Compatible systems along the boundary

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References

- K. Fujiwara. Independence of ℓ for intersection cohomology (after Gabber). In *Algebraic geometry 2000, Azumino (Hotaka)*, 2002.
- I. Vidal. Critères valuatifs (d'après O. Gabber). Appendice à "Courbes nodales et ramification sauvage virtuelle", *Manuscripta Math.*, 2005.

Plan of the talk

1. Serre's conjectures on ℓ-independence
2. Compatible systems along the boundary
3. Relation with wild ramification

References

- J.-P. Serre, J. Tate. Good reduction of abelian varieties. *Ann. Math.* (1968).
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Serre proposed conjectures C1–C8 related to the definition of the Hasse-Weil zeta functions of projective smooth varieties over global fields.

Arithmetic zeta function

- Riemann zeta function:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

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- Let $\mathcal{X}$ be a scheme of finite type over Spec($\mathbf{Z}$). Arithmetic zeta function:

$$\zeta_{\mathcal{X}}(s) = \sum_{C\in Z_0^{\mathrm{eff}}(\mathcal{X})} \frac{1}{(NC)^s} = \prod_{x\in|\mathcal{X}|} \frac{1}{1-(Nx)^{-s}}$$
$$= \prod_{v\in|V|} Z_{\mathcal{X}_v}((Nv)^{-s})$$

for $\mathcal{X}$ over V of finite type over Spec($\mathbf{Z}$).

Cohomological interpretation

Let X be a variety (= scheme separated of finite type) over a field k. For each $\ell \neq \mathrm{char}(k)$, Grothendieck defined a finite-dimensional $\mathbf{Q}_\ell$-vector space $H^i_{\ell,c} = H^i_c(X_{\bar{k}}, \mathbf{Q}_\ell)$, equipped with a continuous action of $\mathrm{Gal}(\bar{k}/k)$.

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Theorem (Grothendieck)

Let X be a variety over $k = \mathbf{F}_q$. For each $\ell \nmid q$,

$$Z_X(t) = \prod_i P_{i,\ell}(t)^{(-1)^{i+1}},$$

where

$$P_{i,\ell}(t) = \det(1 - \mathrm{Fr}t, H^i_{\ell,c}).$$

Weil conjectures (continued)

Let X be a proper smooth variety over $k = \mathbf{F}_q$.

Theorem (Deligne, C2)

The reciprocal roots of $P_{i,\ell}$ are of weight i (algebraic numbers with all complex conjugates of absolute value $q^{i/2}$).

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Corollary (C1)

$P_{i,\ell} \in \mathbf{Z}[t]$ *and is independent of* ℓ.

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Corollary (C1)

$P_{i,\ell} \in \mathbf{Z}[t]$ and is independent of ℓ.

Corollary

Let X be a proper smooth variety over an arbitrary field k. Then the Betti number $\dim H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$ is independent of $\ell \neq \mathrm{char}(k)$.

Hasse-Weil zeta function

Let X be a proper smooth variety over a global field F.

$$\zeta_X(s) = \prod_i L_i(s)^{(-1)^{i+1}},$$

$$L_i(s) = \prod_v \det(1 - \mathrm{Fr} q_v^{-s}, (H^i_\ell)^{I_v}),$$

where v runs over finite places of F, and I_v denotes the inertia group at v.

ℓ-independence

Let K be a local field: a complete discrete valuation field of finite residue field $\mathbf{F}_q$. Let X be a proper smooth variety over K.

Conjecture

- *(Serre, C5)* $\det(1 - \mathrm{Fr}t, (H^i_\ell)^{I_K}) \in \mathbf{Z}[t]$ *and is independent of* $\ell \nmid q$.

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- *(Serre-Tate, C8) For each lifting* $F \in \mathrm{Gal}(\bar{K}/K)$ *of* Fr, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \nmid q$.

Monodromy Weight Conjecture

Let M denote the monodromy filtration.

Conjecture

Eigenvalues of F lifting Fr on $\mathrm{gr}_n^M H_\ell^i$ are of weight $i+n$.

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C8 + Monodromy Weight Conjecture

$\Rightarrow \det(1 - Ft, \mathrm{gr}_n^M H_\ell^i) \in \mathbf{Z}[t]$ and is independent of ℓ

$\Rightarrow$ C5

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(Monodromy Weight Conjecture $\Rightarrow$ C6 + C7)

General residue field

Let K be a complete discrete valuation field of residue field k. Let X be a proper smooth variety over K.

Conjecture (Serre-Tate, C4)

For each $F \in I_K$, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \neq \mathrm{char}(k)$.

Local monodromy theorem

Let X be a variety over K.

Theorem

- *(Grothendieck) An open subgroup of I_K acts on $H^i_{\ell,c}$ unipotently.*

Local monodromy theorem

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Theorem

- *(Grothendieck) An open subgroup of I_K acts on $H^i_{\ell,c}$ unipotently.*
- *(Deligne, Gabber, Illusie) There exists an open subgroup I' of I_K, independent of ℓ, such that for every $g \in I'$, $(g-1)^{i+1}$ acts by 0 on H^i_ℓ and $H^i_{\ell,c}$.*

Equal characteristic case

Theorem (Deligne, Terasoma, Ito)

Monodromy Weight Conjecture holds in equal characteristic.

Equal characteristic case (continued)

Let K be a complete discrete valuation field of residue field k, both of characteristic $p > 0$. Let X be a proper smooth variety over K.

Theorem

- *(Lu-Z., C4) For each* $F \in I_K$, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \neq p$.
- *(Deligne, Terasoma, Lu-Z., C8) Assume* $k = \mathbf{F}_q$. *For each lifting* $F \in \mathrm{Gal}(\bar{K}/K)$ *of* Fr, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \neq p$.

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- *(Deligne, Terasoma, Lu-Z., C8) Assume $k = \mathbf{F}_q$. For each lifting $F \in \mathrm{Gal}(\bar{K}/K)$ of Fr, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ and is independent of $\ell \neq p$.*

Corollary (C5)

Assume $k = \mathbf{F}_q$. For each lifting $F \in \mathrm{Gal}(\bar{K}/K)$ of Fr, $\det(1 - Ft, (H^i_\ell)^{I_K}) \in \mathbf{Z}[t]$ and is independent of $\ell \nmid q$.

General characteristic: alternating sums

Let X be a variety over a field K.

Theorem

- *(Gabber, C1') Assume* $K = \mathbf{F}_q$. *For each* $F \in W(\bar{K}/K)$, $\sum_i (-1)^i \mathrm{tr}(F, H^i_\ell) \in \mathbf{Q}$ *and is independent of* $\ell \nmid q$.

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- *(Vidal, C4') Assume* K *is a complete discrete valuation field of residue characteristic* $p > 0$. *For each* $F \in I_K$, $\sum_i (-1)^i \mathrm{tr}(F, H^i_\ell) \in \mathbf{Z}$ *and is independent of* $\ell \neq p$.
- *(Ochiai, Z., C8') Assume* K *is a local field of residue field* $\mathbf{F}_q$. *For each* $F \in W(\bar{K}/K)$, $\sum_i (-1)^i \mathrm{tr}(F, H^i_\ell) \in \mathbf{Q}$ *and is independent of* $\ell \nmid q$.

Plan of the talk

Spreading out

Let X be a proper smooth variety over a field F of characteristic $p > 0$. There exists a scheme B of finite type over $\mathbf{F}_p$ and a Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(F) & \longrightarrow & B \end{array}$$

with f proper smooth. We have

$$H^i(X_{\bar{F}}, \mathbf{Q}_\ell) \simeq (R^i f_* \mathbf{Q}_\ell)_{\bar{F}}$$

This leads us to study the system $(R^i f_* \mathbf{Q}_\ell)_\ell$ of (lisse) $\mathbf{Q}_\ell$-sheaves on B.

Compatible systems

Let $\mathcal{O}_K$ be an excellent Henselian discrete valuation ring of residue field $k = \mathbf{F}_q$ (no restriction on the characteristic of the fraction field K). Let X be a scheme of finite type over $S = \mathrm{Spec}(\mathcal{O}_K)$. Let $K(X, \overline{\mathbf{Q}_\ell})$ be the Grothendieck group of $\overline{\mathbf{Q}_\ell}$-sheaves on X. Fix ℓ_i, $i \in I$.

Definition

$(L_i) \in \prod_i K(X, \overline{\mathbf{Q}_{\ell_i}})$ is compatible if for every $x \in |X|$, and every $F \in W(\bar{x}/x)$, $\mathrm{tr}(F, (L_i)_{\bar{x}}) \in \mathbf{Q}$ and is independent of i. Here $|X| := |X_K| \cup |X_k|$ denotes the set of locally closed points of X.

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More general notion with fixed embeddings $Q \hookrightarrow \overline{\mathbf{Q}_{\ell_i}}$.

Gabber's theorem

Theorem (Gabber, Z.)

Over S, compatible systems are preserved by duality and Grothendieck's six operations:

$$f^*, f_*, f_!, f^!, \otimes, R\mathcal{H}om.$$

Local fundamental groups

Let $\bar{C}$ be a smooth curve over $\mathbf{F}_q$ and let $C \subseteq \bar{C}$ be a Zariski dense open. For $x \in \bar{C} \backslash C$, we have $\mathrm{Spec}(K_x) = \bar{C}_{(x)} \times_{\bar{C}} C \to C$, where $\bar{C}_{(x)}$ denotes the Henselization of $\bar{C}$ at x. Short exact sequence:

$$1 \to I_x \to \mathrm{Gal}(\overline{K_x}/K_x) \to \mathrm{Gal}(\bar{x}/x) \to 1.$$

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$$1 \to I_x \to \mathrm{Gal}(\overline{K_x}/K_x) \to \mathrm{Gal}(\bar{x}/x) \to 1.$$

More generally, let $\bar{X}$ be a normal scheme of finite type over S and let $X \subseteq \bar{X}$ be a Zariski dense open. For $x \in \bar{X}$, the open immersion $\bar{X}_{(x)} \times_{\bar{X}} X \subseteq \bar{X}_{(x)}$ induces a surjection

$$\pi_1(\bar{X}_{(x)} \times_{\bar{X}} X) \to \pi_1(\bar{X}_{(x)}) \simeq \mathrm{Gal}(\bar{x}/x).$$

Compatible systems along the boundary

Definition

$(L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ is compatible on $\bar{X}$ if for every $x \in |\bar{X}|$, for every $F \in W(\bar{X}_{(x)} \times_{\bar{X}} X, \bar{a})$ (where $\bar{a}$ is a geometric point), $\operatorname{tr}(F, (L_i)_{\bar{a}}) \in \mathbf{Q}$ and is independent of i.

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Question

Assume $(L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ *compatible on* X*. Is* (L_i) *compatible on* $\bar{X}$*?*

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Question

Assume $(L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible on X. Is (L_i) compatible on $\bar{X}$?

Yes up to stratification or modification.

Compatible ⇒ Compatible along the boundary up to ...

Theorem (Lu-Z.)

Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i\in I} K_{\mathrm{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible with I finite. There exists a partition $X = \bigcup_\alpha X_\alpha$ into locally closed subschemes such that each X_α admits a normal compactification $X_\alpha \subseteq \bar{X}_\alpha$ over S with $(L_i|_{X_\alpha})$ compatible on $\bar{X}_\alpha$.

Compatible $\Rightarrow$ Compatible along the boundary up to ...

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Theorem (Lu-Z.)

Let $\bar{X}$ be a reduced scheme separated of finite type over S and let $X \subseteq \bar{X}$ be a Zariski dense open. Let $(L_i) \in \prod_{i\in I} K_{\mathrm{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible with I finite. There exists a proper birational transformation $f\colon \bar{Y} \to \bar{X}$ such that $(L_i|_{f^{-1}(X)})$ is compatible on $\bar{Y}$.

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Due to Deligne in the case where X is a curve over $\mathbf{F}_q$.

Valuative criterion

Corollary

Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbf{Q}_{\ell_i}})$. Consider commutative squares

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\mathcal{O}_L) & \longrightarrow & S, \end{array}$$

where $\mathcal{O}_L$ is a Henselian valuation ring and $L = \mathrm{Frac}(\mathcal{O}_L)$.

1. $(L_i)_{i \in I}$ *compatible* $\Leftrightarrow$ *for every square with closed point of* $\mathrm{Spec}(\mathcal{O}_L)$ *quasi-finite over* S, $\mathrm{tr}(F, (L_i)_{\bar{L}}) \in \mathbf{Q}$ *and is independent of* ℓ *for all* $F \in W(\bar{L}/L)$.
2. $(L_i)_{i \in I}$ *compatible* $\Rightarrow$ *for every square with* $\mathcal{O}_L$ *strictly Henselian,* $\mathrm{tr}(F, (L_i)_{\bar{L}}) \in \mathbf{Q}$ *and is independent of* ℓ *for all* $F \in \mathrm{Gal}(\bar{L}/L)$.

Serre's conjectures in equal characteristic

Let $\mathcal{O}_L$ be a Henselian (not necessarily discrete) valuation field ring of residue field k and characteristic $p > 0$. Let $L = \mathrm{Frac}(\mathcal{O}_L)$. Let X be a proper smooth variety over L.

Corollary

- *(C4) For each* $F \in I_L$, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \neq p$.
- *(C8) Assume* $k = \mathbf{F}_q$. *For each lifting* $F \in \mathrm{Gal}(\bar{L}/L)$ *of* Fr, $\det(1 - Ft, H^i_\ell) \in \mathbf{Z}[t]$ *and is independent of* $\ell \neq p$.

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The valuative criterion was inspired by Gabber's valuative criterion for the ramified part of π_1.

Plan of the talk

Ramified part of π_1

Let $\mathcal{O}_K$ be an encellent Henselian discrete valuation ring of residue characteristic $p > 0$.

Definition (Vidal)

Let X be a integral normal scheme separated of finite type over $S = \mathrm{Spec}(\mathcal{O}_K)$. Closed subsets $\pi_1^{\mathrm{wr}}(X) \subseteq \pi_1^{\mathrm{r}}(X) \subseteq \pi_1(X)$:

- For any normal compactification $X \subseteq \bar{X}$ over S, $\pi_1^{\mathrm{r}}(X)_{\bar{X}}$ is the closure of the union of the conjugates of $\mathrm{Im}(\pi_1(\bar{X}_{(\bar{x})} \times_{\bar{X}} X) \to \pi_1(X))$, where $\bar{x}$ runs through geometric points of $\bar{X}$.
- (ramified part) $\pi_1^{\mathrm{r}}(X) = \bigcap_{\bar{X}} \pi_1^{\mathrm{r}}(X)_{\bar{X}}$.

Ramified part of π_1

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- For any normal compactification $X \subseteq \bar{X}$ over S, $\pi_1^{\mathrm{r}}(X)_{\bar{X}}$ is the closure of the union of the conjugates of $\mathrm{Im}(\pi_1(\bar{X}_{(\bar{x})} \times_{\bar{X}} X) \to \pi_1(X))$, where $\bar{x}$ runs through geometric points of $\bar{X}$.
- (ramified part) $\pi_1^{\mathrm{r}}(X) = \bigcap_{\bar{X}} \pi_1^{\mathrm{r}}(X)_{\bar{X}}$.
- (wildly ramified part) $\pi_1^{\mathrm{wr}}(X) = \pi_1^{\mathrm{r}}(X) \cap \bigcup_H H$, where H runs through pro-p-Sylows of $\pi_1(X)$.

Gabber's valuative criterion

Theorem (Gabber)

$\pi_1^{\mathrm{r}}(X)$ is the closure of the union of the conjugates of $\mathrm{Im}(\mathrm{Gal}(\bar{L}/L) \to \pi_1(X))$, indexed by commutative squares

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\mathcal{O}_L) & \longrightarrow & S \end{array}$$

where $\mathcal{O}_L$ is a strictly Henselian valuation ring.

Compatible wild ramification

Let X be a scheme of finite type over S.

Definition

$(L_i) \in \prod_{i\in I} K(X, \overline{\mathbf{F}_{\ell_i}})$ has compatible wild ramification if for every separated integral normal subscheme Y and every $g \in \pi_1^{\mathrm{wr}}(Y, \bar{a})$ (where $\bar{a}$ is a geometric point), $\mathrm{tr}^{\mathrm{Br}}(g, (L_i)_{\bar{a}}) \in \mathbf{Q}$ and is independent of ℓ (as long as $L_i \in K_{\mathrm{lisse}}$).

Saito-Yatagawa and Yatagawa studied a weaker condition “same wild ramification”.

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Saito-Yatagawa and Yatagawa studied a weaker condition "same wild ramification".

Theorem (Deligne, Vidal, Saito-Yatagawa, Yatagawa, Guo)

- *"Compatible wild ramification" is preserved by* $f^*, f_*, f_!, f^!, \otimes, R\mathcal{H}om$.
- *"Same wild ramification" is preserved by* $f^*, f_*, f_!, f^!$.

Compatible $\Rightarrow$ Compatible wild ramification

Assume that the residue field of $\mathcal{O}_K$ is finite. The decomposition map d_ℓ is the composition

$$K(X, \overline{\mathbf{Q}_\ell}) \xleftarrow{\sim} K(X, \overline{\mathbf{Z}_\ell}) \to K(X, \overline{\mathbf{F}_\ell}),$$

where both arrows are given by extension of scalars. Combining Gabber's valuative criterion with ours, we get:

Corollary

$(L_i) \in \prod_i K(X, \overline{\mathbf{Q}_{\ell_i}})$ *compatible* $\Rightarrow (d_{\ell_i} L_i) \in \prod_i K(X, \overline{\mathbf{F}_{\ell_i}})$ *has compatible wild ramification.*

The End

Thank you!