, is a quasi-isomorphism.

Example 2.7.17. Let $f\colon X \to Y$ be a continuous map of topological spaces. The left exact functor $f_*\colon \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ defined by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ admits an exact left adjoint $f^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ defined by $(f^*\mathcal{G})(U) = \mathrm{colim}_{f(U)\subseteq V}\, \mathcal{G}(V)$, where the colimit runs through the filtered category $(U \downarrow f^{-1})^{\mathrm{op}}$. Thus f^* extends to a functor

$$f^*\colon D(\mathrm{Shv}(Y)) \to D(\mathrm{Shv}(X)).$$

Moreover f_* admits a right derived functor $Rf_*\colon D^+(\mathrm{Shv}(X)) \to D^+(\mathrm{Shv}(Y))$. The full subcategory of $\mathrm{Shv}(X)$ spanned by flabby sheaves satisfies (a) and (b′) for f_*. In the special case where Y is a point, $f_* = \Gamma(X, -)$.

For a sequence of continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have natural isomorphisms $f^*g^* \simeq (gf)^*$ and $R(gf)_* \simeq Rg_*Rf_*$.

Remark 2.7.18. We have already seen one important case (second part of Remark 2.7.4) where derived functors exist between unbounded derived categories. There are other such cases. We refer to [KS2, Chapters 14] for details.

2.8 Derived Hom

The goal of this section and the following is to study derive functors of the Hom and tensor functors. Since these functors have two variables, we need some generalities on double complexes.

Double complexes

Let $\mathcal{A}$ be an additive category.

Definition 2.8.1. We define the category of *double complexes* in $\mathcal{A}$ to be $C^2(\mathcal{A}) = C(C(\mathcal{A}))$. Thus a double complex consists of objects $X^{i,j}$ for $i, j \in \mathbb{Z}$ and differentials $d_I\colon X^{i,j} \to X^{i+1,j}$, $d_{II}\colon X^{i,j} \to X^{i,j+1}$ such that $d_I^{i+1,j}d_I^{i,j} = 0$, $d_{II}^{i,j+1}d_{II}^{i,j} = 0$, $d_I^{i,j+1}d_{II}^{i,j} = d_{II}^{i+1,j}d_I^{i,j}$.

Definition 2.8.2. Let X be a double complex in $\mathcal{A}$. We define two complexes in $\mathcal{A}$ with $(\mathrm{tot}_\oplus X)^n = \oplus_{i+j=n} X^{i,j}$ (if the coproducts exist) and $(\mathrm{tot}_\Pi X)^n = \prod_{i+j=n} X^{i,j}$ (if the products exist), called *total complex* of X with respect to coproducts and products, respectively. The differentials are defined as follows. Let $i+j=n$. The composition $X^{i,j} \to (\mathrm{tot}_\oplus X)^n \xrightarrow{d^n} (\mathrm{tot}_\oplus X)^{n+1}$ is given by

$$d_I^{i,j} + (-1)^i d_{II}^{i,j}. \tag{2.8.1}$$

The composition $(\mathrm{tot}_\Pi X)^{n-1} \xrightarrow{d^{n-1}} (\mathrm{tot}_\Pi X)^n \to X^{i,j}$ is given by

$$d_I^{i-1,j} + (-1)^i d_{II}^{i,j-1}. \tag{2.8.2}$$

Remark 2.8.3. The sign in (2.8.1) and (2.8.2) ensures that $d^2=0$. If Y is the transpose of X defined by $Y^{i,j}=X^{j,i}$ and by swapping the two differentials, then we have an isomorphism $\mathrm{tot}_\oplus X \simeq \mathrm{tot}_\oplus Y$ given by $(-1)^{ij}\mathrm{id}_{X^{i,j}}$. The same holds for tot_Π.

In the literature, a variant of Definition 2.8.1 with $d_I d_{II} + d_{II} d_I = 0$ is sometimes used. If we adopt this variant, then (2.8.1) can be simplified to $d = d_I + d_{II}$. The two definitions correspond to each other by multiplying $d_{II}^{i,j}$ by the sign $(-1)^i$.

Definition 2.8.4. We say that a double complex X is *biregular* if for every n, $X^{i,j}=0$ for all but finitely many pairs (i,j) with $i+j=n$. We let $C^2_{\mathrm{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A})$ denote full subcategory consisting of biregular double complexes. It is an additive subcategory.

If $X^{i,j}=0$ for $i<a$ or $j<b$ (X concentrated in a (translated) first quadrant) or $X^{i,j}=0$ for $i>a$ or $j>b$ (X concentrated in a (translated) third quadrant), then X is biregular. If $X^{i,j}=0$ for $|i|\gg 0$ (concentrated in a vertical stripe) or $X^{i,j}=0$ for $|j|\gg 0$ (concentrated in a horizontal stripe), then X is biregular.

Remark 2.8.5. If X is a biregular double complex, then $\mathrm{tot}_\oplus X$ and $\mathrm{tot}_\Pi X$ exist and we have $\mathrm{tot}_\oplus X \xrightarrow{\sim} \mathrm{tot}_\Pi X$. We will simply write $\mathrm{tot}(X)$. We get an additive functor $\mathrm{tot}\colon C^2_{\mathrm{reg}}(\mathcal{A}) \to C(\mathcal{A})$.

Example 2.8.6. Let $f\colon L\to M$ be a morphism of complexes in $\mathcal{A}$. We define a double complex X by $X^{-1,j}=L^j$, $X^{0,j}=M^j$, $X^{i,j}=0$ for $i\neq -1,0$, $d_I^{-1,j}=f^j$, d_{II} given by d_L and d_M. Then $\mathrm{tot}(X)=\mathrm{Cone}(f)$.

Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be additive categories. Let $F\colon \mathcal{A}\times\mathcal{A}' \to \mathcal{A}''$ be a functor that is additive in each variable. Then F extends to a functor $C^2(F)\colon C(\mathcal{A})\times C(\mathcal{A}') \to C^2(\mathcal{A}'')$ additive in each variable. For $X\in C(\mathcal{A})$, $Y\in C(\mathcal{A}')$, the double complex $C^2(F)(X,Y)$ is defined by $C^2(F)(X,Y)^{i,j} = F(X^i,Y^j)$, with $d_I^{i,j} = F(d_X^i, \mathrm{id}_{Y^j})$, $d_{II}^{i,j} = F(\mathrm{id}_{X^i}, d_Y^j)$.

Example 2.8.7. Let R be a ring. The functor $-\otimes_R -\colon \mathbf{Mod}\text{-}R \times R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is additive in each variable. Thus it extends to

$$-\otimes_R -\colon C(\mathbf{Mod}\text{-}R)\times C(R\text{-}\mathbf{Mod}) \to C^2(\mathbf{Ab}).$$

Example 2.8.8. Let $\mathcal{A}$ be an additive category with small Hom sets. The functor $\mathrm{Hom}_\mathcal{A}\colon \mathcal{A}^{\mathrm{op}}\times\mathcal{A} \to \mathbf{Ab}$ is additive in each variable. We have an isomorphism $C(\mathcal{A})^{\mathrm{op}} \simeq C(\mathcal{A}^{\mathrm{op}})$, carrying (X,d) to $((X^{-n}),(-1)^n d^{-n-1})$. Thus $\mathrm{Hom}_\mathcal{A}$ extends to a functor

$$\mathrm{Hom}^{\bullet\bullet}_\mathcal{A}\colon C(\mathcal{A})^{\mathrm{op}}\times C(\mathcal{A}) \to C^2(\mathbf{Ab}),$$

additive in each variable. For $X, Y \in C(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}^{\bullet\bullet}(X,Y)^{i,j} = \mathrm{Hom}_{\mathcal{A}}(X^{-j}, Y^i)$, with

$$d_I^{i,j} = \mathrm{Hom}_{\mathcal{A}}(X^{-j}, d_Y^i), \quad d_{II}^{i,j} = \mathrm{Hom}_{\mathcal{A}}((-1)^j d_X^{-j-1}, Y^i).$$

We define $\mathrm{Hom}_{\mathcal{A}}^{\bullet}$ as the composite functor

$$C(\mathcal{A})^{\mathrm{op}} \times C(\mathcal{A}) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}^{\bullet\bullet}} C^2(\mathbf{Ab}) \xrightarrow{\mathrm{tot}_{\Pi}} C(\mathbf{Ab}).$$

We have

$$\mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y)^n = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^j, Y^{n+j}),$$

and for $f = (f^j) \in \mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y)^n$,

$$(d^n f)^j = d_Y^{j+n} f^j + (-1)^{n+1} f^{j+1} d_X^j.$$

Proposition 2.8.9. *We have*

$$\begin{aligned}
Z^0\mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y) &\simeq \mathrm{Hom}_{C(\mathcal{A})}(X,Y),\\
B^0\mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y) &\simeq \mathrm{im}(\mathrm{Ht}(X,Y) \to \mathrm{Hom}_{C(\mathcal{A})}(X,Y)),\\
H^0\mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y) &\simeq \mathrm{Hom}_{K(\mathcal{A})}(X,Y).
\end{aligned}$$

Proof. We have $d^0(f) = df - fd$, so that $d^0(f) = 0$ if and only if $f\colon X \to Y$ is a morphism of complexes. We have $\mathrm{Ht}(X,Y) = \mathrm{Hom}_{\mathcal{A}}^{\bullet}(X,Y)^{-1}$, and for $h \in \mathrm{Ht}(X,Y)$, $d^{-1}(h) = dh + hd$. ☐

Definition 2.8.10. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be triangulated categories. A *triangulated bifunctor* is a functor $F\colon \mathcal{D}\times\mathcal{D}' \to \mathcal{D}''$ equipped with natural isomorphisms $F(X[1],Y) \simeq F(X,Y)[1]$, $F(X,Y[1]) \simeq F(X,Y)[1]$, such that the following diagram *anticommutes*

$$\begin{tikzcd}
F(X[1],Y[1]) \arrow[r] \arrow[d] & F(X,Y[1])[1] \arrow[d] \\
F(X[1],Y)[1] \arrow[r] & F(X,Y)[2]
\end{tikzcd}$$

and such that F is triangulated in each variable.

A natural transformation of triangulated bifunctors $F \to F'$ is a natural transformation $\alpha\colon F \to F'$ such that for all X an Y,

$$\alpha_{X,-}\colon F(X,-) \to F'(X,-), \quad \alpha_{-,Y}\colon F(-,Y) \to F'(-,Y)$$

are natural transformations of triangulated functor.

Triangulated bifunctors $\mathcal{D}\times\mathcal{D}' \to \mathcal{D}''$ form a category $\mathrm{TrBiFun}(\mathcal{D},\mathcal{D}';\mathcal{D}'')$.

Note that $\mathrm{Hom}^{\bullet}$ factorizes through a triangulated bifunctor

$$K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathbf{Ab}).$$

Derived Hom

Definition 2.8.11. Let $F\colon \mathcal{K}\times\mathcal{K}' \to \mathcal{K}''$ be a triangulated bifunctor and let $\mathcal{N}\subseteq\mathcal{K}$, $\mathcal{N}\subseteq\mathcal{K}'$, $\mathcal{N}''\subseteq\mathcal{K}''$ be full triangulated subcategories. Let $Q\colon \mathcal{K}\to\mathcal{K}/\mathcal{N}$, $Q'\colon \mathcal{K}'\to\mathcal{K}'/\mathcal{N}'$, and $Q''\colon \mathcal{K}''\to\mathcal{K}''/\mathcal{N}''$ be the localization functors. A *right derived bifunctor* of F with respect to $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}'$ is an initial object of $(Q''F\downarrow -(Q\times Q'))$, where $-(Q\times Q')\colon \mathrm{TrBiFun}(\mathcal{K}/\mathcal{N},\mathcal{K}'/\mathcal{N}';\mathcal{K}''/\mathcal{N}'')\to \mathrm{TrBiFun}(\mathcal{K},\mathcal{K}';\mathcal{K}''/\mathcal{N}'')$ is the functor induced by composition with $Q\times Q'$. In other words, a right derived bifunctor is a pair (RF,ϵ), where $RF\colon \mathcal{K}/\mathcal{N}\times\mathcal{K}'/\mathcal{N}'\to\mathcal{K}''/\mathcal{N}''$ is a triangulated bifunctor and $\epsilon\colon Q''F\to(RF)(Q\times Q')$ is a natural transformation of triangulated bifunctors, as shown in the diagram

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{K}' \arrow[r, "Q\times Q'"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\epsilon}{\Rightarrow}"] & \mathcal{K}/\mathcal{N}\times\mathcal{K}'/\mathcal{N}' \arrow[d, "RF"] \\
\mathcal{K}'' \arrow[r, "Q''"] & \mathcal{K}''/\mathcal{N}'',
\end{tikzcd}$$

such that for every pair (G,η), where $G\colon \mathcal{K}/\mathcal{N}\times\mathcal{K}'/\mathcal{N}'\to\mathcal{K}''/\mathcal{N}''$ is a triangulated bifunctor and $\eta\colon Q''F\to G(Q\times Q')$ is a natural transformation of triangulated bifunctors, there exists a unique natural transformation of triangulated bifunctors $\alpha\colon RF\to G$ such that $\eta=(\alpha(Q\times Q'))\epsilon$.

Dually, a *left derived functor* of F with respect to $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}''$ is a final object of $(-(Q\times Q')\downarrow Q''F)$.

Right (resp. left) derived bifunctors of F with respect to $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}'$ are unique up to unique natural isomorphism.

Remark 2.8.12. The above definition only depends on the category $\mathcal{K}''$ via its localization $\mathcal{K}''/\mathcal{N}''$.

Proposition 2.8.13. *Let $F\colon \mathcal{K}\times\mathcal{K}'\to\mathcal{K}''$ be a triangulated bifunctor and let $\mathcal{N}\subseteq\mathcal{K}$, $\mathcal{N}'\subseteq\mathcal{K}''$, $\mathcal{N}'\subseteq\mathcal{K}''$ be full triangulated subcategories such that* $\mathrm{Ob}(\mathcal{N})$ *stable under isomorphisms. Let $\mathcal{J}\subseteq\mathcal{K}$ be a full triangulated subcategory satisfying the following conditions:*

(1) For every $X\in\mathcal{K}$, there exists $X\to Y$ in $S_{\mathcal{N}}$ with $Y\in\mathcal{J}$.
(2) For $Y\in\mathcal{J}\cap\mathcal{N}$ and $X'\in\mathcal{K}'$, we have $F(Y,X')\in\mathcal{N}''$.
(3) For $Y\in\mathcal{J}$ and $X'\in\mathcal{N}'$, we have $F(Y,X')\in\mathcal{N}''$.

Then the right derived bifunctor $(RF\colon \mathcal{K}/\mathcal{N}\times\mathcal{K}'/\mathcal{N}'\to\mathcal{K}''/\mathcal{N}'',\epsilon)$ exists and the restriction of ϵ to $\mathcal{J}\times\mathcal{K}'$ is a natural isomorphism.

Under the assumptions of the proposition, for every $X'\in\mathcal{K}'$, $RF(-,X')\colon \mathcal{K}/\mathcal{N}\to\mathcal{K}''/\mathcal{N}''$ is a right derived functor of $F(-,X')\colon \mathcal{K}\to\mathcal{K}''$.

Proof. The proof is very similar to the proof of Proposition 2.7.3. By (1) and Lemma 2.5.20, the inclusion $\iota\colon \mathcal{J}\to\mathcal{K}$ induces an equivalence of triangulated categories $\phi\colon \mathcal{J}/\mathcal{J}\cap\mathcal{N}\to\mathcal{K}/\mathcal{N}$. By (2) and (3), $F(\iota\times\mathrm{id}_{\mathcal{K}'})$ induces a triangulated bifunctor $\bar{F}\colon \mathcal{J}/\mathcal{J}\cap\mathcal{N}\times\mathcal{K}'/\mathcal{N}'\to\mathcal{K}''/\mathcal{N}''$. We define $RF=\bar{F}\psi$, where ψ is a quasi-inverse of ϕ. $\square$

Let $\mathcal{A}$ be an abelian category with small Hom sets.

Proposition 2.8.14. *Assume that $\mathcal{A}$ admits enough injectives. Then the triangulated bifunctor*

$$\mathrm{Hom}^\bullet_{\mathcal{A}}\colon K(\mathcal{A})^{\mathrm{op}} \times K^+(\mathcal{A}) \to K(\mathbf{Ab})$$

admits a right derived bifunctor

$$R\mathrm{Hom}_{\mathcal{A}}\colon D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathbf{Ab})$$

such that, for $M \in K^+(\mathcal{A})$ with injective components and $L \in K(\mathcal{A})$, we have

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{A}}(L, M).$$

Proof. We denote by $\mathcal{I} \subseteq \mathcal{A}$ the full subcategory spanned by injective objects. We apply Proposition 2.8.13 to

$$\mathcal{K} = \mathcal{K}^+(\mathcal{A}), \quad \mathcal{K}' = \mathcal{K}(\mathcal{A})^{\mathrm{op}}, \quad \mathcal{K}'' = \mathcal{K}(\mathbf{Ab}), \quad \mathcal{J} = \mathcal{K}^+(\mathcal{I}),$$

and $\mathcal{N} \subseteq \mathcal{K}$, $\mathcal{N}' \subseteq \mathcal{K}'$, $\mathcal{N}'' \subseteq \mathcal{K}''$ the full subcategories spanned by acyclic complexes. Let us check the three assumptions.

(1) This follows from Theorem 2.5.19.

(2) We need to show that for $L \in K(\mathcal{A})$, $M \in K^+(\mathcal{I})$ acyclic, $\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M)$ is acyclic. By Proposition 2.5.23, $M = 0$ in $K(\mathcal{A})$. Thus

$$H^n\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \simeq \mathrm{Hom}_{K(\mathcal{A})}(L, M[n]) = 0.$$

(3) We need to show that for $L \in K(\mathcal{A})$ acyclic, $M \in K^+(\mathcal{I})$, $\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M)$ is acyclic. By Proposition 2.5.28, we have

$$H^n\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \simeq \mathrm{Hom}_{K(\mathcal{A})}(L, M[n]) = 0.$$

□

Remark 2.8.15. Assume that $\mathcal{A}$ has enough injectives. For $L \in D(\mathcal{A})$, $M \in D^+(\mathcal{A})$, we have

$$\begin{aligned} H^n R\mathrm{Hom}_{\mathcal{A}}(L, M) \simeq H^n\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M') &\simeq \mathrm{Hom}_{K(\mathcal{A})}(L, M'[n]) \\ &\simeq \mathrm{Hom}_{D(\mathcal{A})}(L, M'[n]) \simeq \mathrm{Ext}^n(L, M), \end{aligned}$$

where we have taken a quasi-isomorphism $M \to M' \in K^+(\mathcal{A})$ such that M' has injective components, and in the third isomorphism we used the fact that M' is homotopically injective (Proposition 2.5.28). In particular, for $X \in \mathcal{A}$, $\mathrm{Ext}^n(X, -)$ is the n-th right derived functor of $\mathrm{Hom}(X, -)$.

Remark 2.8.16. If there are enough homotopically injective complexes, then the triangulated bifunctor

$$\mathrm{Hom}^\bullet_{\mathcal{A}}\colon K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathbf{Ab})$$

admits a right derived bifunctor

$$R\mathrm{Hom}_{\mathcal{A}}\colon D(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathbf{Ab})$$

such that, for $M \in K_{\mathrm{hi}}(\mathcal{A})$ and $L \in K(\mathcal{A})$, we have

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{A}}(L, M).$$

Dually, we have the following.

Proposition 2.8.17. *Assume that $\mathcal{A}$ admits enough projectives. Then the triangulated bifunctor*

$$\mathrm{Hom}^\bullet_{\mathcal{A}}\colon K^-(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathbf{Ab}).$$

admits a right derived bifunctor

$$R\mathrm{Hom}_{\mathcal{A}}\colon D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathbf{Ab})$$

such that for $L \in K^-(\mathcal{A})$ with projective components and $M \in K(\mathcal{A})$, we have

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(L, M) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{A}}(L, M).$$

Remark 2.8.18. In the case where $\mathcal{A}$ admits enough injectives and enough projectives, the functors $R\mathrm{Hom}$ defined in Propositions 2.8.14 and 2.8.17 are isomorphic when restricted to $D^-(\mathcal{A}) \times D^+(\mathcal{A})$, by the universal property of right derived functors. Here is another proof. For $L \in D^-(\mathcal{A})$ and $M \in D^+(\mathcal{A})$, $R\mathrm{Hom}(L, M)$ can be computed by finding quasi-isomorphisms $L' \to L$ and $M \to M'$ such that $L' \in C^-(\mathcal{A})$ has projective components and $M' \in C^+(\mathcal{A})$ has injective components and taking $\mathrm{Hom}^\bullet(L', M')$.

Remark 2.8.19. (1) If $\mathcal{A}$ admits enough injectives or enough projectives, then $R\mathrm{Hom}_{\mathcal{A}}(-,-)$ carries $(D^{\le m}(\mathcal{A}))^{\mathrm{op}} \times D^{\ge n}(\mathcal{A})$ to $D^{\ge n-m}(\mathbf{Ab})$.
(2) If $\mathcal{A}$ admits enough projectives, then an object X of $\mathcal{A}$ satisfies $\mathrm{proj.dim}(X) \le d$ if and only if $R\mathrm{Hom}_{\mathcal{A}}(X, -)$ carries $D^{\le 0}(\mathcal{A})$ to $D^{\le d}(\mathbf{Ab})$.
(3) If $\mathcal{A}$ admits enough injectives, then an object Y of $\mathcal{A}$ satisfies $\mathrm{inj.dim}(Y) \le d$ if and only if $R\mathrm{Hom}_{\mathcal{A}}(-, Y)$ carries $D^{\ge 0}(\mathcal{A})$ to $D^{\le d}(\mathbf{Ab})$.

Remark 2.8.20. Assume that $\mathcal{A}$ admits enough injectives. We have an isomorphism $R\mathrm{Hom}_{\mathcal{A}}(L, M) \simeq R\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M, L)$, natural in $L \in D(\mathcal{A})$ and $M \in D^+(\mathcal{A})$. Here on the right hand side we regard $L \in D(\mathcal{A}^{\mathrm{op}})$ and $M \in D^-(\mathcal{A}^{\mathrm{op}})$.

Proposition 2.8.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories admitting enough injectives and let $F\colon \mathcal{B} \to \mathcal{A}$ be an exact functor admitting a right adjoint $G\colon \mathcal{A} \to \mathcal{B}$. Then for $X \in D(\mathcal{B})$ and $Y \in D^+(\mathcal{A})$, we have*

$$R\mathrm{Hom}_{D(\mathcal{A})}(FX, Y) \simeq R\mathrm{Hom}_{D(\mathcal{B})}(X, RGY),$$
$$\mathrm{Hom}_{D(\mathcal{A})}(FX, Y) \simeq \mathrm{Hom}_{D(\mathcal{B})}(X, RGY).$$

In particular, $RG\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is a right adjoint of the functor $F\colon D^+(\mathcal{B}) \to D^+(\mathcal{A})$.

Proof. We may replace Y by a complex in $K^+(\mathcal{A})$ with injective components. Then RGY is computed by GY. Since F is exact, G preserves injectives, so that GY is in $K^+(\mathcal{B})$ with injective components. The first isomorphism is given by

$$\mathrm{Hom}^{\bullet\bullet}_{\mathcal{A}}(FX, Y) \simeq \mathrm{Hom}^{\bullet\bullet}_{\mathcal{B}}(X, GY), \quad \mathrm{Hom}^{\bullet}_{\mathcal{A}}(FX, Y) \simeq \mathrm{Hom}^{\bullet}_{\mathcal{B}}(X, GY).$$

Applying H^0, we get the second isomorphism. □

Example 2.8.22. Let $f\colon X \to Y$ be a continuous map. The functor

$$Rf_*\colon D^+(\mathrm{Shv}(X)) \to D^+(\mathrm{Shv}(Y))$$

is a right adjoint of the functor $f^*\colon D^+(\mathrm{Shv}(Y)) \to D^+(\mathrm{Shv}(X))$.

Examples

Let $\mathcal{A} = R$-**Mod**, where R is a ring.

Example 2.8.23. Let R be a ring. Let $r \in R$ be an element that is not a right zero-divisor. In other words, $R \xrightarrow{\times r} R$ is an injection. Then the R-module R/Rr has the following projective resolution: $0 \to R \xrightarrow{\times r} R \to R/Rr \to 0$. For any R-module M, $R\mathrm{Hom}_R(R/Rr, M)$ is computed by the complex $M \xrightarrow{r\times} M$ put in degrees 0 and 1.

If R is a PID and $s \in R$ is nonzero, then

$$\mathrm{Ext}^0_R(R/rR, R/sR) \simeq \frac{s}{(r,s)} R/sR \simeq R/(r,s)R \simeq \mathrm{Ext}^1_R(R/rR, R/sR)$$

and, by Proposition 2.6.29, we have $R\mathrm{Hom}(R/rR, R/sR) \simeq R/(r,s)R \oplus (R/(r,s)R)[-1]$. Here (r,s) denotes a greatest common divisor of r and s, namely $(r,s)R = rR + sR$.

Example 2.8.24. Let $R = \mathbb{Z}$. Then $\mathbb{Z}$ has the following injective resolution: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Thus $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$ is computed by the complex $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ put in degrees 0 and 1. We have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) &= 0, \\ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\operatorname*{colim}_{n \in \mathbb{N}^\times} \frac{1}{n}\mathbb{Z}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) &\simeq \lim_{n \in (\mathbb{N}^\times)^{\mathrm{op}}} \mathrm{Hom}_{\mathbb{Z}}(\frac{1}{n}\mathbb{Z}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \\ &\simeq \lim_{n \in (\mathbb{N}^\times)^{\mathrm{op}}} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}. \end{aligned}$$

Here $\mathbb{N}^\times$ is the set of positive integers, ordered by divisibility, the third isomorphism is induced by the pairing

$$\mathbb{Z}/n\mathbb{Z} \times \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

sending $(a \mod n, b \mod 1)$ to $ab \mod 1$, the transition map $\frac{1}{m}\mathbb{Z}/\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ for $m \mid n$ is the inclusion, and the transition map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$ sends $a \mod n$ to $a \mod m$. $\hat{\mathbb{Z}}$ is called the *profinite completion* of $\mathbb{Z}$. In summary, $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \simeq \hat{\mathbb{Z}}[-1]$.

Example 2.8.25. Let A be a ring and let $R = A[x,y]$. Consider the bilateral ideal $\mathfrak{m} = Rx + Ry$. The quotient $R/\mathfrak{m}$ admits a free resolution

$$0 \to R \xrightarrow{(y,-x)} R^2 \xrightarrow{(x,y)} R \to R/\mathfrak{m} \to 0.$$

Thus $R\mathrm{Hom}_R(R/\mathfrak{m}, R)$ is computed by the complex $R \xrightarrow{(x,y)} R^2 \xrightarrow{(y,-x)} R$ in degrees $0, 1, 2$. Therefore, $R\mathrm{Hom}_R(R/\mathfrak{m}, R) \simeq R/\mathfrak{m}[-2]$. In the commutative case, the free resolution above is a special case of the Koszul complex. See the next heading.

Now consider $\mathfrak{m}^2 = Rx^2 + Rxy + Ry^2$. The quotient $R/\mathfrak{m}^2$ admits a free resolution

$$0 \to R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{(x^2, xy, y^2)} R \to R/\mathfrak{m}^2 \to 0.$$

Thus $R\mathrm{Hom}_R(R/\mathfrak{m}^2, R/\mathfrak{m})$ is computed by the complex

$$A \xrightarrow{0} A^3 \xrightarrow{0} A^2$$

in degrees $0, 1, 2$.

Example 2.8.26. Let A be a ring and $R = A[t]/(t^n)$. For $a \leq n$, the R-module R/Rt^a admits a free resolution

$$\cdots \to R \xrightarrow{t^a} R \xrightarrow{t^{n-a}} R \xrightarrow{t^a} R \to R/t^aR \to 0. \tag{2.8.3}$$

Thus, for $b \leq n$, $R\mathrm{Hom}_R(R/Rt^a, R/Rt^b)$ is computed by the complex

$$\cdots \to 0 \to R/Rt^b \xrightarrow{t^a} R/Rt^b \xrightarrow{t^{n-a}} R/Rt^b \xrightarrow{t^a} R/Rt^b \to \cdots$$

in degrees ≥ 0. For $i \geq 0$ even,

$$\ker(d^i) = Rt^{\max(b-a,0)}/Rt^b, \quad \mathrm{im}(d^i) = Rt^{\min(a,b)}/Rt^b.$$

For $i \geq 1$ odd,

$$\ker(d^i) = Rt^{\max(a+b-n,0)}/Rt^b, \quad \mathrm{im}(d^i) = Rt^{\min(n-a,b)}/Rt^b.$$

Thus

$$\mathrm{Ext}^i_R(R/Rt^a, R/Rt^b) \simeq \begin{cases} R/Rt^{\min(a,b)} & i = 0 \\ R/Rt^{\min(a,b)-\max(a+b-n,0)} & i \geq 1 \\ 0 & i < 0. \end{cases}$$

In particular, for A nonzero and $0 < a < n$, $\mathrm{proj.dim}_R(R/Rt^a) = \mathrm{inj.dim}_R(R/Rt^a) = \infty$.

That all Ext^i, $i \geq 1$ are isomorphic has the following explanation, at least when $A = k$ is a field. If L is a complex in an abelian category with L^i of finite length, then

$$\lg(H^nL) = \lg(Z^nL) - \lg(B^nL) = \lg(L^n) - \lg(B^nL) - \lg(B^{n+1}L).$$

Example 2.8.27. Let A be a ring and $R = A[x,y]/(xy)$. The R-module R/Rx admits a free resolution

$$\cdots \to R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/Rx \to 0. \tag{2.8.4}$$

Thus $R\mathrm{Hom}_R(R/Rx, R/Ry)$ is computed by the complex

$$\cdots \to 0 \to R/Ry \xrightarrow{x} R/Ry \xrightarrow{0} R/Ry \xrightarrow{x} \cdots$$

in degrees ≥ 0 and $R\mathrm{Hom}_R(R/Rx, R/Ry) \simeq \bigoplus_{i=0}^{\infty} R/\mathfrak{m}[-(1+2i)]$, where $\mathfrak{m} = Rx + Ry$.

Note that $\mathfrak{m} = Rx \oplus Ry \simeq R/Ry \oplus R/Rx$. Thus the R-module $R/\mathfrak{m}$ admits a free resolution

$$\cdots \to R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{(x,y)} R \to R/\mathfrak{m} \to 0.$$

It follows that $R\mathrm{Hom}_R(R/\mathfrak{m}, R/\mathfrak{m})$ is computed by the complex

$$\cdots \to 0 \to R/\mathfrak{m} \xrightarrow{0} (R/\mathfrak{m})^2 \xrightarrow{0} (R/\mathfrak{m})^2 \xrightarrow{0} (R/\mathfrak{m})^2 \to \cdots$$

in degrees ≥ 0.

The resolutions (2.8.3) and (2.8.4) admit the following partial generalization. Let A be a unique factorization domain and $R = A/(xy)$, where x, y are nonzero elements of A. Then the R-module R/Rx admits a free resolution

$$\cdots \to R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/Rx \to 0.$$

Koszul complexes

Let R be a commutative ring, E an R-module, and $\phi\colon E \to R$ an R-module homomorphism. The *Koszul complex* $K(\phi)$ associated to ϕ is the complex in $C^{\le 0}(R\text{-}\mathbf{Mod})$ defined by $K(\phi)^{-n} = \bigwedge_R^n E$ with differentials given by

$$d^{-n}(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n} (-1)^{i-1} \phi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n.$$

Here $e_1, \dots, e_n \in E$. The differentials are well defined, because the right-hand side of the above formula is zero whenever $e_i = e_j$ for some $i < j$. The complex has the form

$$\cdots \to \bigwedge_R^2 E \xrightarrow{d^{-2}} E \xrightarrow{\phi} R \to 0 \cdots .$$

We are mostly interested in the case where $E = R^r$ is a free module of finite rank. In this case, $\phi\colon R^r \to R$ is given by $f_1, \dots, f_r \in R$ and we denote $K(\phi)$ by $K(f_1, \dots, f_r)$ or $K(R, f_1, \dots, f_r)$. Up to isomorphism, this does not depend on the order of $f_1, \dots, f_r$. We have $K(f_1, \dots, f_r)^{-n} = \bigwedge_R^n R^r \simeq R^{\binom{r}{n}}$ and $K(f_1, \dots, f_r) \in C^{[-r,0]}(R\text{-}\mathbf{Mod})$.

Example 2.8.28. In the case $r = 1$, for $f \in R$, $K(f)$ is the complex $R \xrightarrow{\times f} R$ put in degrees -1 and 0.

Lemma 2.8.29. *Let E be an R-module and $\phi\colon E \to R$ an R-module homomorphism. Let $E' = E \oplus R$ and $\phi' = (\phi, f)\colon E' \to R$, where $f \in R$. Then $K(\phi') \simeq \mathrm{Cone}(K(\phi) \xrightarrow{\times f} K(\phi))$. In other words, $K(\phi') \simeq \mathrm{tot}(K(\phi) \otimes_R K(f))$.*

Proof. Let C denote the cone and let $e_0 = (0, 1) \in E'$. By definition, we have

$$C^{-n} = K(\phi)^{-n+1} \oplus K(\phi)^{-n} = \bigwedge^{n-1} E \oplus \bigwedge^{n} E \xrightarrow[\sim]{\alpha^{-n}} \bigwedge^{n} E' = K(\phi')^{-n},$$

where $\alpha^{-n}(e_1 \wedge \cdots \wedge e_{n-1}, 0) = e_0 \wedge e_1 \wedge \cdots \wedge e_{n-1}$ and $\alpha^{-n}|_{\wedge^n E}$ is induced by the inclusion $E \hookrightarrow E'$. It remains to check that the isomorphism commutes with differentials, namely $\alpha^{-n+1} d_C^{-n} = d_{K(\phi')}^{-n} \alpha^{-n}$. This is clear on $\bigwedge^n E$. Moreover,

$$\begin{aligned}
\alpha^{-n+1}(d_C^{-n}(e_1 \wedge \dots e_{n-1}, 0)) &= \alpha^{-n+1}(-d_{K(\phi)^{-n+1}}(e_1 \wedge \dots e_{n-1}), f e_1 \wedge \dots e_{n-1}) \\
&= f e_1 \wedge \dots e_{n-1} - \sum_{i=1}^{n} \phi(e_i) e_0 \wedge \cdots \wedge \widehat{e_i} \wedge e_{n-1} \\
&= d_{K(\phi')}(e_0 \wedge e_1 \wedge \cdots \wedge e_{n-1}) = d_{K(\phi')}(\alpha^{-n}(e_1 \wedge \dots e_{n-1}, 0)).
\end{aligned}$$

□

Remark 2.8.30. By induction, $K(f_1,\dots,f_r)$ is obtained from $K(f_1)$ by successively tensoring with $K(f_2),\dots,K(f_r)$ and taking tot after each tensoring. In fact, we have

$$K(f_1,\dots,K_r)\simeq \operatorname{tot}(K(f_1)\otimes_R\cdots\otimes_R K(f_r)),$$

where tot denotes the totalization of m-uple complexes. We leave it to the reader to work out the sign rule.

Definition 2.8.31. Let R be a ring and let M be an R-module. A sequence of central elements $f_1,\dots,f_r$ of R is called an *M-regular sequence* if $M/(f_1,\dots,f_r)M$ is not zero and the map $M/(f_1,\dots,f_{i-1})M\xrightarrow{f_i\times}M/(f_1,\dots,f_{i-1})M$ is an injection for each $i=1,\dots,r$. An R-regular sequence is called a *regular sequence*.

Proposition 2.8.32. *Let R be a commutative ring, M an R-module, and $f_1,\dots,f_r$ an M-regular sequence in R. Then $H^i(K(f_1,\dots,f_r)\otimes_R M)=0$ for $i<0$ and $K(f_1,\dots,f_r)\otimes_R M$ is a left resolution of $M/(f_1,\dots,f_r)M$.*

Proof. We proceed by induction. The case $r=0$ is trivial. Assume that $K(f_1,\dots,f_{r-1})\otimes_R M$ is a resolution of $M/(f_1,\dots,f_{r-1})M$. By Lemma 2.8.29, $K(f_1,\dots,f_r)$ is isomorphic to the cone of $K(f_1,\dots,f_{r-1})\otimes_R M\xrightarrow{f_r\times}K(f_1,\dots,f_{r-1})\otimes_R M$. Moreover, by M-regularity, we have a short exact sequence

$$0\to M/(f_1,\dots,f_{r-1})M\to M/(f_1,\dots,f_{r-1})M\to M/(f_1,\dots,f_r)M\to 0.$$

Thus we have a morphism of distinguished triangles in $D(R\text{-}\mathbf{Mod})$

$$\begin{tikzcd}
K(f_1,\dots,f_{r-1})\otimes_R M \arrow[r, "f_r\times"] \arrow[d] & K(f_1,\dots,f_{r-1})\otimes_R M \arrow[r] \arrow[d] & K(f_1,\dots,f_r)\otimes_R M \arrow[r, "+1"] \arrow[d, dashed] & K(f_1,\dots,f_{r-1})\otimes_R M[1] \arrow[d] \\
M/(f_1,\dots,f_{r-1})M \arrow[r, "f_r\times"] & M/(f_1,\dots,f_{r-1})M \arrow[r] & M/(f_1,\dots,f_r)M \arrow[r, "+1"] & M/(f_1,\dots,f_{r-1})M[1]
\end{tikzcd}$$

where the dashed arrow is an isomorphism in $D(R\text{-}\mathbf{Mod})$. □

Remark 2.8.33. The converse of Proposition 2.8.32 holds under the additional assumptions that R is a commutative Noetherian local ring, $f_1,\dots,f_r$ belong to the maximal ideal of R, and $M\neq 0$ is a finitely generated R-module. Under these assumptions, $H^{-1}(K(f_1,\dots,f_r)\otimes_R M)=0$ implies that $f_1,\dots,f_r$ is M-regular. See [M1, Theorem 16.5]. Thus, under these assumptions, any permutation of an M-regular sequence is M-regular.

Warning 2.8.34. In general a permutation of an M-regular sequence is not necessarily M-regular. For example, for k a field and $R=k[x,y,z]$, the sequence $xy,x-1,xz$ is regular but $xy,xz,x-1$ is not, because the image of xz in $R/(xy)$ is a zero-divisor.

Corollary 2.8.35. *Let R be a commutative ring and I an ideal generated by a regular sequence $f_1,\dots,f_r$ in R. Let N be an R/I-module. Then we have $R\mathrm{Hom}_R(R/I,N)\simeq\bigoplus_{n=0}^r N^{\binom{r}{n}}[-n]$. Moreover,* $\operatorname{proj.dim}_R(R/I)=r$.

Proof. By Proposition 2.8.32, $R\mathrm{Hom}_R(R/I,N) \simeq \mathrm{Hom}^\bullet_R(K(f_1,\dots,f_r),N)$. We have $\mathrm{Hom}^\bullet_R(K(f_1,\dots,f_r),N)^n = \mathrm{Hom}_R(\bigwedge^n E^r,N) \simeq N^{\binom{r}{n}}$ and the differentials are zero.

By Proposition 2.8.32, $\mathrm{proj.dim}_R(R/I) \le r$. Taking $N = R/I$ (or any other nonzero N), we get $\mathrm{proj.dim}_R(R/I) = r$. □

The above result on projective dimension extends to the noncommutative case. See Corollary 2.9.31.

Remark 2.8.36. Let R be a commutative ring, E a free R-module of rank r, and $\phi\colon E \to R$ an R-module homomorphism. We have the following form of Poincaré duality. Choose an isomorphism $\int\colon \bigwedge^r E \xrightarrow{\sim} R$. Then we have an isomorphism of complexes

$$K(\phi)[-r] \simeq \mathrm{Hom}^\bullet_R(K(\phi),R)$$

sending $a \in K(\phi)[-r]^n = \bigwedge^{r-n} E$ to the element of $\mathrm{Hom}^\bullet_R(K(\phi),R)^n = \mathrm{Hom}_R(\bigwedge^n E,R)$ given by $b \mapsto (-1)^n \int(a\wedge b)$. To check that this is a morphism of complexes, note that for $c \in \bigwedge^{n+1} E$, we have $0 = d(a\wedge c) = da\wedge c + (-1)^{r-n} a \wedge dc$.

More generally, for any R-module M, we have an isomorphism of complexes

$$K(\phi)\otimes_R M[-r] \simeq \mathrm{Hom}^\bullet_R(K(\phi),M).$$

In degree n, this is the isomorphism

$$\bigwedge^{r-n} E\otimes_R M \simeq \mathrm{Hom}_R(\bigwedge^n E,R)\otimes_R M \simeq \mathrm{Hom}_R(\bigwedge^n E,M).$$

2.9 Derived tensor product

Double complexes and acyclicity

Let $\mathcal{A}$ be an abelian category. For a double complex X in $\mathcal{A}$, we put

$$H_I(X)^{i,j} = \ker(d_I^{i,j})/\mathrm{im}(d_I^{i-1,j}),\quad H_{II}(X)^{i,j} = \ker(d_{II}^{i,j})/\mathrm{im}(d_{II}^{i,j-1}).$$

The full additive subcategory $C^2_{\mathrm{reg}}(\mathcal{A}) \subseteq C^2(\mathcal{A})$ is stable under subobjects and quotients. Thus $C^2_{\mathrm{reg}}(\mathcal{A})$ is an abelian category and the inclusion functor is exact. The functor $\mathrm{tot}\colon C^2_{\mathrm{reg}}(\mathcal{A}) \to C(\mathcal{A})$ is exact.

Proposition 2.9.1. *Let X be a biregular double complex such that $H_I^{i,\bullet}(X)$ is acyclic for every i. Then* $\mathrm{tot}(X)$ *is acyclic.*

A similar statement holds for H_{II}, which generalizes the fact that the cone of a quasi-isomorphism is acyclic.

Proof. For each m, there exists N such that $H^m(\mathrm{tot}(X)) = H^m\mathrm{tot}(\tau_I^{\le n}X)$ for all $n \ge N$. It suffices to show that $H^m\mathrm{tot}(\tau_I^{\le n}X) = 0$ for all n. We proceed by induction on n (for a fixed m). For $n \ll 0$, $(\mathrm{tot}(\tau_I^{\le n}X))^m = 0$. Assume that $H^m\tau_I^{\le n-1}(X) = 0$ and consider the short exact sequence of double complexes

$$0 \to \tau_I^{\le n-1}X \to \tau_I^{\le n}X \to Y \to 0,$$

where $Y = (B_I^{n,\bullet}X \xrightarrow{f} Z_I^{n,\bullet}X)$ is concentrated on the columns $n-1$ and n. Applying tot, we get an exact sequence of complexes

$$0 \to \operatorname{tot}(\tau_I^{\leq n-1}X) \to \operatorname{tot}(\tau_I^{\leq n}X) \to \operatorname{tot}(Y) \to 0. \tag{2.9.1}$$

We have a quasi-isomorphism $\operatorname{tot}(Y)[n] \simeq \operatorname{Cone}((-1)^n f) \to H_I^{n,\bullet}(X)$. It follows that $\operatorname{tot}(Y)$ is acyclic. Taking the long exact sequence associated to (2.9.1), we get

$$H^m\operatorname{tot}(\tau_I^{\leq n}X) \simeq H^m\operatorname{tot}(\tau_I^{\leq n-1}X) = 0.$$

□

Corollary 2.9.2. *Let X be a biregular double complex such that $X^{\bullet,j}$ is acyclic for every j (namely, every row of X is acyclic). Then $\operatorname{tot}(X)$ is acyclic.*

A similar statement holds for columns of X: if $X^{i,\bullet}$ is acyclic for every i, then $\operatorname{tot}(X)$ is acyclic.

Proof. By assumption, $H_I^{i,j}(X) = 0$ and the proposition applies. □

Corollary 2.9.3. *Let $f\colon X \to Y$ be a morphism of biregular double complexes such that $H_I^{i,\bullet}(f)\colon H_I^{i,\bullet}(X) \to H_I^{i,\bullet}(Y)$ is a quasi-isomorphism for each i. Then $\operatorname{tot}(f)\colon \operatorname{tot}(X) \to \operatorname{tot}(Y)$ is a quasi-isomorphism.*

Proof. We let $W = \operatorname{Cone}_{II}(f)$ with $W^{i,j} = X^{i,j+1} \oplus Y^{i,j}$. Then $H_I^{i,\bullet}(W) \simeq \operatorname{Cone}(H_I^{i,\bullet}(f))$ is acyclic. By the proposition applied to W, $\operatorname{tot}(W) \simeq \operatorname{Cone}(\operatorname{tot}(f))$ is acyclic. □

Corollary 2.9.4. *Let $f\colon X \to Y$ be a morphism of biregular double complexes such that $f^{\bullet,j}\colon X^{\bullet,j} \to Y^{\bullet,j}$ is a quasi-isomorphism for each j. Then $\operatorname{tot}(f)\colon \operatorname{tot}(X) \to \operatorname{tot}(Y)$ is a quasi-isomorphism.*

Proof. By assumption, $H_I^{i,j}(f)$ is an isomorphism and Corollary 2.9.3 applies. □

Derived tensor product

Let R be a ring.

The composite functor

$$C(\mathbf{Mod}\text{-}R) \times C(R\text{-}\mathbf{Mod}) \xrightarrow{-\otimes_R -} C^2(\mathbf{Ab}) \xrightarrow{\operatorname{tot}_\oplus} C(\mathbf{Ab})$$

induces a triangulated bifunctor

$$\operatorname{tot}_\oplus(- \otimes_R -)\colon K(\mathbf{Mod}\text{-}R) \times K(R\text{-}\mathbf{Mod}) \to K(\mathbf{Ab}).$$

Proposition 2.9.5. *The triangulated bifunctor*

$$\operatorname{tot}_\oplus(- \otimes_R -)\colon K(\mathbf{Mod}\text{-}R) \times K^-(R\text{-}\mathbf{Mod}) \to K(\mathbf{Ab}).$$

admits a left derived bifunctor

$$- \otimes_R^L -\colon D(\mathbf{Mod}\text{-}R) \times D^-(R\text{-}\mathbf{Mod}) \to D(\mathbf{Ab})$$

such that for all $M \in K^-(R\text{-}\mathbf{Mod})$ with projective components and $L \in K(\mathbf{Mod}\text{-}R)$, we have

$$L \otimes_R^L M \xrightarrow{\sim} \operatorname{tot}_\oplus(L \otimes_R M).$$

Proof. Note that R-**Mod** admits enough projectives. By Proposition 2.8.13, it suffices to show that for $L \in C(\mathbf{Mod}\text{-}R)$, $M \in C^-(R\text{-}\mathbf{Mod})$, M^n projective for all n, with L or M acyclic, $\mathrm{tot}_\oplus(L \otimes_R M)$ is acyclic. If M is acyclic, then the image of M in $K^-(R\text{-}\mathbf{Mod})$ is zero and the assertion is trivial. Assume now that L is acyclic. We have $L \simeq \mathrm{colim}_{n\in(\mathbb{Z},\le)} \tau^{\le n} L$. It follows that

$$\mathrm{tot}_\oplus(L \otimes_R M) \simeq \operatorname*{colim}_n \mathrm{tot}_\oplus(\tau^{\le n} L \otimes_R M).$$

Since filtered colimits are exact in **Ab**, we may assume that $L \in C^-(\mathbf{Mod}\text{-}R)$. Then $L \otimes_R M$ is biregular. For each n, since M^n is projective, and hence flat, $L \otimes M^n$ is acyclic. Thus $\mathrm{tot}_\oplus(L \otimes_R M)$ is acyclic by Corollary 2.9.2. □

By duality, we get the following.

Proposition 2.9.6. *The triangulated bifunctor*

$$\mathrm{tot}_\oplus(- \otimes_R -)\colon K^-(\mathbf{Mod}\text{-}R) \times K(R\text{-}\mathbf{Mod}) \to K(\mathbf{Ab}).$$

admits a left derived bifunctor

$$- \otimes_R^L -\colon D^-(\mathbf{Mod}\text{-}R) \times D(R\text{-}\mathbf{Mod}) \to D(\mathbf{Ab})$$

such that for all $L \in K^-(\mathbf{Mod}\text{-}R)$ with projective components and $M \in K(R\text{-}\mathbf{Mod})$, we have

$$L \otimes_R^L M \xrightarrow{\sim} \mathrm{tot}_\oplus(L \otimes_R M).$$

The functors defined in Propositions 2.9.5 and 2.9.6 are isomorphic when restricted to $D^-(\mathbf{Mod}\text{-}R) \times D^-(R\text{-}\mathbf{Mod})$. Moreover, for $L \in D^{\le a}(\mathbf{Mod}\text{-}R)$, $M \in D^{\le b}(R\text{-}\mathbf{Mod})$, $L \otimes^L M \in D^{\le a+b}(\mathbf{Ab})$.

Definition 2.9.7. For $L \in D(\mathbf{Mod}\text{-}R)$, $M \in D(R\text{-}\mathbf{Mod})$, with $L \in D^-$ or $M \in D^-$, we define the hyper Tor by

$$\mathrm{Tor}_n^R(L, M) = H^{-n}(L \otimes_R^L M).$$

As usual, we drop the word "hyper" when L and M are concentrated in degree 0. For $X \in \mathbf{Mod}\text{-}R$ and $Y \in R\text{-}\mathbf{Mod}$, $\mathrm{Tor}_n^R(X, -)$ is the n-th left derived functor of $X \otimes_R -$ and $\mathrm{Tor}_n^R(-, Y)$ is the n-th left derived functor of $- \otimes_R Y$. We have $\mathrm{Tor}_n^R(X, Y) = 0$ for $n < 0$ and $\mathrm{Tor}_0^R(X, Y) = X \otimes_R Y$.

Proposition 2.9.8. *Let Y be a left R-module. Then the following conditions are equivalent:*

(1) Y is flat;
(2) $\mathrm{Tor}_1^R(X, Y) = 0$ for all right R-module X;
(3) $\mathrm{Tor}_n^R(X, Y) = 0$ for all right R-module X and all $n \ge 1$.

Proof. (3) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (1). Since $\mathrm{Tor}_1^R(-, Y) = 0$, the long exact sequence implies that Y is flat.

(1) $\Longrightarrow$ (3). Let $X' \to X$ be a projective resolution of X, giving rise to the exact sequence
$$\cdots \to X'^{-1} \to X'^{0} \to X \to 0.$$
Since Y is flat, the induced sequence
$$\cdots \to X'^{-1} \otimes_R Y \to X'^{0} \otimes_R Y \to X \otimes_R Y \to 0.$$
is exact. Thus $X \otimes_R^L Y \simeq X' \otimes_R Y \simeq X \otimes_R Y$ in $D(\mathbf{Ab})$. It follows that $\mathrm{Tor}_n^R(X,Y) = 0$. □

Corollary 2.9.9. *Let*
$$0 \to F' \to F \to F'' \to 0$$
be a short exact sequence of R-modules.

(1) If F'' is flat, then for any right R-module X, the induced sequence
$$0 \to X \otimes_R F' \to X \otimes_R F \to X \otimes_R F'' \to 0$$
is exact.

(2) If F and F'' are flat, then F' is flat.

(3) If F' and F'' are flat, then F is flat.

Proof. This follows from Propositions 2.7.12 and 2.9.8. We recall the proof for the convenience of the reader. The long exact sequence has the form
$$\mathrm{Tor}_{n+1}^R(X,F'') \to \mathrm{Tor}_n^R(X,F') \to \mathrm{Tor}_n^R(X,F) \to \mathrm{Tor}_n^R(X,F'').$$

(1) For $n = 0$, we get the injectivity of $X \otimes_R F' \to X \otimes_R F$.

(2) For $n \geq 1$, we have $\mathrm{Tor}_{n+1}^R(X,F'') = \mathrm{Tor}_n^R(X,F) = 0$, which implies $\mathrm{Tor}_n^R(X,F') = 0$ by the long exact sequence. Hence F' is flat.

(3) For $n \geq 1$, $\mathrm{Tor}_n^R(X,F') = \mathrm{Tor}_n^R(X,F'') = 0$, which implies $\mathrm{Tor}_n^R(X,F) = 0$ by the long exact sequence. Hence F is flat. □

Warning 2.9.10. If F' and F are flat, F'' is *not* flat in general.

Remark 2.9.11. By (1) and (2) above and Proposition 2.7.8, the full subcategory of R-**Mod** spanned by flat R-modules is $(X \otimes_R -)$-projective for every right R-module X.

Corollary 2.9.12. *Let $L \in C(\mathbf{Mod}\text{-}R)$, $M \in C(R\text{-}\mathbf{Mod})$. If $L \in C^-(\mathbf{Mod}\text{-}R)$ with flat components, or $M \in C^-(R\text{-}\mathbf{Mod})$ with flat components, then we have*
$$L \otimes_R^L M \xrightarrow{\sim} \mathrm{tot}_\oplus(L \otimes M).$$

Proof. We treat the case $M \in C^-(R\text{-}\mathbf{Mod})$, the other case being similar. Choose a quasi-isomorphism $f\colon M' \to M$, where $M' \in C^-(R\text{-}\mathbf{Mod})$ has projective components. Then $L \otimes_R^L M \simeq \mathrm{tot}_\oplus(L \otimes_R M')$ and the cone of morphism $\mathrm{tot}_\oplus(L \otimes_R f)\colon \mathrm{tot}_\oplus(L \otimes_R M') \to \mathrm{tot}_\oplus(L \otimes_R M)$ is isomorphic to $\mathrm{tot}_\oplus(L \otimes_R \mathrm{Cone}(f))$, which is acyclic by the lemma below. Thus $\mathrm{tot}_\oplus(L \otimes_R f)$ is a quasi-isomorphism. □

Lemma 2.9.13. *Let $L \in C(\mathbf{Mod}\text{-}R)$, $M \in C^-(R\text{-}\mathbf{Mod})$. Assume M acyclic and M^n flat for all n. Then $\mathrm{tot}_\oplus(L \otimes_R M)$ is acyclic.*

Proof. We have $L \simeq \mathrm{colim}_{n\in(\mathbb{Z},\leq)} \tau^{\leq n}L$. It follows that

$$\mathrm{tot}_\oplus(L \otimes_R M) \simeq \operatorname*{colim}_n \mathrm{tot}_\oplus(\tau^{\leq n}L \otimes_R M).$$

Since filtered colimits are exact in $\mathbf{Ab}$, we may assume that $L \in C^-(\mathbf{Mod}\text{-}R)$. Then $L \otimes_R M$ is biregular. Since M is acyclic, M splits into short sequences

$$0 \to Z^n M \to M^n \to Z^{n+1}M \to 0.$$

We prove that Z^nM is flat for all n by descending induction. For $n \gg 0$, $M^n = 0$. If $Z^{n+1}M$ is flat, then, by Corollary 2.9.9, Z^nM is flat. It follows that for each i, $L^i \otimes_R M$ splits into short exact sequences, and hence is acyclic. Thus $\mathrm{tot}_\oplus(L \otimes_R M)$ is acyclic by Corollary 2.9.2. □

Proposition 2.9.14. *We have an isomorphism $L \otimes^L_R M \simeq M \otimes^L_{R^{\mathrm{op}}} L$, natural in $L \in D(\mathbf{Mod}\text{-}R)$ and $M \in D^-(R\text{-}\mathbf{Mod})$. Here on the right hand side we regard $L \in D(R^{\mathrm{op}}\text{-}\mathbf{Mod})$ and $M \in D(\mathbf{Mod}\text{-}R)$.*

Proof. We may assume $M \in C^-(R\text{-}\mathbf{Mod})$ with flat components. Then the isomorphism is given by the isomorphism of double complexes $L \otimes_R M \simeq M \otimes_{R^{\mathrm{op}}} L$. □

Example 2.9.15. Let R be a ring. Let $r \in R$ be an element that is not a left zero-divisor. In other words, $R \xrightarrow{r\times} R$ is an injection. Then the right R-module R/rR has the following projective resolution: $0 \to R \xrightarrow{r\times} R \to R/rR \to 0$. For any left R-module M, $R/rR \otimes^L_R M$ is computed by the complex $M \xrightarrow{r\times} M$ put in degrees -1 and 0. Thus $\mathrm{Tor}^R_1(R/rR, M) \simeq \{m \in M \mid rm = 0\}$ is the r-torsion subgroup of M, which explains the notation Tor. If r is neither a left zero-divisor nor a right zero-divisor, then we get

$$R/rR \otimes^L_R M[-1] \simeq R\mathrm{Hom}_R(R/Rr, M)$$

by comparing with Example 2.8.23. In particular, if R is a PID and $r, s \in R$ are nonzero, then

$$R/rR \otimes^L_R R/sR \simeq R/(r,s)R \oplus R/(r,s)R[1].$$

Example 2.9.16. Let A be an abelian group. Then

$$\mathrm{Tor}^R_1(\mathbb{Q}/\mathbb{Z}, A) \simeq \mathrm{Tor}^R_1(\operatorname*{colim}_{n\in\mathbb{N}^\times} \frac{1}{n}\mathbb{Z}, A) \simeq \operatorname*{colim}_{n\in\mathbb{N}^\times} \mathrm{Tor}^R_1(\frac{1}{n}\mathbb{Z}, A) \simeq A_{\mathrm{tor}}$$

is the torsion subgroup of A.

Example 2.9.17. Let R be a commutative ring and $I \subseteq R$ an ideal generated by a regular sequence $f_1, \dots, f_r$. Then $(R/I) \otimes^L_R (R/I) \simeq K(f_1, \dots, f_r) \otimes_R R/I \simeq \bigoplus_{n=0}^r (R/I)^{\binom{r}{n}}[n]$.

Derived tensor product and derived Hom

Let R and S be rings. We have the following derived functors

$$\begin{aligned}
-\otimes^L_S -\colon & D((R,S)\text{-}\mathbf{Mod})\times D^-(S\text{-}\mathbf{Mod})\to D(R\text{-}\mathbf{Mod}),\\
-\otimes^L_R -\colon & D^-(\mathbf{Mod}\text{-}R)\times D((R,S)\text{-}\mathbf{Mod})\to D(\mathbf{Mod}\text{-}S),\\
R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}\colon & D((R,S)\text{-}\mathbf{Mod})\times D^+(R\text{-}\mathbf{Mod})\to D(S\text{-}\mathbf{Mod}),\\
R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}\colon & D^-(R\text{-}\mathbf{Mod})\times D((R,S)\text{-}\mathbf{Mod})\to D(\mathbf{Mod}\text{-}S),\\
R\mathrm{Hom}_{\mathbf{Mod}\text{-}S}\colon & D((R,S)\text{-}\mathbf{Mod})\times D^+(\mathbf{Mod}\text{-}S)\to D(\mathbf{Mod}\text{-}R),\\
R\mathrm{Hom}_{\mathbf{Mod}\text{-}S}\colon & D^-(\mathbf{Mod}\text{-}S)\times D((R,S)\text{-}\mathbf{Mod})\to D(R\text{-}\mathbf{Mod}).
\end{aligned}$$

Theorem 2.9.18. *We have isomorphisms*

$$(L\otimes^L_R M)\otimes^L_S N\simeq L\otimes^L_R(M\otimes^L_S N), \tag{2.9.2}$$

$$R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M\otimes^L_S N,K)\simeq R\mathrm{Hom}_{S\text{-}\mathbf{Mod}}(N,R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M,K)), \tag{2.9.3}$$

$$R\mathrm{Hom}_{\mathbf{Mod}\text{-}S}(L\otimes^L_R M,P)\simeq R\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(L,R\mathrm{Hom}_{\mathbf{Mod}\text{-}S}(M,P)), \tag{2.9.4}$$

natural in $L\in D^-(\mathbf{Mod}\text{-}R)$, $M\in D((R,S)\text{-}\mathbf{Mod})$, $N\in D^-(S\text{-}\mathbf{Mod})$, $K\in D^+(R\text{-}\mathbf{Mod})$, $P\in D^+(\mathbf{Mod}\text{-}S)$.

Proof. We may assume $K,P\in C^+$ with injective components and $L,N\in C^-$ with projective components (for the first isomorphism it suffices to take $L,N\in C^-$ with flat components). Then the isomorphisms are given by isomorphisms of triple complexes. □

Corollary 2.9.19. *We have isomorphisms*

$$\begin{aligned}
\mathrm{Hom}_{D(R\text{-}\mathbf{Mod})}(M\otimes^L_S N,K)&\simeq \mathrm{Hom}_{D(S\text{-}\mathbf{Mod})}(N,R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M,K)),\\
\mathrm{Hom}_{D(\mathbf{Mod}\text{-}S)}(L\otimes^L_R M,P)&\simeq \mathrm{Hom}_{D(\mathbf{Mod}\text{-}R)}(L,R\mathrm{Hom}_{\mathbf{Mod}\text{-}S}(M,P)),
\end{aligned}$$

natural in K,L,M,N,P as in the theorem.

Proof. This follows from the theorem by taking H^0. □

Proposition 2.9.20. *Let $R\to S$ be a ring homomorphism. We have isomorphisms*

$$(L\otimes^L_R S)\otimes^L_S N\simeq L\otimes^L_R N, \tag{2.9.5}$$

$$R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(N,K)\simeq R\mathrm{Hom}_{S\text{-}\mathbf{Mod}}(N,R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(S,K)), \tag{2.9.6}$$

$$R\mathrm{Hom}_{S\text{-}\mathbf{Mod}}(S\otimes^L_R K',N)\simeq R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K',N), \tag{2.9.7}$$

natural in $N\in D(S\text{-}\mathbf{Mod})$, $L\in D^-(\mathbf{Mod}\text{-}R)$, $K\in D^+(R\text{-}\mathbf{Mod})$, $K'\in D^-(R\text{-}\mathbf{Mod})$.

For $N\in D^-$ (resp. D^+), the first and second (resp. third) isomorphism is a special case of the theorem.

Proof. We may assume $L\in C^-$ with flat components, $K\in C^+$ with injective components, and $K'\in C^-$ with projective components. Then $L^n\otimes_R S$ is a flat right S-module, $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(S,K^n)$ is an injective S-module and $S\otimes_R K'^n$ is a projective S-module. □

Remark 2.9.21. The derived functors and natural isomorphisms above all extend to unbounded derived categories. In particular, for $M \in D((R,S)\text{-}\mathbf{Mod})$,

$$M \otimes^L_S -\colon D(R\text{-}\mathbf{Mod}) \to D(S\text{-}\mathbf{Mod})$$

is left adjoint to

$$R\mathrm{Hom}_R(M,-)\colon D(S\text{-}\mathbf{Mod}) \to D(R\text{-}\mathbf{Mod}).$$

Flat dimension

The following generalizes Proposition 2.9.8.

Proposition 2.9.22. *Let Y be a left R-module and let $m \geq 0$ be an integer. The following conditions are equivalent:*
(1) There exists a flat resolution Y' of Y concentrated in $[-m+1,0]$.
(2) $\mathrm{Tor}^R_m(X,Y) = 0$ for every right R-module X.
(3) $\mathrm{Tor}^R_n(X,Y) = 0$ for every right R-module X and $n \geq m$.
(4) $- \otimes^L_R Y$ carries $D^{\geq 0}(\mathbf{Mod}\text{-}R)$ to $D^{\geq -m+1}(\mathbf{Ab})$.

Proof. (1) $\Longrightarrow$ (4). Indeed for $L \in C^{\geq 0}(\mathbf{Mod}\text{-}R)$, $L \otimes^L_R Y \simeq \mathrm{tot}(L \otimes_R Y)$, and the latter is concentrated in $[-m+1,+\infty)$.

(4) $\Longrightarrow$ (3) $\Longrightarrow$ (2). Trivial.

(2) $\Longrightarrow$ (1). For $m = 0$, taking $X = R$, we get $Y = 0$. For $m \geq 1$, we apply the lemma below (with $k = m-1$) to get the flat resolution. □

Lemma 2.9.23 (Dimension shifting)**.** *Let $0 \to Y' \to F^{-k+1} \to \cdots \to F^0 \to Y$ be an exact sequence of left R-modules with F^i flat. Then $\mathrm{Tor}^R_n(X,Y') \simeq \mathrm{Tor}^R_{n+k}(X,Y)$ for $n \geq 1$.*

Proof. Decomposing the exact sequence into short exact sequences, we reduce by induction to the case $k = 1$. In this case, the assertion follows from the long exact sequence. □

Definition 2.9.24. Let X be a right R-module and let Y be a left R-module. The *flat dimensions* (or Tor-dimensions) of X and Y are defined to be

$$\mathrm{fl.dim}(X) = \sup\{n \in \mathbb{Z} \mid \mathrm{Tor}^R_n(X,Y) \neq 0 \text{ for some } Y\},$$
$$\mathrm{fl.dim}(Y) = \sup\{n \in \mathbb{Z} \mid \mathrm{Tor}^R_n(X,Y) \neq 0 \text{ for some } X\}.$$

The *weak dimension* of R is defined to be

$$\mathrm{w.dim}(R) = \sup\{n \in \mathbb{Z} \mid \mathrm{Tor}^R_n(X,Y) \neq 0 \text{ for some } X, Y\}.$$

The above dimensions take values in $\mathbb{Z}_{\geq 0} \cup \{\pm\infty\}$. Proposition 2.9.22 gives equivalent conditions for $\mathrm{fl.dim}(Y) < m$. We have $\mathrm{fl.dim}(Y) = -\infty$ if and only if $Y = 0$. By definition,

$$\mathrm{w.dim}(R) = \sup_{X \in \mathbf{Mod}\text{-}R} \mathrm{fl.dim}(X) = \sup_{Y \in R\text{-}\mathbf{Mod}} \mathrm{fl.dim}(Y),$$

so that $\mathrm{w.dim}(R^{\mathrm{op}}) = \mathrm{w.dim}(R)$.

Remark 2.9.25. Since projective modules are flat, we have

$$\begin{gathered}\mathrm{fl.dim}(X) \leq \mathrm{proj.dim}(X), \quad \mathrm{fl.dim}(Y) \leq \mathrm{proj.dim}(Y),\\ \mathrm{w.dim}(R) \leq \min\{\mathrm{l.gl.dim}(R), \mathrm{r.gl.dim}(R)\}.\end{gathered}$$

The following generalizes the two special cases of Example 2.6.20.

Proposition 2.9.26. *Let $R \to S$ be a ring homomorphism and N an S-module. Then*

$$\begin{gathered}\mathrm{fl.dim}_R(N) \leq \mathrm{fl.dim}_S(N) + \mathrm{fl.dim}_R(S),\\ \mathrm{proj.dim}_R(N) \leq \mathrm{proj.dim}_S(N) + \mathrm{proj.dim}_R(S),\\ \mathrm{inj.dim}_R(N) \leq \mathrm{inj.dim}_S(N) + \mathrm{fl.dim}(S_R).\end{gathered}$$

Proof. For any R-module L, the left-hand side of (2.9.5) belongs to $D^{\leq \mathrm{fl.dim}_S(N)+\mathrm{fl.dim}_R(S)}$. For any R-module K, the right-hand side of (2.9.6) belongs to $D^{\leq \mathrm{proj.dim}_S(N)+\mathrm{proj.dim}_R(S)}$ by Remark 2.8.19. For any R-module K', the left-hand side of (2.9.7) belongs to $D^{\leq \mathrm{inj.dim}_S(N)+\mathrm{fl.dim}(S_R)}$. □

Projective dimension revisited

Proposition 2.9.27. *Let R be a ring and $f \in R$ a central element that is not a zero-divisor. Let $N \neq 0$ be an R/fR-module such that* $\mathrm{proj.dim}_{R/fR}(N) < \infty$. *Then*

$$\mathrm{proj.dim}_R(N) = 1 + \mathrm{proj.dim}_{R/fR}(N).$$

Proof. Let $d = \mathrm{proj.dim}_{R/fR}(N)$. By Proposition 2.9.26, we have $\mathrm{proj.dim}_R(N) \leq 1 + d$. We proceed by induction on d.

(1) Case $d = 0$. Since $fN = 0$, N is not a projective R-module. Thus $\mathrm{proj.dim}_R(N) = 1$.

(2) Case $d = 1$. Assume $\mathrm{proj.dim}_R(N) \leq 1$. Take a projective resolution of the R-module N of length 1:

$$0 \to Q \to P \to N \to 0.$$

Applying $R/fR \otimes^L_R -$ to the above exact sequence, we get the long exact sequence of R/fR-modules

$$0 \to \mathrm{Tor}_1^R(R/fR, N) \to Q/fQ \to P/fP \to N \to 0.$$

Since Q/fQ and P/fP are projective R/fR-modules and $\mathrm{proj.dim}_{R/fR}(N) = 1 \leq 2$, $\mathrm{Tor}_1^R(R/fR, N)$ is a projective R/fR-module. However, $\mathrm{Tor}_1^R(R/fR, N) \simeq N$ by Example 2.9.15. This contradicts the assumption that $d = 1$. Therefore, $\mathrm{proj.dim}_R(N) = 2$.

(3) Case $d \geq 2$. Take a short exact sequence of R/fR-modules

$$0 \to M \to P' \to N \to 0$$

with P' a projective R/fR-module. Then $\mathrm{proj.dim}_{R/fR}(M) = d-1$. Thus $\mathrm{proj.dim}_R(M) = d$ and $\mathrm{proj.dim}_R(P) = 1$ by induction hypothesis. Therefore, $\mathrm{proj.dim}_R(N) = d + 1$ by Lemma 2.9.28 below. □

Lemma 2.9.28. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. We denote the projective dimensions of M', M, M'' by d', d, d'', respectively. Then $d \leq \max(d', d'')$ and equality holds unless $d'' = d' + 1$.*

Proof. This follows easily from the long exact sequence for Ext. Let $m = \max(d', d'')$. Let N be an object of $\mathcal{A}$. For $i > m$, we have an exact sequence

$$0 = \mathrm{Ext}^i(M'', N) \to \mathrm{Ext}^i(M, N) \to \mathrm{Ext}^i(M', N) = 0.$$

Thus $\mathrm{Ext}^i(M, N) = 0$ for all N, or equivalently $d \leq m$.

Assume $d'' \neq d' + 1$. We have an exact sequence

$$\mathrm{Ext}^{m-1}(M', N) \to \mathrm{Ext}^m(M'', N) \xrightarrow{f} \mathrm{Ext}^m(M, N) \xrightarrow{g} \mathrm{Ext}^m(M', N) \to \mathrm{Ext}^{m+1}(M'', N) = 0.$$

Now either $m = d'$ or $m = d'' > d' + 1$. If $m = d'$, then g is nonzero for some N. If $m = d'' > d' + 1$, then the first term of the above sequence is 0 and consequently f is nonzero for some N (in fact f is an isomorphism in this case). In both cases, $\mathrm{Ext}^m(M, N)$ is nonzero for some n. Therefore, $d = m$. □

Remark 2.9.29. By the lemma, the following holds.

(1) If $d' < d$, then $d'' = d$.
(2) If $d' > d$, then $d'' = d' + 1$.
(3) If $d' = d$, then $d'' \leq d' + 1$.

Warning 2.9.30. The assumption $\mathrm{proj.dim}_{R/fR}(N) < \infty$ in Proposition 2.9.27 cannot be dropped. For example, for k a field, $R = k[t]$, we have $\mathrm{proj.dim}_{R/t^2R}(R/tR) = \infty$ (Example 2.8.26) and $\mathrm{proj.dim}_R(R/tR) = 1$.

Corollary 2.9.31. *Let R be a ring and let $f_1, \dots, f_r$ be a regular sequence in R. Let $I_r = (f_1, \dots, f_r)$. Then, for any R/I-module N satisfying $\mathrm{proj.dim}_{R/I_r}(N) < \infty$, we have $\mathrm{proj.dim}_R(N) = r + \mathrm{proj.dim}_{R/I_r}(N)$. In particular, $\mathrm{proj.dim}_R(R/I_r) = r$.*

Proof. We proceed by induction on r. The case $r = 0$ is trivial. For $r > 0$,

$$\mathrm{proj.dim}_R(N) = r - 1 + \mathrm{proj.dim}_{R/I_{r-1}}(N) = r + \mathrm{proj.dim}_{R/I_r}(N)$$

by induction hypothesis and Proposition 2.9.27. □

Künneth formula

Proposition 2.9.32 (Künneth formula for hyper Tor)**.** *Assume that R is left and right hereditary. Let $L \in D(\mathbf{Mod}\text{-}R)$, $M \in D(R\text{-}\mathbf{Mod})$ such that either (a) $L, M \in D^-$, or (b) $L \in D^b$, or (c) $M \in D^b$. Then we have a split short exact sequence*

$$0 \to \bigoplus_{l+m=-n} (H^l L \otimes_R H^m M) \xrightarrow{f} \mathrm{Tor}_n^R(L, M) \xrightarrow{g} \bigoplus_{l+m=1-n} \mathrm{Tor}_1^R(H^l L, H^m M) \to 0.$$

Here f and g are induced by

$$H^l L \otimes_R H^m M \simeq \mathrm{Tor}_n^R(\tau^{\leq l} L, \tau^{\leq m} M) \to \mathrm{Tor}_n^R(L, M),$$
$$\mathrm{Tor}_n^R(L, M) \to \mathrm{Tor}_n^R(\tau^{\geq l} L, \tau^{\geq m} M) \simeq \mathrm{Tor}_1^R(H^l L, H^m M).$$

The splitting is not canonical.

Proof. We may assume $L \in D^b$. By Proposition 2.6.29, we have

$$\mathrm{Tor}_n^R(L, M) \simeq \bigoplus_{l,m} \mathrm{Tor}_{l+m+n}(H^l K, H^m L).$$

□

Remark 2.9.33. Recall that the singular (resp. cellular) (co)homology of a topological space (resp. CW complex) X with coefficients in an abelian group M is defined by

$$H_n(X, M) = H^{-n}(C_\bullet(X) \otimes M), \quad H^n(X, M) = H^n \mathrm{Hom}^\bullet(C_\bullet(X), M).$$

Here $C_\bullet(X) \in C^{(-\infty,0]}(\mathbf{Ab})$ denotes the singular (resp. cellular) chain complex, which is a complex of free abelian groups. In other words,

$$H_n(X, M) = \mathrm{Tor}_n(C_\bullet(X), M), \quad H^n(X, M) = \mathrm{Ext}^n(C_\bullet(X), M).$$

Since $\mathbb{Z}$ is hereditary, we get split short exact sequences

$$0 \to H_n(X) \otimes M \to H_n(X, M) \to \mathrm{Tor}_1(H_{n-1}(X), M) \to 0,$$
$$0 \to \mathrm{Ext}^1(H_{n-1}(X), M) \to H^n(X, M) \to \mathrm{Hom}(H_n(X), M) \to 0,$$

where $H_n(X) = H_n(X, \mathbb{Z})$. These sequences are known as universal coefficient theorems.

For topological spaces X and Y, the Eilenberg-Zilber theorem provides an isomorphism $C_\bullet(X \times Y) \simeq \mathrm{tot}(C_\bullet(X) \otimes C_\bullet(Y))$ in $K(\mathbf{Ab})$. (For CW complexes, we have an isomorphism in $C(\mathbf{Ab})$.) Thus we have $H_n(X \times Y) \simeq \mathrm{Tor}_n(C_\bullet(X), C_\bullet(Y))$. Applying the Künneth formula, we get a split short exact sequence

$$0 \to \bigoplus_{l+m=n} H_l(X) \otimes H_m(Y) \to H_n(X \times Y) \to \bigoplus_{l+m=n-1} \mathrm{Tor}_1(H_l(X), H_m(X)) \to 0.$$

This is also called the Künneth formula.

More flatness tests

First we present Lambek's theorem. Let R be a ring. Given a left R-module M, the *character module* of M is $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$,[4] which is a right R-module. Recall that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. It has the following "cogenerator" property.

Lemma 2.9.34. *For any abelian group A and nonzero element $x \in A$, there exists a homomorphism of abelian groups $f\colon A \to \mathbb{Q}/\mathbb{Z}$ with such that $f(x) \neq 0$.*

Proof. There exists a nonzero homomorphism $\mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, this extends to a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$. □

Remark 2.9.35. It follows that every abelian group can be embedded into a product of $\mathbb{Q}/\mathbb{Z}$. Such a product is sometimes said to be "cofree".

[4] In the literature the character module is often denoted by M' rather than M^*.

Lemma 2.9.36. *A sequence of R-modules $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if $Z^* \xrightarrow{g^*} Y^* \xrightarrow{f^*} X^*$ is exact.*

Proof. The "only if" part follows from the fact that the functor $M \mapsto M^*$ is exact. Next we show the "if" part.

Let us first show that $gf = 0$. Otherwise there exists $x \in X$ such that $gf(x) \neq 0$, so that by Lemma 2.9.34 there exists $h \in Z^*$ such that $hgf(x) \neq 0$, which implies $f^*g^*(h) = hgf \neq 0$, contradicting the assumption $f^*g^* = 0$.

Thus $gf = 0$, or equivalently, $\mathrm{im}(f) \subseteq \ker(g)$. Assume $\mathrm{im}(f) \supsetneq \ker(g)$. Applying Lemma 2.9.34 to $Y/\mathrm{im}(f)$ we get $i \in Y^*$ such that $i(\mathrm{im}(f)) = 0$ but $i(\ker(g)) \neq 0$. Then $f^*(i) = 0$ so that $i = g^*(h) = hg$, which implies $i(\ker(g)) = 0$. Contradiction. □

Theorem 2.9.37 (Lambek)**.** *A left R-module M is flat if and only if M^* is an injective right R-module.*

Proof. This follows from the natural isomorphism $\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(-, M^*) \simeq (- \otimes_R M)^*$ of functors $\mathbf{Mod}\text{-}R \to \mathbf{Ab}$ and the preceding lemma. □

Proposition 2.9.38. *Let M be a left R-module M. The following conditions are equivalent:*

(1) M is flat.
(2) For every right ideal $I \subseteq R$, the map $I \otimes_R M \to M$ is an injection.
(3) For every finitely generated right ideal $I \subseteq R$, the map $I \otimes_R M \to M$ is an injection.

Note that the injectivity in (2) and (3) means that $I \otimes_R M \to IM$ is a bijection.

Proof. (1) $\iff$ (2). This follows from the natural isomorphism $\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(-, M^*) \simeq (- \otimes_R M)^*$ together with Baer's test. By Lambek's theorem, M is flat if and only if M^* is injective. By Baer's test, this is equivalent to the surjectivity of $\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(R, M^*) \to \mathrm{Hom}_{\mathbf{Mod}\text{-}R}(I, M^*)$ for each right ideal $I \subseteq R$. The surjectivity means that $M^* \to (I \otimes_R M)^*$ is surjective, or equivalently, that $I \otimes_R M \to M$ is an injection.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (2). I is a filtered colimit of finitely generated right R-modules. □

Taking $S = \mathbb{Z}$ and $P = \mathbb{Q}/\mathbb{Z}$ in (2.9.4), we get the following.

Proposition 2.9.39. *We have a canonical isomorphism*

$$R\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(N, M^*) \simeq (N \otimes^L_R M)^*,$$

functorial in $N \in D^-(\mathbf{Mod}\text{-}R)$ and $M \in D(R\text{-}\mathbf{Mod})$. In particular, $\mathrm{Ext}^n_{R^{\mathrm{op}}}(N, M^*) \simeq \mathrm{Tor}^R_n(N, M)^*$.

Corollary 2.9.40. *Let R be a ring, let M be a left R-module, and let $n \geq 0$ be an integer. Then $\mathrm{fl.dim}(M) \leq n$ if and only if $\mathrm{Tor}^R_{n+1}(R/I, M) = 0$ for every finitely generated right ideal I of R.*

Proof. By Proposition 2.9.39, $\mathrm{fl.dim}(M) \leq n$ if and only if $\mathrm{inj.dim}(M^*) \leq n$. By Proposition 2.6.25, this is equivalent to $\mathrm{Ext}^n_{R^{\mathrm{op}}}(R/I, M^*) = 0$ for every right ideal $I \subseteq R$. By Proposition 2.9.39 again, this is equivalent to $\mathrm{Tor}_n^R(R/I, M) = 0$ for every right ideal $I \subseteq R$. Finally, every right ideal is a filtered colimit of finitely generated right ideals. Thus $\mathrm{Tor}_n^R(R/I, M) = 0$ for every finitely generated right ideals I implies that the same holds for every right ideal I. □

Alternative proof. The "only if" part is clear. For the "if" part, take an exact sequence $0 \to N \to F^{-n+1} \to \cdots \to F^0 \to M \to 0$ with F^i flat. Then $\mathrm{Tor}_1^R(R/I, N) \simeq \mathrm{Tor}_{n+1}^R(R/I, M) = 0$. We have an exact sequence

$$0 = \mathrm{Tor}_1^R(R/I, N) \to I \otimes_R N \to N.$$

Thus the map $I \otimes_R N \to N$ is an injection. It follows from Proposition 2.9.38 that N is flat. □

Corollary 2.9.41. *For any ring R, we have*

$$\mathrm{w.dim}(R) = \sup_I \mathrm{fl.dim}(R/I) = \sup_J \mathrm{fl.dim}(R/J),$$

where I (resp. J) runs through finitely generated left (resp. right) ideals of R.

Definition 2.9.42. We say that an R-module M is *finitely presented* if there exists an exact sequence $R^m \to R^n \to M \to 0$ with $m, n \geq 0$.

Finitely presented R-modules are finitely generated. Conversely, R is left Noetherian if and only if every finitely generated R-module is finitely presented [L1, Proposition 4.29].

Proposition 2.9.43. *Let M be a finitely generated R-module. Then M is projective if and only if it is flat and finitely presented.*

In particular, if M is finitely presented, then M is projective if and only if M is flat. We refer the reader to [L1, Theorem 4.30] for a generalization.

Lemma 2.9.44. *For left R-modules X and Y, the homomorphism $Y^* \otimes_R X \to \mathrm{Hom}_R(X, Y)^*$ carrying $g \otimes x$ to $f \mapsto gf(x)$ is an isomorphism whenever X is finitely presented.*

Proof. Since both functors $Y^* \otimes_R -$ and $\mathrm{Hom}_R(-, Y)^*$ are right exact, we reduce to the trivial case where $X = R^n$. □

Proof of Proposition 2.9.43. The "only if" part. It is clear that M is flat. Consider an epimorphism $R^n \to M$. The kernel is a direct summand of R^n, and hence finitely generated. Thus M is finitely presented.

The "if" part. Since $(-)^* \otimes_R M \simeq \mathrm{Hom}(M, -)^*$ is an exact functor, $\mathrm{Hom}(M, -)$ is exact as well. □

To state a derived version of Lemma 2.9.44, we need some terminology.

Definition 2.9.45. We say that an object of $D(R\text{-}\mathbf{Mod})$ is *pseudo-coherent* if it is isomorphic in $D(R\text{-}\mathbf{Mod})$ to a complex X in $C^-(R\text{-}\mathbf{Mod})$ such that X^i is finitely generated and projective for each i. An R-module M is said to be *pseudo-coherent* if $M[0]$ is pseudo-coherent.

Proposition 2.9.46. *For $X \in D^-(R\text{-}\mathbf{Mod})$ pseudo-coherent and $Y \in D^+(R\text{-}\mathbf{Mod})$, we have a canonical isomorphism*

$$Y^* \otimes_R^L X \simeq R\mathrm{Hom}_R(X,Y)^*.$$

In particular, $\mathrm{Tor}_n^R(Y^*, X) \simeq \mathrm{Ext}_R^n(X,Y)^*$ *for all $n \in \mathbb{Z}$.*

Proof. We may assume $X \in C^-(R\text{-}\mathbf{Mod})$ and X^i is finitely generated and projective, and hence finitely presented, for each i. In this case, the isomorphism is the totalization of the isomorphism biregular double complexes

$$Y^* \otimes_R X \simeq \mathrm{Hom}_R^{\bullet\bullet}(X,Y)^*$$

given by Lemma 2.9.44. $\square$

Corollary 2.9.47. *Let M be a pseudo-coherent R-module. Then* $\mathrm{fl.dim}(M) = \mathrm{proj.dim}(M)$.

Proof. We have $\mathrm{fl.dim}(M) \le \mathrm{proj.dim}(M)$. By Proposition 2.9.46, $\mathrm{proj.dim}(M) \le \mathrm{fl.dim}(M)$. $\square$

Example 2.9.48. Let R be a left Noetherian ring. Then any finitely generated R-module M is pseudo-coherent. Indeed, M admits a left resolution by free R-modules of finite rank.

Corollary 2.9.49 (Auslander)**.** *Let R be a left Noetherian ring. For any finitely generated left R-module M, we have* $\mathrm{fl.dim}(M) = \mathrm{proj.dim}(M)$. *Moreover,* $\mathrm{w.dim}(R) = \mathrm{l.gl.dim}(R)$.

Proof. Since M is pseudo-coherent, the first assertion is a special case of Corollary 2.9.47. Moreover, we have

$$\mathrm{w.dim}(R) = \sup_I \mathrm{fl.dim}(R/I) = \sup_I \mathrm{proj.dim}(R/I) = \mathrm{l.gl.dim}(R)$$

by Corollaries 2.6.27 and 2.9.41. Here I runs through left ideals of R. (In fact, without using Corollary 2.9.41, we still get $\mathrm{w.dim}(R) \ge \mathrm{l.gl.dim}(R)$, which is enough to conclude.) $\square$

Alternative proof of the first assertion. It suffices to show $\mathrm{proj.dim}(M) \le \mathrm{fl.dim}(M) = n$. Consider an exact sequence $0 \to N \to F^{-n+1} \to \cdots \to F^0 \to M \to 0$ with $F^i = R^{n_i}$. Then $\mathrm{fl.dim}(N) = 0$, namely that N is flat. Since N is finitely generated, it is projective. Thus $\mathrm{proj.dim}(M) \le n$. $\square$

Corollary 2.9.50. *Let R be a left and right Noetherian ring. Then* $\mathrm{w.dim}(R) = \mathrm{l.gl.dim}(R) = \mathrm{r.gl.dim}(R)$.

We make a brief digression on pseudo-coherent complexes.

Lemma 2.9.51. *Let R be a ring. Pseudo-coherent complexes in $D(R\text{-}\mathbf{Mod})$ span a triangulated full subcategory.*

Proof. It suffices to show that for every distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $D(R\text{-}\mathbf{Mod})$ with X and Y pseudo-coherent, Z is pseudo-coherent. For this we may assume that X and Y are in C^- and X^i, Y^i are finitely generated and projective for all i. By Propositions 2.5.26 and 2.5.28, f is represented by a morphism of complexes. Thus $Z \simeq \operatorname{Cone}(f)$ is pseudo-coherent. □

Proposition 2.9.52. *Let R be a left Noetherian ring. For any pseudo-coherent complex $X \in D^-(R\text{-}\mathbf{Mod})$, H^iX is a finitely generated R-module for each i. Conversely, any $X \in D^b(R\text{-}\mathbf{Mod})$ such that H^iX is a finite generated R-module for each i is pseudo-coherent.*

Proof. For the first assertion, we may assume that X^i is finitely generated and projective. The assertion is then clear, since H^i is a quotient of Z^i, which is a submodule of X^i.

For the second assertion, we assume $X \in D^{[a,b]}(R\text{-}\mathbf{Mod})$. We proceed by induction on $b-a$. The case $b = a$ is Example 2.9.48. For $b > a$, consider the distinguished triangle $\tau^{\le a}X \to X \to \tau^{\ge a+1}X \to (\tau^{\le a}X)[1]$. By induction hypothesis, $\tau^{\le a}X$ and $\tau^{\ge a+1}X$ are pseudo-coherent. We conclude by Lemma 2.9.51. □

Remark 2.9.53. With a bit more work, one can show that for R left Noetherian, $X \in D^-(R\text{-}\mathbf{Mod})$ is pseudo-coherent if and only if H^iX is a finite generated R-module for each i.

Definition 2.9.54. Let R be a ring. A finitely generated R-module M is said to be *coherent* if every finitely generated submodule of M is finitely presented. A ring R is said to be *left coherent* if ${}_RR$ is coherent.

Example 2.9.55.
1. Any Noetherian module is coherent. In particular, any left Noetherian ring is left coherent.
2. Any left semi-hereditary ring is left coherent.

Remark 2.9.56. One can show that for R left coherent, $X \in D^-(R\text{-}\mathbf{Mod})$ is pseudo-coherent if and only if H^iX is a coherent R-module for each i.

Remark 2.9.57. By a theorem of Chase [L1, Theorem 4.47], a ring R is left coherent if and only if every product of flat left R-modules is flat. It follows that if R is a left coherent ring, then an R-module is coherent if and only if it is finitely presented.

Now we return to weak dimensions.

Theorem 2.9.58. *Let R be a ring. The following conditions are equivalent:*
1. $\operatorname{w.dim}(R) \le 0$.
2. *For each $r \in R$, there exists $s \in R$ such that $rsr = r$.*
3. *Every principal left ideal of R is generated by an idempotent.*
4. *Every finitely generated left ideal of R is generated by an idempotent.*

Proof. (1) $\implies$ (2). Let $r \in R$. Consider the short exact sequence

$$0 \to Rr \to R \to R/Rr \to 0.$$

By (1), every R-module is flat. Applying the functor $rR \otimes_R -$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & rRr \arrow[r] \arrow[d, hook] & rR \arrow[r, "f"] \arrow[d, hook] & r(R/Rr) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & Rr \arrow[r] & R \arrow[r, "g"] & R/Rr \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical arrows are injections. We have $f(r) = g(r) = 0$. Thus $r \in rRr$.

(2) $\implies$ (3). Let $I = Rr$, $r \in R$. By (2), there exists $s \in R$ such that $rsr = r$. Let $e = sr \in I$. Then $e^2 = e$. Moreover, $r = re \in Re$. Thus $I = Re$.

(3) $\implies$ (4). Let $I \subseteq R$ be a left ideal generated by n elements. We proceed by induction on n to show that I is generated by an idempotent. The case $n = 0$ is trivial. For $n \geq 1$, by induction hypothesis, $I = Re + Rr$ for some $r \in R$ and some idempotent $e \in R$. Then $I = Re + Rr(1-e) = Re + Re'$, where $e' \in R$ is an idempotent such that $Rr(1-e) = Re'$. Then $e'e \in Rr(1-e)e = 0$. Thus $e'(e+e') = e'^2 = e'$. It follows that $I = Re + Re' = R(e+e')$. By (3), I is generated by an idempotent.

(4) $\implies$ (1). Let M be a right R-module. By Proposition 2.9.38, to prove that M is flat, it suffices to show that for every finitely generated left ideal I, the map $M \otimes_R I \to M$ is an injection. By (3), $I = Re$ for some idempotent e, and hence a direct summand of ${}_RR = Re \oplus R(1-e)$. Thus $M \otimes_R I \to M$ is a split injection. □

Rings satisfying the equivalent conditions of the theorem are called *von Neumann regular* rings. By Condition (4), von Neumann regular rings are left (and right) semi-hereditary. By Condition (2), a von Neumann regular domain is a division ring.

Remark 2.9.59. A ring is semisimple if and only if it is von Neumann regular and left (or right) Noetherian. Indeed, the "if" part follows from Corollary 2.9.49 and the "only if" part is clear.

Example 2.9.60. Boolean rings ($r^2 = r$ for all $r \in R$) are von Neumann regular.

Perfect complexes

Let R be a ring.

Definition 2.9.61. A complex K of R-modules is said to be *perfect* if it is isomorphic in $D(R\text{-}\mathbf{Mod})$ to a bounded complex of finitely generated projective R-modules. We let $D_{\mathrm{perf}}(R\text{-}\mathbf{Mod})$ denote the full subcategory of $D(R\text{-}\mathbf{Mod})$ spanned by perfect complexes.

Lemma 2.9.62. *Let P be a finitely generated projective R-module. Then*

(1) $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P,R)$ *is a projective right* R*-module and the canonical homomorphisms*

$$\begin{aligned}\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P,R)\otimes_R M &\to \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P,M)\\ f\otimes m &\mapsto (p\mapsto f(p)m)\end{aligned}\tag{2.9.8}$$

$$\begin{aligned}P &\to \mathrm{Hom}_{\mathbf{Mod}\text{-}R}(\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P,R),R)\\ p &\mapsto (f\mapsto f(p))\end{aligned}\tag{2.9.9}$$

are isomorphisms for every R*-module* M*.*

(2) If R *is commutative, then the canonical homomorphism*

$$\mathrm{Hom}_R(L,\mathrm{Hom}_R(P,R)\otimes_R M)\to\mathrm{Hom}_R(P\otimes_R L,M)$$

induced by (2.9.8) *and Example 1.4.5 is an isomorphism for all* R*-modules* L *and* M*.*

Proof. This is clear for P free of finite rank. The general case follows immediately. □

Remark 2.9.63. (1) One can show that if P is an R-module such that (2.9.8) is an isomorphism for every R-module M, then P is finitely generated projective. In fact, it follows from the isomorphism that $\mathrm{Hom}_R(P,-)$ is right exact, or, in other words, that P is projective.

(2) An R-module such that (2.9.9) is an isomorphism is said to be *reflexive*. Reflexive vector spaces are finite-dimensional. However, reflexive modules are not necessarily finitely generated or projective in general. For example, for $R=\mathbb{Z}$, Specker showed that the canonical homomorphism $\mathbb{Z}^{\oplus\mathbb{N}}\to\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}},\mathbb{Z})$ is an isomorphism, which implies that $\mathbb{Z}^{\oplus\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$ are reflexive. The latter is not a projective $\mathbb{Z}$-module (Remark 1.8.31). The statement that all projective $\mathbb{Z}$-modules are reflexive is consistent with ZFC.

Proposition 2.9.64. *Let* $K\in D_{\mathrm{perf}}(R\text{-}\mathbf{Mod})$*. Then*

(1) $R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K,R)$ *is a perfect complex of right* R*-modules and the canonical morphisms*

$$R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K,R)\otimes_R M\to R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K,M),\tag{2.9.10}$$

$$K\to R\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K,R),R)\tag{2.9.11}$$

are isomorphisms for all $M\in D(R\text{-}\mathbf{Mod})$*.*

(2) If R *is commutative, then the canonical morphism*

$$R\mathrm{Hom}_R(L,R\mathrm{Hom}_R(K,R)\otimes^L_R M)\to R\mathrm{Hom}_R(K\otimes^L_R L,M)$$

induced by (2.9.10) *and* (2.9.3) *is an isomorphism for all* $L,M\in D(R\text{-}\mathbf{Mod})$ *satisfying* $L\in D^-$ *or* $M\in D^+$*.*

Proof. We may assume that K is a bounded complex of finitely generated R-modules and, for (2), that $L\in C^-$ with projective components or $M\in D^+$ with injective components. In this case $R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(K,R)$ is computed by $\mathrm{Hom}^\bullet_{R\text{-}\mathbf{Mod}}(K,R)$, which is perfect by Lemma 2.9.62. Moreover, the morphisms in question are computed by

$$\begin{gathered}\mathrm{Hom}^\bullet_{R\text{-}\mathbf{Mod}}(K,R)\otimes_R M \simeq \mathrm{Hom}^\bullet_{R\text{-}\mathbf{Mod}}(K,M),\\ K \simeq \mathrm{Hom}^\bullet_{\mathbf{Mod}\text{-}R}(\mathrm{Hom}^\bullet_{R\text{-}\mathbf{Mod}}(K,R),R),\\ \mathrm{Hom}^\bullet_R(\mathrm{tot}(K\otimes_R L),M)\to \mathrm{Hom}^\bullet_R(L,\mathrm{tot}(\mathrm{Hom}^\bullet_R(K,R)\otimes_R M)),\end{gathered}$$

which are isomorphisms by Lemma 2.9.62. □

Remark 2.9.65. One can show that if $K \in D(R\text{-}\mathbf{Mod})$ is such that (2.9.10) is an isomorphism for every $M \in D(R\text{-}\mathbf{Mod})$, then K is a perfect complex.

Corollary 2.9.66. *The functor $R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(-,R)$ induces an equivalence of triangulated categories $D_{\mathrm{perf}}(R\text{-}\mathbf{Mod})^{\mathrm{op}} \to D_{\mathrm{perf}}(\mathbf{Mod}\text{-}R)$.*

Proof. Indeed, by (2.9.11) applied to R and R^{op}, $R\mathrm{Hom}_{\mathbf{Mod}\text{-}R}(-,R)$ is a quasi-inverse of $R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(-,R)$. □

Remark 2.9.67. Let R be a commutative ring. By Proposition 2.9.64 (2) (which holds in fact for all $L, M \in D(R\text{-}\mathbf{Mod})$), we have

$$\mathrm{Hom}_{D(R\text{-}\mathbf{Mod})}(K\otimes^L_R L, M) \to \mathrm{Hom}_{D(R\text{-}\mathbf{Mod})}(L, K^\vee \otimes^L_R M),$$

where $K^\vee := R\mathrm{Hom}_R(K,R)$. In other words, $K\otimes^L_R -\colon D(R\text{-}\mathbf{Mod}) \to D(R\text{-}\mathbf{Mod})$ is a left adjoint of $K^\vee\otimes^L_R -\colon D(R\text{-}\mathbf{Mod}) \to D(R\text{-}\mathbf{Mod})$. Since $K \simeq K^{\vee\vee}$, $K^\vee \otimes^L_R -$ is also a left adjoint of $K\otimes^L_R -$.

Example 2.9.68. Let R be a ring and let $f \in R$ be an element that is not a right zero-divisor. Then R/Rf is perfect. Indeed, it is isomorphic in $D(R\text{-}\mathbf{Mod})$ to the complex $R \xrightarrow{\times f} R$ put in degrees -1 and 0. $R\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(R/Rf,R)$ is computed by the complex $R \xrightarrow{f\times} R$ put in degrees 0 and 1, which is isomorphic to $R/fR[-1]$ if f is neither a left zero-divisor or a right zero-divisor. In this case, (2.9.10) recovers (2.9.15).

Example 2.9.69. Let R be a commutative ring and let $I \subseteq R$ be an ideal generated by a regular sequence $f_1,\dots,f_r$. Then R/I is perfect. Indeed, we have a quasi-isomorphism $K(f_1,\dots,f_r) \to R/I$. Moreover, we have an isomorphism $R/I[-r] \simeq R\mathrm{Hom}_R(R/I,R)$, computed by the Poincaré duality isomorphism $K(f_1,\dots,f_r)[-r] \simeq \mathrm{Hom}^\bullet_R(K(f_1,\dots,f_r),R)$ in Remark 2.8.36. By (2.9.10), we have

$$R/I\otimes^L_R M[-r] \simeq R\mathrm{Hom}_R(R/I,M)$$

for every $M \in D(R\text{-}\mathbf{Mod})$. In particular,

$$\mathrm{Tor}^R_{r-n}(R/I,M) \simeq \mathrm{Ext}^n_R(R/I,M).$$

Definition 2.9.70. Let R be a ring and $K \in D(R\text{-}\mathbf{Mod})$. Let $I \subseteq \mathbb{Z}$ be an interval. We say that K has Tor-*amplitude* in I if $N\otimes^L_R K \in D^I(\mathbf{Ab})$ for every right R-module N. We say that K has *finite* Tor-*amplitude* (or finite Tor-dimension) if there exists a finite interval I such that K has Tor-amplitude in I.

Taking $N = R$, we see that K has Tor-*amplitude* in I implies $K \in D^I(R\text{-}\mathbf{Mod})$.

Proposition 2.9.71. *A complex $K \in D(R\text{-}\mathbf{Mod})$ is perfect if and only if K is pseudo-coherent and has finite* Tor-*amplitude.*

Proof. The "only if" part is clear. Now assume that K is pseudo-coherent and has Tor-amplitude in $[a, b]$. To show that K is perfect, we may assume that $K \in C^-$ and K^i is finitely generated and projective for all i. Then $K \to \tau^{\geq a}K$ is a quasi-isomorphism and $(\tau^{\geq a}K)^a$ is flat. Moreover, $(\tau^{\geq a}K)^a = K^a/B^aK$ is finitely presented. Indeed, B^aK is finitely generated and there exists an R-module P such that $K^a \oplus P$ is finitely generated and free. Thus $K^a/B^aK \simeq (K^a \oplus P)/(B^aK \oplus P)$ is finitely presented. By Proposition 2.9.43, it follows that $(\tau^{\geq a}K)^a$ is projective. Thus $\tau^{\geq a}K$ is a bounded complex of finitely generated projective R-modules. □

2.10 Homology and cohomology of groups

Let G be a group. Recall that $H^n(G, -)$ is the n-th right derived functor of $(-)^G$ and $H_n(G, -)$ is the n-th left derived functor of $(-)_G$.

Consider the trivial G-action on $\mathbb{Z}$. For any G-module M, we have

$$M^G \simeq \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M), \quad M_G = M/I_GM \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} M,$$

where $I_G = \ker(\mathbb{Z}G \to \mathbb{Z})$ (the map given by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g$) is the augmentation ideal. Thus

$$H^n(G, M) \simeq \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M), \quad H_n(G, M) \simeq \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M).$$

These groups can be computed using a projective resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$. Note that the isomorphism $\mathbb{Z}G \simeq (\mathbb{Z}G)^{\mathrm{op}}$ carrying g to g^{-1} induces an isomorphism of categories between left G-modules and right G-modules.

Example 2.10.1. For $G = \mathbb{Z}$, we have an isomorphism $\mathbb{Z}[G] \simeq \mathbb{Z}[x, x^{-1}]$ sending a generator g of $\mathbb{Z}$ to x. The trivial G-module $\mathbb{Z}$ admits the free resolution

$$0 \to \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Thus $R\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$ is computed by $M \xrightarrow{g-1} M$ put in degrees 0 and 1 and $\mathbb{Z} \otimes^L_{\mathbb{Z}G} M$ is computed by $M \xrightarrow{g-1} M$ put in degrees -1 and 0. Therefore,

$$\begin{gathered} H^0(G, M) \simeq M^G \simeq H_1(G, M), \\ H^1(G, M) \simeq M/(g-1)M \simeq H_0(G, M), \\ H^n(G, M) = H_n(G, M) = 0, \ n \geq 2. \end{gathered}$$

Example 2.10.2. For $G = \mathbb{Z}/m\mathbb{Z}$, we have an isomorphism $\mathbb{Z}G \simeq \mathbb{Z}[x]/(x^m - 1)$ sending a generator g of $\mathbb{Z}$ to the class of x. Let $N = 1 + g + \cdots + g^{m-1} \in \mathbb{Z}G$. The trivial G-module $\mathbb{Z}$ admits the free resolution

$$\cdots \to \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Therefore,

$$H^n(G,M) \simeq \begin{cases} M^G & n=0 \\ \ker(N_M)/(g-1)M & n>0 \text{ odd} \\ M^G/\operatorname{im}(N_M) & n>0 \text{ even,} \end{cases}$$

$$H_n(G,M) \simeq \begin{cases} M/(g-1)M & n=0 \\ M^G/\operatorname{im}(N_M) & n>0 \text{ odd} \\ \ker(N_M)/(g-1)M & n>0 \text{ even.} \end{cases}$$

Here $N_M\colon M \xrightarrow{N} M$.

Standard resolution

Definition 2.10.3. The *standard resolution* of $\mathbb{Z}$ is the sequence

$$\cdots \to F^{-1} \to F^0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules defined as follows. For each $n \geq 0$, F^{-n} is the free $\mathbb{Z}$-module on the set $G^{n+1} = \{(g_0,\dots,g_n)\}$, with G-action defined by $g(g_0,\dots,g_n) = (gg_0,\dots,gg_n)$. The differentials are defined by $d^{-n}(g_0,\dots,g_n) = \sum_{i=0}^n (-1)^i(g_0,\dots,\hat{g}_i,\dots,g_n)$, where $\hat{g}_i$ means removing g_i.

The sequence is clearly a complex of $\mathbb{Z}G$-modules. It is exact, since the underlying complex of $\mathbb{Z}$-modules is homotopy equivalent to zero: $\mathrm{id} = dh + hd$, where $h^{-n}(g_0,\dots,g_n) = (1,g_0,\dots,g_n)$. Note that each F^{-n} is a free $\mathbb{Z}G$-module. Thus the standard resolution is a free resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$. It follows that $H^n(G,M)$ is the n-th cohomology group of the complex

$$0 \to C^0(G,M) \to C^1(G,M) \to \cdots,$$

and $H_n(G,M)$ is the $-n$-th cohomology group of the complex

$$\cdots \to C_1(G,M) \to C_0(G,M) \to 0,$$

where

$$C^n(G,M) = \operatorname{Hom}_{\mathbb{Z}G}(F^{-n},M), \quad C_n(G,M) = F^{-n} \otimes_{\mathbb{Z}G} M.$$

A basis of the free $\mathbb{Z}G$-module F^{-n} is given by those elements of G^{n+1} whose 0-th component is 1. It is convenient to adopt the *bar notation*:

$$[g_1|g_2|\dots|g_n] = (1, g_1, g_1g_2, \dots, g_1\dots g_n).$$

We have

$$d^{-n}[g_1|g_2|\dots|g_n] = g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] + (-1)^n[g_1|g_2|\dots|g_{n-1}].$$

Thus $C^n(G,M)$ can be identified with the abelian group of maps $f\colon G^n \to M$, with differential given by

$$(-1)^n(d^{n-1}f)(g_1,\dots,g_n) = \\ g_1f(g_2,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_n) + (-1)^n f(g_1,g_2,\dots,g_{n-1}).$$

H^1 and H_1

We have

$$H^1(G,M) = Z^1(G,M)/B^1(G,M),$$

where $Z^1(G,M)$ is the group of *crossed homomorphisms* (or *derivations*) $f\colon G \to M$, namely maps satisfying $f(gh) = f(g)+gf(h)$, and $B^1(G,M)$ is the group of *principal crossed homomorphisms*, namely maps of the form $g \mapsto gm - m$ for some $m \in M$. In particular, for a *trivial* G-module (namely, abelian group with trivial G-action) A, we have $H^1(G,A) \simeq \mathrm{Hom}(G_{\mathrm{ab}}, A)$, where $G_{\mathrm{ab}} = G/[G,G]$ is the abelianization of G.

The short exact sequence

$$0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0 \tag{2.10.1}$$

induces a long exact sequence

$$0 \to H_1(G,M) \to I_G \otimes_{\mathbb{Z}G} M \xrightarrow{f} M \to M_G \to 0,$$

where $f((\sum_{g\in G} a_g g)\otimes m) = \sum_{g\in G} a_g gm$. For a trivial G-module A, we have $f=0$ so that $H_1(G,A) \simeq I_G \otimes_{\mathbb{Z}G} A \simeq I_G \otimes_{\mathbb{Z}G} \mathbb{Z} \otimes_{\mathbb{Z}} A$. Applying $I_G \otimes_{\mathbb{Z}G} -$ to (2.10.1), we get $I_G \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq I_G/I_G^2$. Moreover, we have an isomorphism $I_G/I_G^2 \simeq G_{\mathrm{ab}}$ carrying the class of $\sum_g a_g g$ to $\prod_g \bar{g}^{a_g}$, where $\bar{g}$ denotes the class of g in G_{ab}. The inverse carries $\bar{g}$ to the class of $g-1$, which is well-defined since $gh-1-(g-1)-(h-1) = (g-1)(h-1) \in I_G^2$. Therefore, $H_1(G,A) \simeq G_{\mathrm{ab}} \otimes_{\mathbb{Z}} A$.

Universal coefficients and duality

Theorem 2.10.4 (Universal coefficients)**.** *For any trivial G-module A, we have split short exact sequences*

$$0 \to H_n(G) \otimes_{\mathbb{Z}} A \to H_n(G,A) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G),A) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G),A) \to H^n(G,A) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(G),A) \to 0,$$

functorial in A. Here $H_n(G) = H_n(G,\mathbb{Z})$.

Proof. This follows from Künneth formulas over $\mathbb{Z}$ and the isomorphisms

$$(\mathbb{Z} \otimes^L_{\mathbb{Z}G} \mathbb{Z}) \otimes^L_{\mathbb{Z}} A \simeq \mathbb{Z} \otimes^L_{\mathbb{Z}G} (\mathbb{Z} \otimes^L_{\mathbb{Z}} A) \simeq \mathbb{Z} \otimes^L_{\mathbb{Z}G} A,$$
$$R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \otimes^L_{\mathbb{Z}G} \mathbb{Z}, A) \simeq R\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},A)) \simeq R\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},A).$$

□

Remark 2.10.5. This also follows from the interpretation of $H^n(G,A)$ and $H_n(G,A)$ as the cohomology and homology of any $K(G,1)$-space.

Theorem 2.10.6. *Let G be a finite group. For any G-module M, we have*

$$H_n(G,M^*) \simeq H^n(G,M)^*.$$

Here $M^* = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is a right G-module. For any right G-module N, we set $H_n(G, N) = \mathrm{Tor}_n^{\mathbb{Z}G}(N, \mathbb{Z})$.

Proof. By Proposition 2.9.46 applied to $R = \mathbb{Z}G$ and $X = \mathbb{Z}$, we have

$$\mathrm{Tor}_n^{\mathbb{Z}G}(M^*, \mathbb{Z}) \simeq \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)^*.$$

□

Extensions and crossed extensions

A sequence of homomorphisms of groups $G' \xrightarrow{f} G \xrightarrow{g} G''$ is said to be *exact* at G if $\mathrm{im}(f) = \ker(g)$. Given a short exact sequence of groups

$$0 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1,$$

where M is an abelian group, conjugation in E induces an action of G on M. We call E an *extension* of G by the G-module M.

We say that the extension *splits* if there exists a section of π that is a group homomorphism. In this case the extension can be identified with the semidirect product $M \rtimes G$, with i and π given by the inclusion and projection. The underlying set of $M \rtimes G$ is $M \times G$, with group law given by $(m, g)(n, h) = (m + gn, gh)$.

Theorem 2.10.7. *Let G be a group and M a G-module. There is a canonical bijection between $H^2(G, M)$ and the set of isomorphism classes of extensions of G by M, carrying the class of $f \in Z^2(G, M)$ to the class of the group on the set $M \times G$ with group law given by $(m, g)(n, h) = (m + gn + f(g, h), gh)$.*

In particular, the bijection carries $0 \in H^2(G, M)$ to the class of split extensions. The proof is not hard. See for example [HS, Theorem 10.3].

Example 2.10.8. The class of the extension

$$0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

in $H^2(G, M)$ with $G = \mathbb{Z}/n\mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, and trivial G-action on M, is given by the 2-cocycle $G^2 \to M$ given by the table of carries $(\bar{a}, \bar{b}) \mapsto (\lfloor \frac{a+b}{m} \rfloor \mod n)$, where $0 \le a, b < m$ are representatives of $\bar{a}, \bar{b} \in \mathbb{Z}/m\mathbb{Z}$.

Definition 2.10.9. A *crossed module* is a group homomorphism $f\colon G \to H$, equipped with an action α of H on G, such that the following diagram commutes

$$\begin{tikzcd}
G \times G \arrow[r, "f \times \mathrm{id}_G"] \arrow[d] & H \times G \arrow[r, "\mathrm{id}_H \times f"] \arrow[dl, "\alpha"] & H \times H \arrow[d] \\
G \arrow[rr, "f"] & & H.
\end{tikzcd}$$

Here the vertical arrows are given by conjugation $(g, g') \mapsto gg'g^{-1}$.

Given a crossed module, $\mathrm{im}(f)$ is a normal subgroup of H, and $\ker(f)$ is central in G, equipped with an action of $\mathrm{coker}(f)$. We get an exact sequence of groups

$$0 \to \ker(f) \to G \xrightarrow{f} H \to \mathrm{coker}(f) \to 1.$$

Example 2.10.10. For any group G, the homomorphism $G \to \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ denotes the group of automorphisms of G, given by conjugation is a crossed module. It gives rise to the exact sequence

$$0 \to Z(G) \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1,$$

where $Z(G)$ denotes the center of G and $\mathrm{Out}(G)$ denotes the group of outer automorphisms of G.

Definition 2.10.11. Given a group G, a G-module M, and an integer $n \geq 1$, a *crossed n-extension* of G by M is an exact sequence of groups

$$0 \to M \xrightarrow{d_n} G_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G \to 1,$$

where d_1 is a crossed module, G_i is a G-module for $i \geq 2$, $d_2 \colon G_2 \to \ker(d_1)$ is a homomorphism of G-modules, and d_i is a homomorphism of G-modules for $i \geq 3$.

Theorem 2.10.12. *Let G be a group and M a G-module. For each integer $n \geq 1$, there is a canonical bijection between $H^{n+1}(G, M)$ and the set of equivalence classes of n-extensions of G by M.*

The equivalence relation is defined similarly to Yoneda n-extensions.

The theorem was discovered independently by several people in the 1970s. We refer to [ML1] for a sketch of the history and references.

2.11 Spectral objects and spectral sequences

Let $\tilde{\mathbb{Z}}$ be the totally ordered set $\mathbb{Z} \cup \{\pm\infty\}$. Let $\mathcal{C}_2$ be the partially ordered set of pairs $(p, q) \in \tilde{\mathbb{Z}}^2$, $p \leq q$, corresponding to the category of morphisms of $\tilde{\mathbb{Z}}$. In other words, $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$. Let $\mathcal{C}_3$ be set of triples $(p, q, r) \in \tilde{\mathbb{Z}}^3$, $p \leq q \leq r$, with the following partial order: $(p, q, r) \leq (p', q', r')$ if and only if $p \leq p'$, $q \leq q'$, and $r \leq r'$.

Definition 2.11.1. Let $\mathcal{D}$ be a triangulated category. A *spectral object* in $\mathcal{D}$ consists of a functor $X \colon \mathcal{C}_2 \to \mathcal{D}$ and morphisms $\delta(p, q, r) \colon X(q, r) \to X(p, q)[1]$, functorial in $(p, q, r) \in \mathcal{C}_3$, such that for each (p, q, r),

$$X(p, q) \to X(p, r) \to X(q, r) \xrightarrow{\delta(p,q,r)} X(p, q)[1]$$

is a distinguished triangle in $\mathcal{D}$.

Definition 2.11.2. Let $\mathcal{A}$ be an abelian category. A *spectral object* in $\mathcal{A}$ consists of functors $H^n \colon \mathcal{C}_2 \to \mathcal{A}$, $n \in \mathbb{Z}$, and morphisms $\delta^n(p, q, r) \colon H^n(q, r) \to H^{n+1}(p, q)$, functorial in $(p, q, r) \in \mathcal{C}_3$, such that for each (p, q, r), the sequence

$$H^n(p, r) \to H^n(q, r) \xrightarrow{\delta^n(p,q,r)} H^{n+1}(p, q) \to H^{n+1}(p, r)$$

is exact.

Example 2.11.3. Let X be a complex in $\mathcal{A}$, equipped with an increasing filtration

$$0 = X(-\infty) \hookrightarrow \cdots \hookrightarrow X(p) \hookrightarrow X(p+1) \hookrightarrow \cdots \hookrightarrow X(\infty) = X.$$

Set $X(p,q) = X(q)/X(p)$. The short exact sequence

$$0 \to X(q)/X(p) \to X(r)/X(p) \to X(r)/X(q) \to 0$$

induces a distinguished triangle

$$X(p,q) \to X(p,r) \to X(q,r) \xrightarrow{\delta(p,q,r)} X(p,q)[1]$$

in $D(\mathcal{A})$. We thus obtain a spectral object in $D(\mathcal{A})$.

For any complex X, the following spectral objects induced by truncation are particularly useful. The spectral object associated to the filtration $X(p) = \sigma^{\geq -p}X$ is called the first spectral object of X, satisfying $X(-p-1,-p) \simeq X^p[-p]$. The spectral object associated to the filtration $X(p) = \tau^{\leq p}X$ is called the second spectral object of X, satisfying $X(p-1,p) \simeq (H^pX)[-p]$.[5] Note that the first spectral object of X is functorial in $X \in C(\mathcal{A})$ and the second spectral object of X is functorial in $X \in D(\mathcal{A})$.

Remark 2.11.4. Let X be a spectral object in $\mathcal{D}$. Any triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ induces a spectral object FX in $\mathcal{D}'$. Any cohomological functor $H\colon \mathcal{D} \to \mathcal{A}$ induces a spectral object $H(X[n])$ in $\mathcal{A}$.

Given a spectral object (H^n, δ^n) in $\mathcal{A}$, consider the increasing filtration

$$F^q = F^qH^n(-\infty,\infty) = \operatorname{im}(H^n(-\infty,q) \to H^n(-\infty,\infty)).$$

We approximate $\mathrm{gr}^q_F H^n(-\infty,\infty) = F^q/F^{q-1}$ by

$$E^{p,q}_{r+1} = \operatorname{im}(H^n(q-r,q) \to H^n(q-r,q+r-1))/\operatorname{im}(H^n(q-r,q-1) \to H^n(q-r,q+r-1))$$

for $r \geq 1$, where $p = n-q$. We have $E^{p,q}_2 \simeq H^n(q-1,q)$ and $E^{p,q}_\infty = \mathrm{gr}^q_F H^n(-\infty,\infty)$. It follows from the exact sequence

$$H^n(q-r,q-1) \to H^n(q-r,q+r-1) \to H^n(q-1,q+r-1)$$

that $E^{p,q}_{r+1} \simeq \operatorname{im}(H^n(q-r,q) \to H^n(q-1,q+r-1))$. The commutative diagram

$$\begin{tikzcd}
H^n(q-r,q) \arrow[r] \arrow[d, "\delta^n"'] & H^n(q-1,q+r-1) \arrow[d, "\delta^n"] \\
H^{n+1}(q-2r,q-r) \arrow[r] & H^{n+1}(q-r-1,q-1)
\end{tikzcd}$$

induces a morphism $d^{p,q}_{r+1}\colon E^{p,q}_{r+1} \to E^{p+r+1,q-r}_{r+1}$.

[5] Our convention for the second spectral object differs from [V2, III.4.3.1] by a shift by 1.

Definition 2.11.5. Let $a \in \mathbb{Z}$. A *spectral sequence* $(E_r^{p,q})_{r\geq a}$ in $\mathcal{A}$ is a family of objects $E_r^{p,q}$ in $\mathcal{A}$ for $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq a}$ equipped with *differentials* $d_r = d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r d_r = 0$, and isomorphisms of $E_{r+1}^{p,q}$ with the cohomology of $E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$ at $E_r^{p,q}$. For each r, the collection $(E_r^{p,q}, d_r^{p,q})$ is sometimes called a *page* of the spectral sequence.

Given a spectral sequence $(E_r^{p,q})_{r\geq a}$ and objects $(H^n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$, an *abutment*, usually denoted by $E_a^{p,q} \Rightarrow H^n$, consists of an increasing filtration $F^q H^n$ on each H^n and an identification of $E_\infty^{p,q} = \mathrm{gr}_F^q H^n = F^q H^n / F^{q-1} H^n$ with a subquotient of $E_r^{p,q}$, compatible with the identification of $E_{r+1}^{p,q}$ with a subquotient of $E_r^{p,q}$. Here $p = n - q$.

We say that the spectral sequence with abutment *converges* if for every pair (p, q), there exists $b \geq a$ such that $E_b^{p,q} = E_\infty^{p,q}$. We say that the spectral sequence with abutment *degenerates at* E_b if $E_b^{p,q} = E_\infty^{p,q}$ for all p, q, which is equivalent to $d_r^{p,q} = 0$ for all $r \geq b$, p, q in addition to convergence.

Theorem 2.11.6. *Let (H^n, δ^n) be a spectral object in $\mathcal{A}$. The construction above provides a spectral sequence with abutment*

$$E_2^{p,q} = H^{p+q}(q-1, q) \Rightarrow H^n(-\infty, \infty).$$

We refer to [V2, Sections II.4.2, II.4.3] for a proof (cf. [CE, Section XV.7]).

Remark 2.11.7 (Page shift)**.** Given a spectral sequence $(E_r^{p,q})_{r\geq a}$ and $c \in \mathbb{Z}$, we can produce a spectral sequence $(E_r'^{p,q})_{r\geq a-c}$ by $E_r'^{p,q} = E_{r+c}^{p+cn,q-cn}$, where $n = p + q$. Moreover, an abutment $E_a^{p,q} \Rightarrow H^n$ is the same as an abutment $E_{a-c}'^{p,q} \Rightarrow H^n$. Applying this to the spectral sequence in the theorem, we get a spectral sequence with abutment

$$E_1'^{p,q} = H^{p+q}(-p-1, -p) \Rightarrow H^n(-\infty, \infty).$$

Example 2.11.8. Let $F \colon D^*(\mathcal{A}) \to D(\mathcal{B})$ be a triangulated functor and let $X \in C^*(\mathcal{A})$, where $*$ is either empty or one of $+, -, b$. Then the first and second spectral objects of X induce, via F and the cohomological functor $H^0 \colon D(\mathcal{B}) \to \mathcal{B}$, spectral sequences with abutments:

$$E_1^{p,q} = H^q F(X^p) \Rightarrow H^n(FX), \tag{2.11.1}$$

$$E_2'^{p,q} = H^p F(H^q X) \Rightarrow H^n(FX). \tag{2.11.2}$$

In the first spectral sequence, $d_1^{p,q} = H^q F(d_X^p)$.

Example 2.11.9 (Grothendieck spectral sequence)**.** Let $F \colon \mathcal{A} \to \mathcal{B}$, $G \colon \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories. Let $\mathcal{I} \subseteq \mathcal{A}$ be an F-injective subcategory and let $\mathcal{J} \subseteq \mathcal{B}$ be a G-injective subcategory. Assume that F carries $\mathcal{I}$ into $\mathcal{J}$. By Proposition 2.7.13, $R(GF) \simeq (RG)(RF)$. As a special case of (2.11.2), for $X \in \mathcal{A}$ (or more generally $X \in D^+(\mathcal{A})$), we get a converging spectral sequence

$$E_2^{p,q} = R^p G R^q F(X) \Rightarrow R^n(GF)(X).$$

Example 2.11.10 (Hochschild–Serre spectral sequence)**.** Let G be a group and H a normal subgroup. Given a G-module M, M^H is equipped with an action of G/H.

This defines a functor $F\colon \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbb{Z}(G/H)\text{-}\mathbf{Mod}$, that fits into a commutative square

$$\begin{array}{ccc}
\mathbb{Z}G\text{-}\mathbf{Mod} & \xrightarrow{F} & \mathbb{Z}(G/H)\text{-}\mathbf{Mod} \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\phi} \\
\mathbb{Z}H\text{-}\mathbf{Mod} & \xrightarrow{F'} & \mathbf{Ab},
\end{array}$$

where $F' = (-)^H$, ρ is restriction of scalars and ϕ is the forgetful functor. Since $\mathbb{Z}G$ is a free right $\mathbb{Z}H$-module, ρ preserves injectives. Thus the above commutative square induces a commutative square

$$\begin{array}{ccc}
D^+(\mathbb{Z}G\text{-}\mathbf{Mod}) & \xrightarrow{RF} & D^+(\mathbb{Z}(G/H)\text{-}\mathbf{Mod}) \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\phi} \\
D^+(\mathbb{Z}H\text{-}\mathbf{Mod}) & \xrightarrow{RF'} & D^+(\mathbf{Ab}).
\end{array}$$

In particular, the abelian group underlying R^nFM can be identified with $H^n(H, M)$.

The composite of $\mathbb{Z}G\text{-}\mathbf{Mod} \xrightarrow{F} \mathbb{Z}(G/H)\text{-}\mathbf{Mod} \xrightarrow{(-)^{G/H}} \mathbf{Ab}$ is $(-)^G$. The Grothendieck spectral sequence for this composition takes the following form: for every G-module M (or more generally $M \in D^+(G\text{-}\mathbf{Mod})$), we have

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \Rightarrow H^n(G, M).$$

Example 2.11.11. Let $m > 0$ be an odd integer and let $D_m = C_m \rtimes C_2$ be the dihedral group, where $C_m = \mathbb{Z}/m\mathbb{Z}$ and $C_2 = \mathbb{Z}/2\mathbb{Z}$. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(C_2, H^q(C_m, \mathbb{Z})) \Rightarrow H^n(D_m, \mathbb{Z}).$$

We have

$$H^q(C_m, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/m\mathbb{Z} & q > 0 \text{ even} \\ 0 & q \text{ odd.} \end{cases}$$

Let $C_2 = \{1, \sigma\}$. One can check that σ acts on $H^{2j}(C_m, \mathbb{Z})$ by multiplication by $(-1)^j$. It follows that

$$E_2^{p,q} = \begin{cases} \mathbb{Z} & p = q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q = 0,\ p > 0 \text{ even} \\ \mathbb{Z}/m\mathbb{Z} & p = 0,\ 4 \mid q > 0 \\ 0 & \text{otherwise.} \end{cases}$$

part of which is shown below:

$$
\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 & 0 \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 & 0 \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 & 0 \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 & 0 \\
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}
$$

Thus the spectral sequence degenerates at E_2. Since extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/m\mathbb{Z}$ are trivial, we have

$$
H^n(D_m, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4 \\ \mathbb{Z}/2m\mathbb{Z} & 4 \mid n \\ 0 & n \text{ odd.} \end{cases}
$$

Summary of properties of rings and modules

Properties of rings

$$
\begin{tikzcd}
 & \text{field} \arrow[r, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \substack{\text{division}\\ \text{ring}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{semisimple} \arrow[r, Rightarrow, shift left] \arrow[dd, Rightarrow] & \text{von Neumann regular} \arrow[l, Rightarrow, dotted, shift left, "\substack{\text{left}\\ \text{Noetherian}}"] \arrow[ddd, Rightarrow, bend left] \\
\substack{\text{valuation}\\ \text{ring}} \arrow[d, Rightarrow] & \text{PID} \arrow[r, Rightarrow] \arrow[dl, Rightarrow, shift left] \arrow[d, Rightarrow] & \text{PLID} \arrow[ddd, Rightarrow] \arrow[dr, Rightarrow] & & \\
\substack{\text{Bézout}\\ \text{domain}} \arrow[ur, Rightarrow, dotted, shift left, "\substack{\text{left}\\ \text{Noeth.}}"] \arrow[dr, Rightarrow] & \text{Dedekind} \arrow[rr, Rightarrow] \arrow[d, Rightarrow, shift right] & & \substack{\text{left hereditary}\\ \text{left Noetherian}} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{left hereditary} \arrow[d, Rightarrow] \\
 & \substack{\text{Prüfer}\\ \text{domain}} \arrow[u, Rightarrow, dotted, shift right, "\substack{\text{left}\\ \text{Noetherian}}"'] \arrow[rrr, Rightarrow] \arrow[d, Rightarrow] & & & \text{left semi-hereditary} \arrow[ul, Rightarrow, dotted, "\text{left Noetherian}"] \arrow[d, Rightarrow] \\
 & \substack{\text{commutative}\\ \text{domain}} \arrow[r, Rightarrow] & \text{domain} & \text{left Noetherian} \arrow[r, Rightarrow] & \text{left coherent}
\end{tikzcd}
$$

Properties of R-modules

$$
\begin{tikzcd}
\text{free} \arrow[r, Rightarrow, shift left] & \text{projective} \arrow[l, Rightarrow, dotted, shift left, "R \text{ PLID or } R = S[x_1, \ldots, x_n]"] \arrow[r, Rightarrow, shift left] \arrow[d, Leftrightarrow, dotted, "R \text{ quasi-Frobenius}"'] & \text{flat} \arrow[l, Rightarrow, dotted, shift left, "\text{finitely presented}"] \arrow[r, Rightarrow, shift left] & \text{torsion-free} \arrow[l, Rightarrow, dotted, shift left, "R \text{ Prüfer domain}"] \\
 & \text{injective} \arrow[r, Rightarrow, shift left] & \substack{\text{divisible}\\ (R \text{ domain})} \arrow[l, Rightarrow, dotted, shift left, "R \text{ Dedekind or PLID}"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Noetherian} \arrow[r, Rightarrow] & \text{coherent} \arrow[r, Rightarrow] & \substack{\text{finitely}\\ \text{presented}} \arrow[r, Rightarrow, shift left] & \substack{\text{finitely}\\ \text{generated}} \arrow[l, Rightarrow, dotted, shift left, "\text{projective}"]
\end{tikzcd}
$$

Here S denotes a PID.

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