Mod ℓ cohomology algebras of quotient stacks

Analogues of Quillen's theory

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Plan of the talk

Introduction

Fix a prime number ℓ. $\mathbb{F}_\ell := \mathbb{Z}/\ell\mathbb{Z}$. Let G be a compact Lie group, BG be a classifying space of G. Consider the graded $\mathbb{F}_\ell$-algebra

$$H_G^*(\mathbb{F}_\ell) := H^*(BG, \mathbb{F}_\ell),$$

satisfying

$$a \cup b = (-1)^{ij} b \cup a$$

for $a \in H_G^i(\mathbb{F}_\ell)$, $b \in H_G^j(\mathbb{F}_\ell)$. The $\mathbb{F}_\ell$-algebra $H_G^{\epsilon *}(\mathbb{F}_\ell)$ is commutative, where

$$\epsilon = \begin{cases} 1 & \ell = 2, \\ 2 & \ell > 2. \end{cases}$$

Definition

An elementary abelian ℓ-group is a finite dimensional $\mathbb{F}_\ell$-vector space. The rank of the group is the dimension of the vector space.

Fact

Let $A \simeq (\mathbb{Z}/\ell\mathbb{Z})^r$.

$$H_A^*(\mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \ell = 2, \\ \wedge(\mathbb{F}_\ell x_1 \oplus \cdots \oplus \mathbb{F}_\ell x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \ell > 2, \end{cases}$$

where $x_1, \dots, x_r$ *form a basis of* $H^1 = \mathrm{Hom}(A, \mathbb{F}_\ell)$, $y_1, \dots, y_r \in H^2$. *In particular,* $\mathrm{Spec}(H_A^{\epsilon *}(\mathbb{F}_\ell))$ *is homeomorphic to* $\mathbb{A}^r_{\mathbb{F}_\ell}$.

Quillen's structure theorem

Let $\mathcal{A}$ be the category of elementary abelian ℓ-subgroups of G. A morphism $A \to A'$ in $\mathcal{A}$ is an element $g \in G$ such that $g^{-1}Ag \subset A'$.

Theorem (Quillen)

The homomorphism

$$H^*_G(\mathbb{F}_\ell) \to \varprojlim_{A \in \mathcal{A}} H^*_A(\mathbb{F}_\ell)$$

is a uniform F-isomorphism.

A homomorphism of $\mathbb{F}_\ell$-algebras is called a uniform F-isomorphism if $F^N = 0$ on the kernel and cokernel for N large enough. Here $F\colon a \mapsto a^\ell$.

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Corollary

The Krull dimension of $H_G^{\epsilon *}(\mathbb{F}_\ell)$ is equal to the maximum rank of the elementary abelian ℓ-subgroups of G.

This was conjectured by Atiyah and Swan.

More generally, Quillen considered the equivariant cohomology algebra $H^*_G(X, \mathbb{F}_\ell)$, where X is a topological space acted on by G.

Theorem (Quillen)

Assume X is paracompact and of finite ℓ-cohomological dimension. Then the homomorphism

$$H^*_G(X, \mathbb{F}_\ell) \to \varprojlim_{(A,C)} H^*_A(\mathbb{F}_\ell)$$

is a uniform F-isomorphism. Here the limit is taken over pairs (A, C), where A is an elementary abelian ℓ-subgroup of G, C is a connected component of the fixed point set X^A.

Algebraic setting

Fix an algebraically closed base field k of characteristic $\neq \ell$.

- Structure theorem for $H^*([X/G], \mathbb{F}_\ell)$, where X is a scheme over k, G is an algebraic group over k acting on X, $[X/G]$ is the quotient stack.

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- Structure theorem for $H^*([X/G], \mathbb{F}_\ell)$, where X is a scheme over k, G is an algebraic group over k acting on X, $[X/G]$ is the quotient stack.
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- Stacky interpretation of $H^*(\mathcal{M}, \mathbb{F}_\ell)$, where $\mathcal{M}$ is a moduli stack over k.
- $H^*(\mathcal{M}, R^*f_*\mathbb{F}_\ell)$, where $f\colon \mathcal{T} \to \mathcal{M}$ is a universal family.

Plan of the talk

Plan of the talk

Cartesian sheaves

Let $\mathcal{X}$ be an Artin stack.

$$\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell) := \varprojlim \mathrm{Mod}(X_{\text{ét}}, \mathbb{F}_\ell),$$

where the limit is taken over smooth morphisms $X \to \mathcal{X}$, where X is a scheme. If $X_0 \to \mathcal{X}$ is a smooth presentation (i.e. a smooth surjection such that X_0 is a scheme),

$$\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell) \simeq \varprojlim_n \mathrm{Mod}((X_n)_{\text{ét}}, \mathbb{F}_\ell),$$

where $X_\bullet = \mathrm{cosk}_0(X_0/\mathcal{X})$ (X_n is the fiber product of $n+1$ copies of X_0 above $\mathcal{X}$).

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Example

If $\mathcal{X}$ is a Deligne-Mumford stack, $\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell) \simeq \mathrm{Mod}(\mathcal{X}_{\text{ét}}, \mathbb{F}_\ell)$.

Example

Let X be a scheme over k, G be an algebraic group over k acting on X. The quotient stack $[X/G]$ is an Artin stack and $\mathrm{Mod}_{\mathrm{Cart}}([X/G], \mathbb{F}_\ell)$ is the category of G-equivariant $\mathbb{F}_\ell$-sheaves on $X_{\text{ét}}$.

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Example

$BG = [\mathrm{Spec}(k)/G]$. $\mathrm{Mod}_{\mathrm{Cart}}(BG, \mathbb{F}_\ell)$ is the category of $\mathbb{F}_\ell$-representations of G. In particular,

$$\mathrm{Mod}_{\mathrm{Cart}}(BG, \mathbb{F}_\ell) \simeq \mathrm{Mod}_{\mathrm{Cart}}(B\pi_0(G), \mathbb{F}_\ell).$$

$\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$ does not determine $H^*(\mathcal{X}, \mathbb{F}_\ell)$.

Derived category of Cartesian sheaves

Two approaches:

1 (Laumon-Moret-Bailly) Consider the site whose objects are smooth morphisms $X \to \mathcal{X}$ where X is a scheme and whose covering families are smooth surjective families. It defines a topos $\mathcal{X}_{\mathrm{sm}}$. $\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$ is a full subcategory of $\mathrm{Mod}(\mathcal{X}_{\mathrm{sm}}, \mathbb{F}_\ell)$. Define $D_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$ to be the triangulated full subcategory of $D(\mathcal{X}_{\mathrm{sm}}, \mathbb{F}_\ell)$ consisting of complexes with cohomology sheaves in $\mathrm{Mod}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$.

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- (Behrend, Gabber) $\mathcal{X}_{\mathrm{sm}}$ is not functorial. For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Artin stacks, $f^*\colon \mathcal{Y}_{\mathrm{sm}} \to \mathcal{X}_{\mathrm{sm}}$ is not left exact in general.
- (Olsson, Laszlo-Olsson) Define

$$f^*\colon D_{\mathrm{Cart}}(\mathcal{Y}, \mathbb{F}_\ell) \to D_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$$

using smooth presentations and cohomological descent.

2 (Liu-Zheng in progress)

$$\mathcal{D}_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell) := \varprojlim_n \mathcal{D}((X_n)_{\text{ét}}, \mathbb{F}_\ell)$$

is a presentable stable ∞-category, independent (up to equivalences) of the choice of the smooth presentation $X_0 \to \mathcal{X}$. Here $\mathcal{D}((X_n)_{\text{ét}}, \mathbb{F}_\ell)$ is the derived ∞-category of $\mathrm{Mod}((X_n)_{\text{ét}}, \mathbb{F}_\ell)$ defined by Lurie.

- Advantages:
  base change in derived categories (instead of on the level of sheaves); fewer finiteness assumptions

We define $D_c(\mathcal{X}, \mathbb{F}_\ell)$ to be the full subcategory of $D_{\mathrm{Cart}}(\mathcal{X}, \mathbb{F}_\ell)$ consisting of complexes with constructible cohomology sheaves. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks of finite presentation (*a fortiori* quasi-separated) over k. We have functors

$$f^*\colon D_c^+(\mathcal{Y}, \mathbb{F}_\ell) \to D_c^+(\mathcal{X}, \mathbb{F}_\ell),$$
$$Rf_*\colon D_c^+(\mathcal{X}, \mathbb{F}_\ell) \to D_c^+(\mathcal{Y}, \mathbb{F}_\ell),$$
$$-\otimes -\colon D_c^+(\mathcal{X}, \mathbb{F}_\ell) \times D_c^+(\mathcal{X}, \mathbb{F}_\ell) \to D_c^+(\mathcal{X}, \mathbb{F}_\ell).$$

Proposition

Let $G = GL_{n,k}$, $T = \mathbb{G}_m^n \subset G$. *Then*

$$R\Gamma(BT, \mathbb{F}_\ell) = \oplus_q H^{2q}(BT, \mathbb{F}_\ell)[-2q],$$
$$H^*(BT, \mathbb{F}_\ell) = \mathbb{F}_\ell[t_1, \dots, t_n],$$

where $t_i = c_1(\mathcal{L}_i) \in H^2(BT, \mathbb{F}_\ell)$, $\mathcal{L}_i$ *is the i-th tautological line bundle on* BT, *and*

$$R\Gamma(BG, \mathbb{F}_\ell) = \oplus_q H^{2q}(BG, \mathbb{F}_\ell)[-2q],$$
$$H^*(BG, \mathbb{F}_\ell) = (\mathbb{F}_\ell[t_1, \dots, t_n])^{\mathfrak{S}_n} = \mathbb{F}_\ell[x_1, \dots, x_n],$$

where $x_i = c_i(\mathcal{E}) \in H^{2i}(BG, \mathbb{F}_\ell)$, $\mathcal{E}$ *is the tautological vector bundle on* BG.

This follows from approximation by finite Grassmannians (Deligne).

A finiteness theorem

Theorem

Let X be a scheme of finite type over k, G be a linear algebraic group over k acting on X, $K \in D_c^b([X/G], \mathbb{F}_\ell)$. Then $H^*(BG, \mathbb{F}_\ell)$ is a finitely generated $\mathbb{F}_\ell$-algebra and $H^*([X/G], K)$ is a finite $H^*(BG, \mathbb{F}_\ell)$-module.

This is an analogue of Quillen's finiteness theorem (for G a compact Lie group).

Plan of the talk

Equivariant version with constant coefficients

Let X be a separated scheme of finite type over k, G be a linear algebraic group over k acting on X. Let $\mathcal{B}$ be the category of pairs (A, C), where A is an elementary abelian ℓ-subgroup of G, $C \in \pi_0(X^A)$. A morphism $(A, C) \to (A', C')$ in $\mathcal{B}$ is an element $g \in G(k)$ such that $g^{-1}Ag \subset A'$, $Cg \supset C'$.

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$$\begin{array}{ccc} BA \times C' & \longrightarrow & BA \times C \\ \downarrow & & \downarrow \\ BA' \times C' & \longrightarrow & [X/G] \end{array}$$

which in turn induces a commutative diagram

$$\begin{array}{ccccc} H^*([X/G], \mathbb{F}_\ell) & \longrightarrow & H^*(BA' \times C', \mathbb{F}_\ell) & \longrightarrow & H^*(BA', \mathbb{F}_\ell) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(BA \times C, \mathbb{F}_\ell) & \longrightarrow & H^*(BA \times C', \mathbb{F}_\ell) & \longrightarrow & H^*(BA, \mathbb{F}_\ell) \end{array}$$

Theorem

The homomorphism

$$H^*([X/G], \mathbb{F}_\ell) \to \varprojlim_{(A,C)\in\mathcal{B}} H^*(BA, \mathbb{F}_\ell)$$

is a uniform F-isomorphism.

Finiteness of orbit types

Let G be an algebraic group over k, A be a finite group, $X = \mathrm{Hom}(A, G)$ (a closed subscheme of $\prod_{a\in A} G$). G acts on X by conjugation.

Theorem (Serre)

Assume that the order of A is indivisible by the characteristic of k. Then the orbits of X are open and the number of orbits is finite.

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Corollary

There are only finitely many conjugacy classes of elementary abelian ℓ-subgroups of G.

It follows that there are only finitely many isomorphism classes of objects of $\mathcal{B}$. Moreover, the limit in the preceding structure theorem is isomorphic to a limit indexed by a finite category.

Toward general coefficients

In $BA \times X^A$, BA is covariant with respect to A and X^A is contravariant with respect to A.

Ends

Let $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor.

Definition

- Let E be an object of $\mathcal{D}$. A wedge $w\colon E \to F$ is a family $(w_A\colon E \to F(A,A))_{A\in\mathcal{C}}$ of morphisms in $\mathcal{D}$ such that for every morphism $f\colon A \to A'$ in $\mathcal{C}$, the following square commutes

$$\begin{array}{ccc} E & \xrightarrow{w_A} & F(A,A) \\ \downarrow{\scriptstyle w_{A'}} & & \downarrow{\scriptstyle F(1,f)} \\ F(A',A') & \xrightarrow{F(f,1)} & F(A,A') \end{array}$$

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- An end of F is an object $E = \int_{A\in\mathcal{C}} F(A,A)$ of $\mathcal{D}$ endowed with a wedge $w\colon E \to F$ such that for every wedge $w'\colon E' \to F$ there exists a unique morphism $h\colon E' \to E$ such that $w' = w \circ h$.

Define a category $\mathcal{C}^\flat$ as follows. An object of $\mathcal{C}^\flat$ is a morphism $A \to A'$ in $\mathcal{C}$. A morphism in $\mathcal{C}^\flat$ from $A \to A'$ to $B \to B'$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \uparrow \\ B & \longrightarrow & B' \end{array}$$

We have

$$\int_{A \in \mathcal{C}} F(A, A) \simeq \varprojlim_{(A \to A') \in \mathcal{C}^\flat} F(A, A').$$

Reformulation of the structure theorem

Let $\mathcal{A}$ be the category of elementary abelian ℓ-subgroups of G. A morphism $A \to A'$ in $\mathcal{A}$ is an element $g \in G(k)$ such that $g^{-1}Ag \subset A'$.

- We have

$$\varprojlim_{(A,C)\in\mathcal{B}} H^*(BA, \mathbb{F}_\ell) \simeq \int_{A\in\mathcal{A}} H^0(X^A, R^*\pi_*\mathbb{F}_\ell),$$

  where $\pi\colon BA \times X^A \to X^A$ is the projection.

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  where $\pi\colon BA \times X^A \to X^A$ is the projection.

- The structure theorem for constant coefficients is equivalent to the assertion that the homomorphism

$$H^*([X/G], \mathbb{F}_\ell) \to \int_{A\in\mathcal{A}} H^*(BA\times X^A, \mathbb{F}_\ell) \simeq \varprojlim_{(A\to A')\in\mathcal{A}^\flat} H^*(BA\times X^{A'}, \mathbb{F}_\ell)$$

  is a uniform F-isomorphism.

Let $\mathcal{A}^\natural$ be the category of pairs (A, A'), where $A \subset A' \subset G$ are elementary abelian ℓ-subgroups. A morphism $(A, A') \to (Z, Z')$ in $\mathcal{A}^\natural$ is an element $g \in G(k)$ such that $g^{-1}Ag \subset Z$, $g^{-1}A'g \supset Z'$. The inclusion $\mathcal{A}^\natural \subset \mathcal{A}^\flat$ is cofinal, so that for any functor $F\colon (\mathcal{A}^\flat)^{\mathrm{op}} \to \mathcal{D}$, we have

$$\varprojlim_{(A\to A')\in\mathcal{A}^\flat} F(A \to A') \simeq \varprojlim_{(A,A')\in\mathcal{A}^\natural} F(A \subset A').$$

Equivariant version with general coefficients

A morphism $(A, A') \to (Z, Z')$ in $\mathcal{A}^\natural$ induces a 2-commutative diagram:

$$\begin{array}{ccc} BA \times X^{A'} & \longrightarrow & BZ \times X^{Z'} \\ & \searrow & \downarrow \\ & & [X/G] \end{array}$$

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Theorem

Let $K \in D_c^+([X/G], \mathbb{F}_\ell)$, endowed with a ring structure $K \otimes K \to K$. Then the homomorphism

$$H^*([X/G], K) \to \varprojlim_{(A,A') \in \mathcal{A}^\natural} H^*(BA \times X^{A'}, K)$$

is a uniform F-isomorphism.

Geometric points of Artin stacks

Let X be a scheme. The category of points of $X_{\text{ét}}$ is equivalent to the category of geometric points of X. A geometric point of X is a morphism $x \to X$, where x is the spectrum of a separably closed field. A morphism of geometric points from $x \to X$ to $y \to X$ is an X-morphism $X_{(x)} \to X_{(y)}$ of the strict henselizations. This construction extends trivially to Deligne-Mumford stacks. For Artin stacks we proceed as follows.

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Definition

Let $\mathcal{X}$ be an Artin stack. We denote by $\mathcal{P}'_{\mathcal{X}}$ the category of morphisms $S \to \mathcal{X}$ where S is a strictly local scheme (spectrum of a strictly henselian local ring). The category $\mathcal{P}_{\mathcal{X}}$ of geometric points of $\mathcal{X}$ is the category obtained from $\mathcal{P}'_{\mathcal{X}}$ by inverting local morphisms.

Example

Let G be an algebraic group scheme over k. Then $\mathcal{P}_{BG} \simeq \mathcal{P}_{B\pi_0(G)}$ is a connected groupoid of fundamental group $\pi_0(G)$.

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Proposition

Let $\mathcal{X}$ be an Artin stack, $\mathcal{F} \in \mathrm{Mod}_c(\mathcal{X}, \mathbb{F}_\ell)$. The homomorphism

$$H^0(\mathcal{X}, \mathcal{F}) \to \varprojlim_{x \in \mathcal{P}_{\mathcal{X}}} \mathcal{F}_x$$

is an isomorphism.

We didn't find any reference even for the case of a scheme.

Stacky version

Let $\mathcal{X}$ be an Artin stack of finite presentation over k.

- A morphism $\mathcal{Y} \to \mathcal{X}$ of Artin stacks is representable if for every geometric point y of $\mathcal{Y}$, the group homomorphism $\mathrm{Aut}_{\mathcal{Y}}(y) \to \mathrm{Aut}_{\mathcal{X}}(y)$ is a monomorphism.

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- We denote by $\mathcal{Q}'_{\mathcal{X}}$ the category of representable morphisms $\mathcal{S} \to \mathcal{X}$, where $\mathcal{S} \simeq [S/A]$, S is a strictly local scheme, A is an elementary abelian ℓ-group acting on S and acting trivially on the closed point s of S. An $\mathcal{X}$-morphism $\mathcal{S} \to \mathcal{S}'$ induces a monomorphism of groups

$$A = \mathrm{Aut}_{\mathcal{S}}(s) \to \mathrm{Aut}_{\mathcal{S}'}(s).$$

We denote by $\mathcal{Q}_{\mathcal{X}}$ the category obtained from $\mathcal{Q}'_{\mathcal{X}}$ by inverting local morphisms whose induced monomorphisms of groups are isomorphisms.

Theorem

Assume that either $\mathcal{X}$ has finite inertia, or $\mathcal{X} \simeq [X/G]$, where X is a separated scheme of finite type over k and G is a linear algebraic group over k acting on G. Let $K \in D_c^+(\mathcal{X}, \mathbb{F}_\ell)$, endowed with a ring structure $K \otimes K \to K$. Then the homomorphism

$$H^*(\mathcal{X}, K) \to \varprojlim_{\mathcal{S} \in \mathcal{Q}_\mathcal{X}} H^*(\mathcal{S}, K)$$

is a uniform F-isomorphism.

Note that $H^*(\mathcal{S}, K) = H^*(BA_s, K_s)$.

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Remark

The three structure theorems also hold for $[X/G]$, where G is an abelian variety.

The inertia stack $I_{\mathcal{X}}$ of $\mathcal{X}$ is the fiber product

$$\begin{array}{ccc} I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \Delta_{\mathcal{X}} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ at a geometric point $x \to \mathcal{X}$ is the group scheme $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$. We say $\mathcal{X}$ has finite inertia if $I_{\mathcal{X}} \to \mathcal{X}$ is finite. If $\mathcal{X}$ is a separated Deligne-Mumford stack, then $\mathcal{X}$ has finite inertia.

One step of the proof

Assume $\mathcal{X}$ has finite inertia. Let $\pi\colon \mathcal{X} \to Y$ be the projection to the coarse moduli space. The edge homomorphism

$$H^*(\mathcal{X}, K) \to H^0(Y, R^*\pi_*K)$$

of the Leray spectral sequence of π

$$E_2^{pq} = H^p(Y, R^q\pi_*K) \Rightarrow H^{p+q}(\mathcal{X}, K)$$

is a uniform F-isomorphism.

Plan of the talk

A localization theorem

Let k be an algebraically closed field of characteristic $p \geq 0$ (possibly equal to ℓ), X be a separated scheme of finite type over k, $A \simeq (\mathbb{Z}/\ell\mathbb{Z})^r$. Recall

$$H^*(BA, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \ell = 2, \\ \wedge(\mathbb{F}_\ell x_1 \oplus \cdots \oplus \mathbb{F}_\ell x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \ell > 2, \end{cases}$$

where $x_1, \dots, x_r$ form a basis of $H^1 = \mathrm{Hom}(A, \mathbb{F}_\ell)$; $y_i = \beta x_i \in H^2$, $\beta\colon H^1 \to H^2$ is the Bockstein.

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Let k be an algebraically closed field of characteristic $p \geq 0$ (possibly equal to ℓ), X be a separated scheme of finite type over k, $A \simeq (\mathbb{Z}/\ell\mathbb{Z})^r$. Recall

$$H^*(BA, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \ell = 2, \\ \wedge(\mathbb{F}_\ell x_1 \oplus \cdots \oplus \mathbb{F}_\ell x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \ell > 2, \end{cases}$$

where $x_1, \dots, x_r$ form a basis of $H^1 = \mathrm{Hom}(A, \mathbb{F}_\ell)$; $y_i = \beta x_i \in H^2$, $\beta \colon H^1 \to H^2$ is the Bockstein.

Theorem

For any action of A on X, the homomorphism

$$H^*([X/A], \mathbb{F}_\ell)[e^{-1}] \to H^*(X^A \times BA, \mathbb{F}_\ell)[e^{-1}]$$

is an isomorphism, where $e = \prod_{0 \neq x \in H^1} \beta x \in H^{2\ell^r - 2}$.

Definition

X is mod ℓ acyclic if $H^q(X, \mathbb{F}_\ell) = 0$ for $q \neq 0$ and $H^0(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$.

If $\ell \neq p$, X is mod ℓ acyclic if and only if $H^q(X, \mathbb{Z}_\ell) = 0$ for $q \neq 0$ and $H^0(X, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell$.

Corollary

Assume X is mod ℓ acyclic. Let G be a finite group acting on X. Then

- *X/G is mod ℓ acyclic;*
- *(Serre) X^G is mod ℓ acyclic if G is an ℓ-group.*

The reduced cohomology $\tilde{H}^q(X, \mathbb{F}_\ell)$ is defined by $\tilde{H}^q(X, \mathbb{F}_\ell) = H^q(X, \mathbb{F}_\ell)$ for $q \neq -1, 0$ and the exact sequence

$$0 \to \tilde{H}^{-1}(X, \mathbb{F}_\ell) \to \mathbb{F}_\ell \to H^0(X, \mathbb{F}_\ell) \to \tilde{H}^0(X, \mathbb{F}_\ell) \to 0.$$

Definition

X is a cohomological sphere of dimension N if $\tilde{H}^q(X, \mathbb{F}_\ell) = 0$ for $q \neq N$ and $\tilde{H}^N(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$.

X is a cohomological sphere of dimension -1 if and only if X is empty.

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Corollary

Assume that an ℓ-group G acts on X and X is a cohomological sphere of dimension N. Then X^G is a cohomological sphere of dimension $M \leq N$ and $\ell(N - M)$ is even.

This is an analogue of a theorem of Borel (which generalizes a theorem of Smith).

The end

Thank you!